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Concise Fuzzy System Modeling Integrating Soft Subspace Clustering and Sparse Learning

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***Abstract*—The superior interpretability and uncertainty modeling ability of Takagi-Sugeno-Kang fuzzy system (TSK FS) make it possible to describe complex nonlinear systems intuitively and efficiently. However, classical TSK FS usually adopts the whole feature space of the data for model construction, which can result in lengthy rules for high-dimensional data and lead to degeneration in interpretability. Furthermore, for highly nonlinear modeling task, it is usually necessary to use a large number of rules which further weakens the clarity and interpretability of TSK FS. To address these issues, a concise zero-order TSK FS construction method, called ESSC-SL-CTSK-FS, is proposed in this paper by integrating the techniques of enhanced soft subspace clustering (ESSC) and sparse learning (SL). In this method, ESSC is used to generate the antecedents and various sparse subspace for different fuzzy rules, whereas SL is used to optimize the consequent parameters of the fuzzy rules, based on which the number of fuzzy rules can be effectively reduced. Finally, the proposed ESSC-SL-CTSK-FS method is used to construct concise zero-order TSK FS that can explain the scenes in high-dimensional data modeling more clearly and easily. Experiments are conducted on various real-world datasets to confirm the advantages.**

***Index Terms*—Interpretability; high-dimensional data; sparse learning; TSK fuzzy system; enhanced soft subspace clustering;**

I. Introduction`

Fuzzy system (FS) is a kind of rule-based systems which use fuzzy logic and fuzzy inference to realize knowledge representation and uncertain inference. The core part of FS is the knowledge base which is composed of IF-THEN fuzzy rules. FS has better ability in handling uncertainty than other rule-based systems. It can transform vague human language into fuzzy rules and simulate the uncertain inference. In various existing FSs, TSK FS [1-4] is the one that is most commonly used because of two major advantages: the intrinsic interpretability of its rule-based form and the data-driven learning ability.

Most of the recent researches focus on the improvement of the learning ability of TSK FS. However, as model complexity increases, the interpretability of TSK FS decreases despite increase in learnability [5-7]. It is therefore imperative to revisit the interpretability of TSK FS, which is among the emerging topics on the interpretabilities of machine learning methods [8]. There are mainly two classes of interpretable methods in the field of machine learning [9]. The first class is intrinsically interpretable models, where linear models, decision trees and rule-based systems are common models of this class. The second class is post-hoc interpretability methods that usually applies interpretable methods to extract information from the learned black-box models. Common models of the second class include visualizations [10], explanations by examples [11] and model-agnostic methods. The TSK FS concerned in this paper belongs to the first class. On the other hand, it is noteworthy to highlight that TSK-FS has stronger learning abilities than other types of FSs and is more capable of dealing with uncertain information than other non-fuzzy rule-based systems [12].

As pointed out in [9], given the limited capacity of human cognition, neither rule-based systems nor decision trees are interpretable with increasing complexity of the models. While TSK FS is interpretable for its rule-based form, the interpretability can be reduced to a great extent as model complexity increases. There are two major factors that could increase the model complexity of TSK FS: 1) *High-dimensional features*: classical data-driven TSK FSs usually use all input features to generate fuzzy rules. In fact, only part of the features are useful for certain fuzzy rules. Besides, fuzzy rules generated using all the features would be very long, and the corresponding linguistic descriptions are tedious, which reduces the interpretability of the fuzzy models. 2) *Excessive rules*: To achieve good performance, many rules are usually required to construct TSK FS, which inevitably increase the model complexity. In fact, some rules may be redundant and can be removed without losing modeling performance. Thus, concise rule base is necessary to improve the interpretability.

To deal with the abovementioned issues, a more interpretable TSK FS construction method is proposed in the paper. The method first introduces an enhanced soft subspace clustering (ESSC) [13] method to perform feature reduction for each fuzzy rule, where ESSC can select different feature subsets for different fuzzy rules. Then, sparse learning (SL) [14] technique is used to optimize the consequent parameters of fuzzy rules to perform rule reduction. Finally, a more concise TSK FS is obtained with the number of features and fuzzy rules

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reduced simultaneously. The proposed is thus called enhanced soft subspace clustering and sparse learning based concise zero-order TSK FS, abbreviated as ESSC-SL-CTSK-FS. It has the following advantages:

1) The TSK FS is trained using ESSC and SL simultaneously so that the fuzzy rules are more linguistically descriptive and the rule base is more compact, thereby increasing the interpretability of the model.

2) The fuzzy model constructed possesses a more human-like inference mechanism that different feature subsets are used in different rules by the ESSC for fuzzy inference, which is analogous to making a decision based on the views of different experts.

3) The ESSC clustering algorithm avoids the use of noisy features in the original feature space by unsupervised learning. Therefore, the proposed method is more robust in handling high-dimensional data where noisy features may be embedded in the original feature space.

The rest of this paper is organized as follows. In section II, the related work on the interpretability of fuzzy systems is briefly reviewed. Zero-order TSK FS, ESSC and SL are then introduced in Section III. In section IV, the proposed method is discussed in detail. The experimental studies are reported in section V. Finally, conclusions and future work are given in Section VI.

II. Related Work

A. Interpretability

As discussed in Section I, intrinsically interpretable models are an important class of interpretable methods. *Transparency* is a key property of these models, which can be considered from three levels [9], i.e., *simulatability* at the level of the entire model, *decomposability* at the level of individual components, and *algorithmic transparency* at the level of the training algorithm. For the TSK FS considered in the paper, the simulatability is clearly evident from the rule-based human-like inference mechanism [15]. The decomposability is also evident from the characteristic that all the components of the TSK FS can provide intuitive explanations. The training algorithm of the TSK FS can converge to a unique solution [1], which exhibits the property of algorithmic transparency of the interpretable models.

Besides investigations on the properties that constitute the transparency of interpretable models as discussed above, another line of research on interpretability is post-hoc explanations of black-box models (e.g., deep neural networks). Here, neural network visualization is an important research direction [10]. Previous researches in [16] and [17] both visualize the individual units of neural networks to understand their representations. In [18] and [19], methods are proposed to disentangle the representations and quantify the interpretability of neural networks. Some researches attempt to interpret the models at the level of examples [20]. Model-agnostic methods are also proposed to improve the interpretability of machine learning algorithms, such as local surrogate models [21] and influence functions [22].

B. Fuzzy Systems

Research has been conducted to improve the interpretability of TSK FS with concise rule base. In [23], Type-2 hierarchical fuzzy system (T2HFS) is proposed for handling high-dimensional data, where principle component analysis (PCA) is used for feature extraction. However, the interpretability is weakened since the physical meaning of the original features is destroyed by PCA. In [24], genetic algorithm and integer programming are integrated to propose a fuzzy rule classifier algorithm (hGA) to tackle the precision and rule reduction problems in the classification of high-dimensional data. The method does not consider the reduction in interpretability caused by high dimensionality. In [15, 25, 26], subsets of features have been used in different rules of TSK FS, but the selection of features for the subsets is often conducted randomly which decreases the effectiveness of the constructed model. In [15], a TSK FS called ETSK-FS is proposed to construct more concise fuzzy model for high-dimensional data by using the extracted feature subsets for different fuzzy rules. However, the removal of redundant rules is not addressed. In summary, it is necessary to develop adaptive methods to improve the performance of TSK FS, both the prediction accuracy and the model conciseness.

III. Backgrounds

A. Zero-order TSK-FS

TSK fuzzy logic system (TSK FS) [27] is a classical fuzzy inference model that has been widely applied because of its flexibility and performance. In this paper, the commonly used zero-order TSK FS is investigated due to its simplicity. For a zero-order TSK FS, the fuzzy rules are defined as follows [28].

$$\begin{aligned}&\text{IF } x_1 \text{ is } A_1^k \wedge x_2 \text{ is } A_2^k \wedge \cdots \wedge x_d \text{ is } A_d^k,\\&\text{THEN } y^k(\mathbf{x}) = p^k, \qquad k = 1,\cdots,K.\end{aligned} \tag{1}$$

The kth rule in the fuzzy rule base is given in (1), where A_i^k is the fuzzy subset associated with the ith feature, p^k is the consequent parameter, y^k is the output of this rule, $k=1,2,...,K$, and K is the number of fuzzy rules in the rule base. For an input vector $\mathbf{x}=(x_1,x_2,...,x_d)^{\mathrm{T}}$, when multiplication is adopted for conjunction and implication, addition for combination, and the center of gravity for defuzzification, the output of the zero-order TSK FS, i.e., $f(\mathbf{x})$, can be expressed as

$$f(\mathbf{x}) = \sum_{k=1}^{K} \frac{u^k(\mathbf{x})}{\sum_{k=1}^{K} u^k(\mathbf{x})} y^k(\mathbf{x}) = \sum_{k=1}^{K} \tilde{u}^k(\mathbf{x}) y^k(\mathbf{x}) . \tag{2}$$

In (2), $u^k(\mathbf{x})$ and $\tilde{u}^k(\mathbf{x})$ are commonly called the firing strength and the normalized firing strength respectively, which are given by

$$u^k(\mathbf{x}) = \prod_{i=1}^{d} u_{A_i^k}(x_i) \text{ and} \tag{3}$$

$$\tilde{u}^k(\mathbf{x}) = u^k(\mathbf{x}) / \sum_{k=1}^{K} u^k(\mathbf{x}) , \tag{4}$$

where $u_{A_i^k}(\cdot)$ is the membership function of fuzzy subset A_i^k associated with the ith feature in the kth rule. Gaussian function is commonly used as the membership function [28], i.e.,

$$u_{A_i^k}(x_i)=\exp\left(\frac{-(x_i-v_i^k)^2}{2\sigma_i^k}\right). \tag{5}$$

The parameters v_i^k and σ_i^k in (5) can be estimated with different strategies, such as clustering techniques. If fuzzy c-mean (FCM) is used, they can be obtained by

$$v_i^k=\sum_{j=1}^{N}u_{jk}x_{ji}\Big/\sum_{j=1}^{N}u_{jk} \text{ and} \tag{6}$$

$$\sigma_i^k=h\sum_{j=1}^{N}u_{jk}(x_{ji}-v_i^k)^2\Big/\sum_{j=1}^{N}u_{jk}, \tag{7}$$

where u_{jk} represents the membership of the input vector $\mathbf{x}_j=(x_{j1},x_{j2},...,x_{jd})^{\mathrm{T}}$ of the kth cluster and h is a manually adjustable scale parameter [29, 30].

Once the antecedent parameters in the zero-order TSK FS are determined, the output can be expressed as a linear model in a mapping new feature space as follows.

$$y=f(\mathbf{x})=\mathbf{p}_g^{\mathrm{T}}\mathbf{x}_g \tag{8}$$

$$\mathbf{x}_g=[\tilde{u}^1(\mathbf{x}),\tilde{u}^2(\mathbf{x}),...,\tilde{u}^K(\mathbf{x})]^{\mathrm{T}} \tag{9}$$

$$\mathbf{p}_g=[p^1,p^2,...,p^K]^{\mathrm{T}} \tag{10}$$

B. Soft Subspace Clustering

Clustering has been widely applied for fuzzy modeling. It is effective for partitioning the training data to generate the appropriate space partition. Based on the partition results, the corresponding parameters of the antecedents and/or consequents in the fuzzy rules can be estimated. For example, a fuzzy rule can be generated based on a group in the clustering results. Traditional clustering algorithms, such as K-means [31] and fuzzy c-means FCM [32], have been used in many fuzzy modeling methods to determine the space partition and generate fuzzy rules. However, these clustering methods usually generate fuzzy rules using the same feature space, which is not reasonable for many practical applications. For a rule generated based on the knowledge of an expert, the rule only contains a certain feature subset that is associated with the view of the expert. When multiple experts are involved, it is therefore more appropriate for the different rules to be associated with different feature subsets. To this end, adaptive clustering techniques are needed. One example is the soft subspace clustering (SSC) technique [13] which is originally proposed to effectively cluster high-dimensional data. A distinctive characteristic of SSC is that it can identify different groups and determine the importance of the features in each group simultaneously. The optimization objective function of different SSC algorithms can be expressed with the generalized form

$$J(\mathbf{U},\mathbf{V},\mathbf{W})=f_1(\mathbf{U},\mathbf{V},\mathbf{W})+\Delta, \tag{14}$$

where $\mathbf{U}$ is a partition matrix of data, $\mathbf{V}$ is a matrix containing K center vectors, $\mathbf{W}$ is a weight matrix that characterizes the importance of features in different clusters. The objective function consists two terms, the loss term and the regularization term,

Theoretically, the more compact a cluster is in some dimensions of the feature space, the larger the corresponding weights. Thus, important feature subsets can be identified based on the clustering results of the SSC algorithms, which can be adopted to generate the fuzzy rules in different feature subspace.

C. Lasso Sparse Learning

Sparse learning has received increasing attention in recent years for intelligent modeling. Lasso is a classical sparse learning method for linear regression model [33]. Its aim is to obtain a sparse vector as the solution to a linear model such that the final decision function of the linear model is concise. Given a training dataset $\{\mathbf{x}_i,y_i\}$, $i=1,2,...,n$ for a linear model $f(\mathbf{x})=\mathbf{w}^{\mathrm{T}}\mathbf{x}$, the optimization objective of the Lasso algorithm is given by

$$\min_{\mathbf{w}}\sum_{i=1}^{n}(y_i-\mathbf{w}^{\mathrm{T}}\mathbf{x}_i)^2+\lambda\|\mathbf{w}\|_1, \tag{15}$$

where $\|\mathbf{w}\|_1$ is the ℓ_1 norm of $\mathbf{w}$ and $\lambda\ (>0)$ is the regularization parameter. The ℓ_1 norm of $\mathbf{w}$ is introduced into the learning criterion to effectively reduce over-fitting which may occur in dealing with small high-dimensional dataset. Besides, it is apt to obtain a sparse solution for the linear model.

IV. Concise TSK FS Construction using ESSC and SL

In this paper, a concise TSK FS is constructed using SSC and SL techniques. The fuzzy rules of the concise TSK FS are designed as follows,

$$\begin{aligned}&\text{IF } x_1^k \text{ is } A_1^k \wedge x_2^k \text{ is } A_2^k \wedge\cdots\wedge x_{m_k}^k \text{ is } A_{m_k}^k,\\&\text{THEN } y^k=f^k(\mathbf{x}^k),\qquad k=1,\cdots,K.\end{aligned} \tag{16}$$

where K is the number of rules, $\mathbf{x}^k=(x_1^k,x_2^k,...,x_{m_k}^k)\in R^{m_k}$ is a vector containing m_k features that are extracted from the full d features of the original input vector $\mathbf{x}$, A_i^k is the fuzzy subset associated with feature x_i^k in the kth fuzzy rule, $f^k(\mathbf{x}^k)$ is the consequent output, which is a constant in the adopted zero-order TSK FS, i.e., $f^k(\mathbf{x}^k)=p^k$.

A. Antecedent Parameter Estimation using ESSC

Many SSC algorithms have been proposed to cluster high-dimensional data and to find the important subspace for each group. Unlike most existing SSC algorithms that only focus on the within-cluster compactness, ESSC exhibits distinctive advantage that it considers not only the within-cluster compactness but also the between-cluster separation simultaneously [13]. Given the training data $\mathbf{X}=[\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_N]$ and corresponding labels $\mathbf{y}=[y_1,y_2,\cdots y_N]^{\mathrm{T}}$, where $\mathbf{x}_i\in R^d$ and $y_i\in R$, the objective function of the ESSC algorithm is

$$J_{\text{ESSC}}(\mathbf{U},\mathbf{V},\mathbf{W})=\sum_{i=1}^{K}\sum_{j=1}^{N}u_{ij}^{m}\sum_{k=1}^{d}w_{ik}(x_{jk}-v_{ik})^2+\varepsilon\sum_{i=1}^{K}\sum_{k=1}^{d}w_{ik}\ln w_{ik}$$
$$-\eta\sum_{i=1}^{K}(\sum_{j=1}^{N}u_{ij})\sum_{k=1}^{d}w_{ik}(v_{ik}-v_{0k})^2 \quad , \quad (17)$$
$$\text{s.t. } 0\le u_{ij}\le 1,\ \sum_{i=1}^{K}u_{ij}=1,\ 0<\sum_{j=1}^{N}u_{ij}<N,\ 0\le w_{ij},\ \sum_{k=1}^{D}w_{ik}=1$$

where $\mathbf{V}=[\mathbf{v}_1,...,\mathbf{v}_K]_{d\times K}$ and $\mathbf{W}=[\mathbf{w}_1,...,\mathbf{w}_K]_{d\times K}$ are the matrix of clustering center and the weighting matrix respectively, and $\mathbf{U}=[\mathbf{u}_1,...,\mathbf{u}_N]_{K\times N}$ is the fuzzy partition matrix. K,N,d are the number of clusters, the number of samples and the number of features respectively. The objective function of ESSC consists of three terms which correspond to the weighted within-cluster compactness, regularization of feature weights, and the weighted between-cluster separation respectively. The first and second terms directly inherit from the objective function of the classical SSC algorithm FSC [34]. The parameters ε, η are used to balance the influence of the different terms.

As a state-of-the-art SSC algorithm, ESSC is very effective for partitioning high-dimensional data, where the corresponding distribution of the importance of the different groups can be also be obtained. In this study, ESSC is used to generate the antecedents of the zero-order TSK FS so that more elastic fuzzy rules with different feature subsets can be constructed. The procedure of the antecedent generation is described as follows.

For a given training dataset, the ESSC method divides the training examples into K groups that correspond to the K fuzzy rules; the weight vector $\mathbf{w}_k=[w_{k1},\cdots,w_{kd}]$ corresponds to the contribution of all the features to the kth group. Given the threshold $\beta\in(0,1)$, for the kth rule, the features with the weights $w_{kl}>\beta$ are selected. When the feature subsets for each fuzzy rule are determined, the antecedent parameters, i.e., the parameters v_i^k and σ_i^k in the Gaussian membership function, can be estimated using (6) and (7) as described in Section II-A.

B. Consequent Parameter Estimation and Rule Reduction using SL

After generating the antecedents, the consequent parameters of the fuzzy rules of the zero-order TSK FS can be estimated based on linear model optimization techniques. According to the descriptions in Section II-A, we construct the following objective function for optimization,

$$\min_{\mathbf{p}_g}\sum_{i=1}^{N}(y_i-\mathbf{p}_g^{\mathrm{T}}\mathbf{x}_{i,g})^2\ , \qquad (18)$$

where $\mathbf{x}_{i,g}\in R^K$ is the input vector in the new feature space, which is mapped from the input vector $\mathbf{x}_i\in R^d$ in the original feature space through (9) with the selected features obtained by ESSC, and $\mathbf{p}_g$ is the linear model parameters in the new feature space, which is composed of the consequent parameters of all the fuzzy rules. Each element in $\mathbf{p}_g$ is associated with a fuzzy rule.

For zero-order TSK FS, the number of fuzzy rules may be large and the resulting model would become more complicated and less interpretable. However, a concise model for TSK FS is generally expected in practical applications. To this end, Lasso algorithm [33] is introduced for both consequent parameters estimation and fuzzy rules reduction. The objective function is given by

$$\min_{\mathbf{p}_g}\frac{1}{2}\left\|\mathbf{X}_g\mathbf{p}_g-\mathbf{y}\right\|^2+\frac{\lambda}{2}\left\|\mathbf{p}_g\right\|_1 . \qquad (19)$$

In (19), $\mathbf{X}_g=[\mathbf{x}_{1,g},\mathbf{x}_{2,g},\cdots,\mathbf{x}_{N,g}]^{\mathrm{T}}$ is obtained by concatenating the data from all the examples in the new feature space. $\left\|\mathbf{p}_g\right\|_1$ denotes the ℓ_1 norm of $\mathbf{p}_g$, which is opt to produce a sparse solution vector, i.e., some elements of $\mathbf{p}_g$ is reduced to zero; λ is a regularization parameter to control the influence of $\left\|\mathbf{p}_g\right\|_1$. When λ is set appropriately, the solution $\mathbf{p}_g^*$ only contains a small number of nonzero elements, i.e., the consequent parameters of many rules in the zero-order TSK FS are zero. Furthermore, when the consequent parameter of a fuzzy rule is equal to zero, the rule can be removed. Note that the objective function in (19) is non-smooth due to the non-smoothness of the ℓ_1 norm regularization term. The accelerated proximal gradient descent method is used to solve this problem, where the solution $\mathbf{p}_g=[p^1,p^2,...,p^K]^{\mathrm{T}}$ can be learned with the following update rules [35]

$$p_{g_s}=\begin{cases}p_{g_s}-\gamma, & p_{g_s}>\gamma\\ 0, & \left|p_{g_s}\right|\le\gamma\\ p_{g_s}+\gamma, & p_{g_s}<-\gamma\end{cases},\qquad (20)$$
$$s=1,2,\cdots,K$$

where $\mathrm{sgn}(.)$ is the sign function and γ is a positive constant. Here, γ is obtained by $\gamma=\lambda/L$ and L is the Lipschitz constant which can be calculated using the approach in [35]. The optimization procedure in (20) is iterative and the initial $\mathbf{p}_g^*$ can be obtained as an analytic solution to $\min_{\mathbf{p}_g}\left\|\mathbf{X}_g\mathbf{p}_g-\mathbf{y}\right\|^2$. The iteration is repeated until $\mathbf{p}_g$ converges.

C. The ESSC-SL-CTSK-FS Algorithm

The ESSC-SL-CTSK-FS algorithm proposed in this paper to improve the interpretability of FS is presented in Table I. First, ESSC is used to select important subspace for the antecedents. Lasso sparse learning is then used to learn the consequent parameters and reduce the number of rules.

D. Multi-Class Classification

TSK FSs have been widely applied for regression and classification. In most cases, fuzzy regression models can be constructed easily, whereas fuzzy classification problems are solved using fuzzy regression models with multiple outputs. Given a multi-class data set $D=\{\mathbf{x}_j, y_j\}, y_i\in\{1,...,m\}$, a regression dataset with multiple outputs $\{\mathbf{x}_i,\tilde{\mathbf{y}}_i\}$ can be constructed, where $y_i\in R^m$ is the output vector for the ith sample. If the label of the ith sample is p $(1\le p\le m)$, y_i is encoded as $\tilde{y}_i=[0,...,\underset{p}{1},0,...,0]^T$, where the pth element of y_i is 1 and the remaining elements are 0. In this way, the original classification problem is transformed into m regression problems and m fuzzy regression models can thus be built. For a test sample, the outputs of the m regression models can be encoded as $\tilde{y}_i^{model}=[y_{i,1}^{model},...,y_{i,m}^{model}]^T$ which is assigned to the class corresponding to the largest output element in y_i^{model}.

E. Complexity Analysis

1) Model Complexity

The model complexity is evaluated by the number of model parameters in the final model. In this paper, the trained zero-order TSK FS contains antecedent parameters v_i^k, σ_i^k in the fuzzy membership functions, and the consequent parameter p^k. The antecedent parameters and the non-zero consequent parameters determine the complexity of the final fuzzy model. For traditional zero-order TSK, the number of parameters in the final model is $(2d+1)K$, where d is the number of features, and K is the number of fuzzy rules. For the proposed ESSC-SL-CTSK-FS, the trained zero-order TSK FS has more elastic rules, i.e., different feature subsets are used for the antecedents in different rules . Note that the consequent parameters of some rules may be zero after Lasso sparse learning. Hence, the number of final rules is smaller than K. Let m_k be the number of selected features for the kth rule, the maximum model complexity is $\sum_{k=1}^{K}(2m_k+1)$.

2) Time Complexity

The proposed ESSC-SL-CTSK-FS algorithm includes two major steps: the acquisition of antecedent parameters of the fuzzy rules using ESSC, and the learning of the consequent parameters using Lasso algorithm. In the first step, the time complexity of ESSC is $O(TNKd)$ where T, N, K and d are the number of iterations, data, clusters and features respectively. The second step is essentially a classical Lasso problem that can be solved by many existing methods [14, 36, 37]. In this paper, the accelerated proximal gradient method [35] is adopted and the time complexity is detailed in [38, 39]. Conclusively, the time complexity of ESSC-SL-CTSK-FS is $O(TNKd+N)$.

TABLE I
FLOWCHART OF THE ESSC-SL-CTSK-FS ALGORITHM

Algorithm: **ESSC-SL-CTSK-FS**

Input: the number of fuzzy rules K; the adjustment parameter h in the Gaussian membership function; the weight threshold β for feature selection; the regularization parameter η in (17); and the training data $D_{tr}=\{x_i,y_i\}$.

Output: antecedent v_i^k and σ_i^k, consequent parameters $\mathbf{p}_g$.

Procedure MV-ITCC:

Stage 1: Generation of concise antecedents using ESSC

Step 1: Implement ESSC on the input dataset $\{\mathbf{x}_i\}$. Divide $\{\mathbf{x}_i\}$ into K clusters and obtain the partition matrix $\mathbf{U}$. Set the cluster center matrix $\mathbf{V}$ and the feature weight matrix $\mathbf{W}$.

Step 2: Match each cluster to a fuzzy rule. Determine the importance of the features for each rule using $\mathbf{W}$ and β.

Step 3: Estimate the parameters of the fuzzy membership functions with (6) and (7).

Stage 2: Consequent parameter learning and rule reduction using LASSO

Step 4: Set the regularization parameter λ in (19).

Step 5: Solve the optimal consequent parameters using (20).

Stage 3: TSK FS construction

Step 6: The final TSK FS is constructed using the antecedent and consequent parameters obtained in Stage 1 and Stage 2.

V. EXPERIMENTS

A. Experimental Setup

To evaluate the effectiveness, the proposed ESSC-SL-CTSK-FS is compared with several classical fuzzy models, including TSK fuzzy classification (TSK-FC) [40], L2-norm penalty-based TSK FS (L2-TSK-FS) [1], TSK FS based on IQP optimization (TSK-IQP) [1], TSK FS based on LSSLI optimization (TSK-LSSLI) [1], and PCA feature extraction based L2-TSK-FS (TSK-FS-PCA) [1].

For all the methods under comparison, the hyper-parameters are optimized using five-folds cross-validation and grid search strategy. Since all the methods are FS based, their parameters are almost the same. The following arrangements are made for optimal parameter setting. The number of fuzzy rules is set to 30. The scale parameter in the Gaussian function is set using the search grid $\{0.01,0.1,1,10,100\}$. The regularization parameter is set using the search grid $\{2^{-10},2^{-9},\cdots 2^{0},\cdots,2^{9},2^{10}\}$. For the proposed ESSC-SL-CTSK-FS, the threshold parameter for feature selection is set using the search grid $\{0.1,0.15,0.2,0.25,0.3\}$ with a step size of 0.05; the regularization parameters ε and η of ESSC in (10) are set using the search grids $\{0.01,0.1,1,10,100\}$ and $\{0.01,0.05,0.1,0.3,0.5\}$ respectively. The regularization parameter λ of SL in (19) is set using the search grid $\{0.1,0.2,\cdots,0.8,0.9\}$.

B. Datasets

Ten real-world medical datasets from the UCI Repository are adopted for performance comparison. In the experiments, all the features in the input vector of the samples are normalized to the interval [0,1]. Table II gives the details of the datasets, including the name of the datasets, the number of samples and features in the dataset, the number of classes and the number of samples in each class.

TABLE II
TEN REAL-WORLD MEDICAL DATASETS

Index	Datasets	Samples	Features	Classes*
1	Breast[a]	699	9	2 (458/241)
2	WDBC[b]	569	30	2 (357/212)
3	WPBC[c]	198	33	2 (47/151)
4	Heart Disease	303	13	2 (139/164)
5	Statlog (Heart)	270	13	2 (150/120)
6	SPECT Heart	267	22	2 (55/212)
7	SPECTF Heart	267	44	2 (55/212)
8	Hepatitis	155	19	2 (32/123)
9	Kidney Disease	400	24	2 (150/250)
10	Thyroid	215	5	3 (150/35/30)

[a]Breast: Wisconsin Original Breast Cancer
[b]WDBC: Wisconsin Prognostic Breast Cancer;
[c]WPBC: Wisconsin Prognostic Breast Cancer.
^The number of samples in each of the classes is given inside the bracket, separated by slash.

C. Evaluation Index

The proposed ESSC-SL-CTSK-FS is evaluated from the perspectives of classification accuracy and model complexity.

1) *Classification Performance*

Five metrics are adopted to evaluate the classification performance, i.e., accuracy, precision (P), recall (R), F-measure and rand index (RI). Accuracy is defined based on the four elements of the confusion matrix, i.e., true positive (TP), false positive (FP), true negative (TN) and false negative (FN), and is given as follows.

$$Accuracy = (TP + TN)/(TP + FP + FN + TN) \tag{21}$$

Since accuracy cannot reflect the situation of imbalanced data [41], F-measure is also adopted. As shown in (22), F-measure combines P and R to provide more insight into the functionality of a classifier.

$$P = TP/(TP + FP) \tag{22a}$$

$$R = TP/(TP + FN) \tag{22b}$$

$$F\text{-}Measure = (2 \times P \times R)/(P + R) \tag{22c}$$

Furthermore, the metric RI is also adopted, which is more robust against imbalanced datasets [42]. RI can be calculated using (23), where a denotes the number of pairs of examples belonging to the same original class that are classified to the same predicted class; b denotes the number of pairs of examples belonging to the different original classes that are classified to different predicted classes. The denominator of (23) is the number of all possible pairs generated by classification.

$$RI = \frac{a + b}{N(N-1)/2} \tag{23}$$

The proposed method is compared with the other algorithms based on the five metrics. The results are presented in terms of the mean and standard deviation of five-fold cross-validation.

2) *Model Complexity*

The model complexity is evaluated by the number of model parameters involved in the resulting model. In the experiments, zero-order TSK FS model is concerned for all the algorithms under comparison. The complexity of the proposed method has been detailed in Section-III-E-1 and the model complexity of the other methods are analyzed as follows.

For TSK-FC, TSK-IQP and TSK-LSSLI, they can be used directly for binary classification. There are $2dK$ parameters for the center v_i^k and the kernel width σ_i^k of the Gaussian membership function in the antecedent part, and K parameters for p^k in the consequent part. Therefore, the number of parameters of these three FSs is $(2d+1)K$ for binary classification. Based on the strategy adopted for multi-class classification, there are m zero-order TSK FSs which share the same antecedent part. Hence, the number of parameters for these three models is $(2d+m)K$ when they are used for multi-class classification.

For L2-TSK-FS and TSK-FS-PCA, they can be used directly for regression. When they are used for classification, there will be m zero-order TSK FSs. Hence, the number of parameters of these two models is $m(2d+1)K$.

D. *Classification Performance Evaluation*

Fig.1 shows the mean of precision and recall of the six methods under comparison on the ten datasets. It can be seen that the proposed method outperforms the other algorithms in terms of both metrics. The accuracy of the methods is given in Table III. The results show that accuracy of the proposed method is better than, or at least comparable with that of the other methods. Furthermore, the results of F-measure and RI are given in Tables IV and V. The proposed method also shows superiority even on imbalanced datasets.

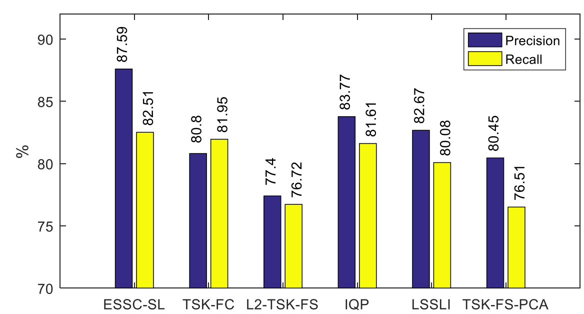


Fig.1. The mean of precision and recall on the ten datasets for the six methods.

TABLE III
CLASSIFICATION ACCURACY OF DIFFERENT METHODS ON TEN REAL-WORLD MEDICAL DATASETS

	TSK-FC	L2-TSK-FS	TSK-IQP	TSK-LSSLI	TSK-FS-PCA	ESSC-SL-TSK
Breast	96.30±.0034	96.44±.0029	96.50±.0025	96.58±.0034	96.50±.0014	**97.14±.0113**
WDBC	91.39±.0017	91.88±.0039	**96.31±.0052**	96.10±.0074	86.17±.0049	95.96±.0148
WPBC	75.89±.0049	76.42±.0035	77.52±.0204	77.64±.00202	76.58±.0099	**81.33±.0408**
Heart Disease	78.91±.0089	82.25±.0084	**82.97±.0112**	82.31±.0144	77.88±.0078	82.50±.0398
Statlog (Heart)	79.59±.0146	83.26±.0106	81.74±.0106	80.85±.0112	79.63±.0152	**83.70±.0591**
SPECT Heart	83.64±.0032	84.35±.0016	82.73±.0173	82.25±.0101	84.35±.0024	**85.39±.0530**
SPECTF Heart	79.40±.0002	79.42±.0002	80.83±.0067	80.19±.0060	79.42±.0002	**80.95±.0760**
Hepatitis	81.03±.0153	79.61±.0062	81.42±.0291	81.61±.0267	80.97±.0123	**85.16±.0489**
Kidney Disease	93.10±.0027	93.43±.0053	94.08±.0076	93.50±.0076	92.75±.0000	**97.75±.0205**
Thyroid	95.16±.0085	95.86±.0056	96.19±.0037	96.00±.0073	96.44±.0040	**97.21±.0255**
Average	85.44±.0063	86.29±.0048	87.03±.0114	86.70±.0111	85.07±.0058	**88.71±.0389**

TABLE IV
CLASSIFICATION F-MEASURE OF DIFFERENT METHODS ON TEN REAL-WORLD MEDICAL DATASETS

	TSK-FC	L2-TSK-FS	TSK-IQP	TSK-LSSLI	TSK-FS-PCA	ESSC-SL-TSK
Breast	96.04±.0034	96.14±.0034	96.18±.0027	96.23±.0036	96.20±.0016	**96.90±.0119**
WDBC	90.67±.0026	91.40±.0042	**96.11±.0043**	95.98±.0080	86.51±.0043	95.74±.0151
WPBC	58.50±.0203	47.29±.0453	66.31±.0397	65.74±.0413	51.58±.0577	**69.52±.0931**
Heart Disease	78.57±.0098	82.25±.0079	**82.99±.0113**	82.19±.0138	77.76±.0083	82.57±.0379
Statlog (Heart)	79.39±.0628	83.19±.0129	81.41±.0100	80.64±.0162	79.61±.0135	**83.66±.0556**
SPECT Heart	73.61±.0057	72.84±.0143	70.43±.0350	68.03±.0496	72.94±.0079	**78.52±.0647**
SPECTF Heart	**67.92±.0278**	44.24±.0001	67.52±.0042	58.48±.0355	61.17±.0172	62.78±.0884
Hepatitis	73.79±.0133	61.94±.0426	70.92±.0195	**75.14±.0507**	64.66±.0438	72.77±.1048
Kidney Disease	92.84±.0030	93.00±.0052	93.91±.0077	93.38±.0079	92.58±.0008	**97.65±.0216**
Thyroid	93.64±.0135	94.14±.0067	94.81±.0091	94.75±.0112	96.33±.0075	**96.90±.0303**
Average	80.50±.0162	76.64±.0143	82.06±.0144	81.04±.0238	77.93±.0163	**83.70±.0523**

TABLE V
RAND INDEX OF DIFFERENT METHODS ON TEN REAL-WORLD MEDICAL DATASETS

	TSK-FC	L2-TSK-FS	TSK-IQP	TSK-LSSLI	TSK-FS-PCA	ESSC-SL-TSK
Breast	92.94±.0051	93.10±.0071	93.50±.0057	93.48±.0060	93.51±.0056	**94.72±.0317**
WDBC	84.01±.0043	85.57±.0111	92.89±.0075	92.49±.0137	76.21±.0122	**93.56±.0428**
WPBC	63.89±.0057	64.06±.0072	66.04±.0269	65.35±.0257	64.22±.0094	**69.28±.0652**
Heart Disease	66.52±.0131	70.20±.0132	71.59±.0165	70.74±.0203	65.29±.0139	**71.61±.0285**
Statlog (Heart)	67.67±.0183	**72.39±.0211**	70.23±.0143	68.88±.0185	66.44±.0187	71.88±.0890
SPECT Heart	72.88±.0088	73.50±.0059	71.54±.0120	70.24±.0233	**73.55±.0044**	73.29±.1390
SPECTF Heart	67.26±.0057	67.39±.0046	69.10±.0104	68.13±.0083	67.45±.0065	**69.53±.1365**
Hepatitis	68.33±.0125	67.54±.0078	70.20±.0401	70.10±.0329	69.64±.0210	**79.01±.1428**
Kidney Disease	87.26±.0025	87.52±.0081	88.84±.0134	87.84±.0140	86.56±.0013	**95.61±.0395**
Thyroid	93.56±.0084	93.49±.0076	93.82±.0120	93.46±.0165	95.28±.0094	**96.26±.0372**
Average	76.43±.0084	77.48±.0094	78.78±.0159	78.08±.0179	75.82±.0102	**81.48±.0752**

E. Interpretability Analysis

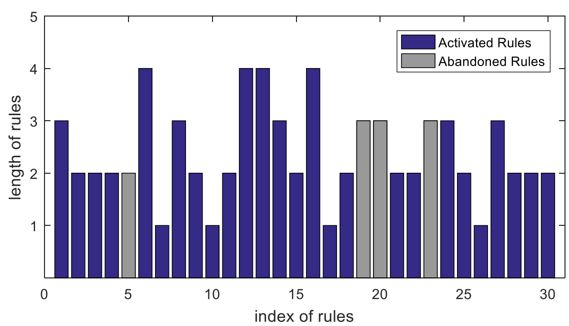
The proposed ESSC-SL-CTSK-FS is expected to have better interpretability. Experiments are conducted to verify this advantage from two aspects: quantitative evaluation of model complexity and demonstration of the intuitiveness of the rule-based form.

For the first aspect, the model complexity of the six methods is calculated based on the analysis in Section V-C-2. The results are shown in Table VI. The model complexity of the proposed ESSC-SL-CTSK-FS is calculated based on the model with the best F-measure value in Table IV. It can be concluded from the results that the proposed method outperforms the other algorithms for all the ten datasets. More importantly, the model complexity of the proposed method is significantly lower, where the number of parameters is only about 10% of the other methods.

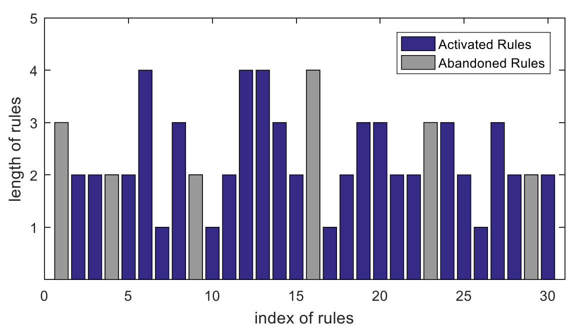
TABLE VI
MODEL COMPLEXITY OF THE MODELS OBTAINED BY DIFFERENT METHODS ON TEN REAL-WORLD DATASETS

---	TSK-FC	L2-TSK-FS	TSK-IQP	TSK-LSSLI	TSK-FS-PCA	ESSC-SL-TSK
1	570	1140	570	570	540	**174**
2	1830	3660	1830	1830	1860	**246**
3	2010	4020	2010	2010	1980	**229**
4	810	1620	810	810	780	**213**
5	810	1620	810	810	780	**197**
6	1350	2700	1350	1350	1380	**240**
7	2670	5340	2670	2670	2700	**217**
8	1170	2340	1170	1170	1140	**176**
9	1470	2940	1470	1470	1500	**204**
10	1530	4410	1530	1530	2250	**356**
---	**1422**	**2979**	**1422**	**1422**	**1491**	**225.2**

For the second aspect, the interpretability of the proposed method can be demonstrated by the intuitiveness resulting from its concise rule-based form and human-like fuzzy inference. Taking the Brest dataset as the example. Using the adopted strategy for multi-classification, two TSK FSs are constructed for binary classification. They share the same antecedent parameters while having different consequent parameters. The rules of the resulting models are demonstrated in Fig. 2. The *x*-axis denotes the indices of all the rules in the rule base and there are 30 rules at the beginning. The *y*-axis denotes the length of rules which is decided by the number of selected features for each rule. It can be seen from Fig. 2(a) and 2(b) that only 1 to 4 features are remained after applying ESSC, i.e., the length of the rules is reduced. Given the capacity of human cognition is limited, the proposed method produces more interpretable results. Besides, some redundant rules are abandoned after applying SL. For the first FS in Fig. 2(a), the indices of the abandoned rules are 5, 19, 20 and 23, whereas for the second FS in Fig. 2(b), the indices of the abandoned rules are 1, 4, 9, 16, 23 and 29. Hence, as demonstrated by Fig. 2, the proposed ESSC-SL-CTSK-FS enables the construction of concise FS and improves interpretability.



(a) The resulting rules of the first TSK FS



(b) The resulting rules of the second TSK FS

Fig. 2. The resulting rules of the TSK-FS after ESSC and SL.

To illustrate the rules of the resulting model more specifically, the linguistic descriptions of the rules are given as follows. Firstly, Fig. 3 shows the activation of the selected feature subsets. The *x*-axis and *y*-axis are the indices of rules and the indices of features respectively. For the Breast dataset, there are 9 features. Each block represents a feature. The colors of the blocks indicate whether a feature is selected or abandoned (i.e. gray). To demonstrate the linguistic descriptions of the rules, the centers of the membership function in (5) are divided into five intervals, each represented with a distinct color as shown in the figure. Since the values of the datasets is normalized into [0,1], the intervals of the centers are also divided within the same range, i.e., the five intervals are [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8] and [0.8,1], which correspond to the five colors and the five vague semantics: *Low*, *Lower*, *Medium*, *Higher* and *High*. It can be seen from Fig. 3 that the activation of the features in different rules is sparse and scattered.

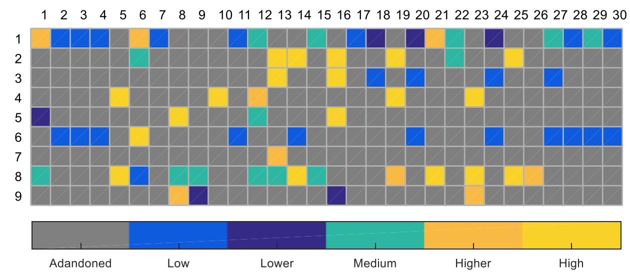


Fig. 3 The degree of activation of the features in the rule base

For the first TSK FS in Fig. 3 that is generated using the Breast dataset, the fuzzy rule base can be described with Table VII which demonstrates that the rules generated with the proposed method are more interpretable.

TABLE VII
THE RULE BASE GENERATED ON BREAST DATASET

The Rule Base of the Breast Dataset
Rule 1: IF: the *first* feature is *Higher*, and the *fifth* feature is *Lower*, and the *eighth* feature is *Medium.* Then: the output of the rule is -0.0112.
Rule 2: IF: the *first* feature is *Low*, and the *sixth* feature is *Low* Then: the output of the rule is 0.3470.
…… (These rules can be obtained according to Fig.3 and the rules 5, 19, 20 and 23 are abandoned according to Fig.2 (a)).
Rule 29: IF: the *first* feature is *Medium*, and the *sixth* feature is *Low* Then: the output of the rule is 1.4885.
Rule 30: IF: the *first* feature is *Low*, and the *sixth* feature is *Low* Then: the output of the rule is 3.9374.

F. *Effectiveness of the ESSC and Sparse Learning*

In this section, the model complexity of the proposed ESSC-SL-CTSK-FS is analyzed under different settings of ESSC and SL. All the experiments conducted on the Breast dataset with $h=10$, ε=0.01 and η=0.01 .

The most important parameter of ESSC is the threshold β used to select the features. Given the weight vector $\mathbf{w}_k=[w_{k1},\cdots,w_{kd}]$ obtained that corresponds to the contribution of all the features to the *k*th rule, the threshold β is usually set to a value between [0,1] and the features with the weights

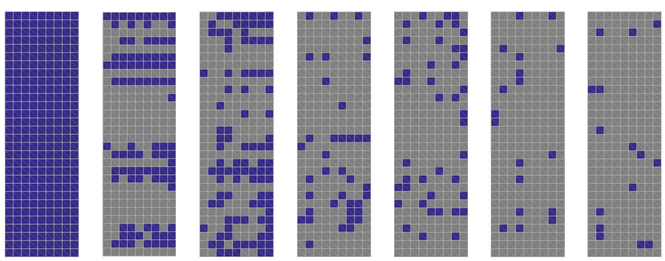
$w_{kl} > \beta$ are selected. In Sections D and E, the experiments are conducted with $\beta \geq 0.1$. Here, β is within [0,1] for a more extensive evaluation of its influence.

The regularization parameter λ is the key to control the sparsity of the consequent parameters of fuzzy systems, i.e., the number of the rules in the rule base. The parameter is optimally set by the search grid $\{0.1, 0.2, \cdots, 0.8, 0.9\}$ in Sections D and E. In this section, the sparsity of the rules is studied more closely with various parameter settings.

In the following three subsections, the analysis of model complexity of FS constructed based on ESSC and SL is conducted from three aspects: FS constructed by the ESSC technique only, FS constructed by the SL technique only, and the FS constructed by ESSC and SL simultaneously.

1) Effectiveness of ESSC

By keeping all the other parameters fixed and setting the sparsity regularization parameter λ to 0, the model complexity and the classification performance are analyzed with different values of the weight threshold of ESSC. The threshold is set as $\{0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3\}$. Fig. 4 shows the activation of the features in different rules under different thresholds. The gray blocks denote the abandoned features and the blue blocks denote the activated features. Note that the format of the subfigures in Fig. 4 is the same as that of Fig. 3; they are rotated and put together for compact display and easy comparison. It can be seen that the sparsity of the activated features increases with the threshold.



(a): 0 (b): 0.05 (c): 0.1 (d): 0.15 (e): 0.2 (f): 0.25 (g): 0.3

Fig. 4. The activation of features by ESSC under different weight threshold

Fig. 5 illustrates the number of parameters and the corresponding F-measure of the obtained FSs using the different values of the weight threshold. The figure shows that the number of the parameters of the resulting model decreases with increasing threshold, which is consistent with the results in Fig. 4. Besides, the value of F-measure decreases with decreasing number of parameters. It can be concluded that the weight threshold can reduce the number of parameters effectively, demonstrating the ability of ESSC in feature selection and concise FS construction.

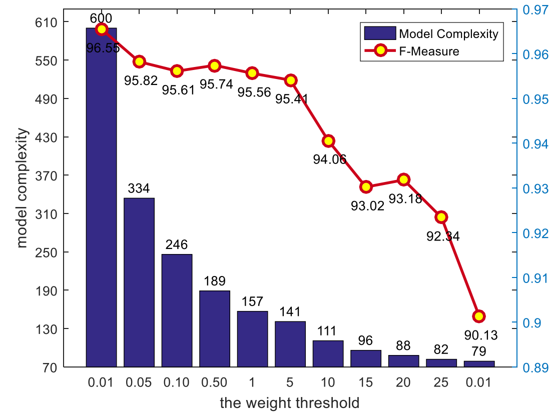


Fig.5. The F-measure and number of parameters under different weight thresholds

2) Effectiveness of SL

To analyze the effectiveness of SL, the sparsity regularization parameter λ is varied while the other parameters fixed. The weight threshold is set to 0, which means that all the features in the rule base are used. Fig. 6 shows the F-measure and the number of parameters under different sparsity regularization parameters. It can be seen that the F-measure almost remains the same or becomes even higher when the sparsity regularization parameter is less than 0.25, whereas the number of parameters is reduced considerably. The above analysis shows that sparse learning can effectively reduce model complexity while maintaining the classification performance.

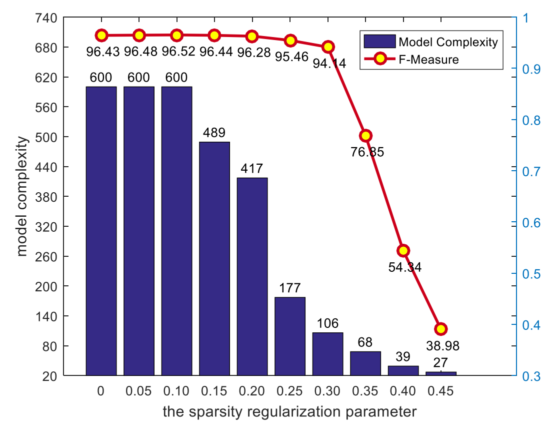


Fig.6. The F-measure and number of parameters under different sparsity regularization parameter

3) Effectiveness of combining ESSC and SL

As discussed in Section I, classification performance usually increases with model complexity. The proposed method aims to reduce the model complexity while keeping the classification performance. The key is to integrate ESSC and SL with an ideal combination of their parameters. An experiment is conducted here to study the effect of different combinations of the weight threshold parameter of ESSC and the sparsity regularization parameter of SL. The weight threshold is taken from the search grid $\{0.01, 0.05, 0.1, 0.5, 1\}$ of 5 values and varied within each group. Similarly, the sparsity regularization parameter is taken from the search grid $\{0.1, 0.5, 1, 5, 10\}$ of 5 values and varied between the groups. That is, a search grid of 25 combinations of the two parameters is used.

Fig. 7 shows the model complexity and F-measure obtained with different parameter combinations. The figure only displays the results of combinations that achieve 95% F-measure or above. The stacked bars are used to display the contributions of ESSC and SL in reducing the model complexity of the resulting FSs. It can be seen in the first group of stacked bars on the left, with indices 1 to 5, that the reduction in model complexity is mostly attributed to ESSC, while SL almost has no effect on fuzzy rule reduction since the sparsity regularization parameter is set to 0.1. In the fourth and fifth groups of stacked bars on the right, SL is dominated in making contribution to model complexity reduction, but the classification performance and model complexity are both low because there are only few rules in the rule base. The second and the third groups of staked bars produce more satisfactory results with a more balanced contribution of ESSC and SL, where low model complexity and high classification performance is achieved. The experiment shows that optimal parameter combinations are a key factor. In practice, the best parameter settings can be decided according to the acceptable degree of trade-off between the classification performance and model complexity.

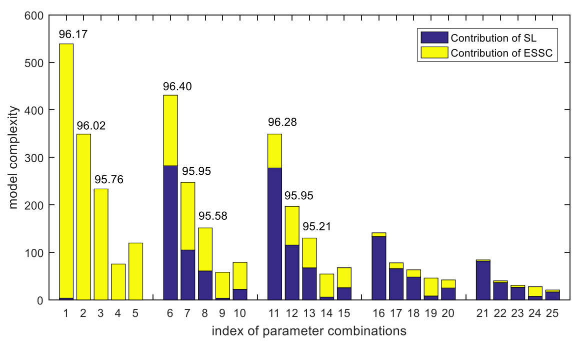


Fig.7. The contributions of ESSC and SL with different parameter combinations

The results show that the proposed method is a flexible TSK FS construction method. The importance of ESSC and SL can be readily adjusted by selecting different parameter settings according to the requirements of the application scenarios. Although there exist some other methods that can also be used to reduce the model complexity of TSK FS, they lack the flexibility of the proposed method. For example, the methods usually attempt to reduce model complexity by employing the strategy of grid search for the number of rules, given the upper limit of the number of rules in the rule base. However, the strategy is deficient in that it can only reduce the model complexity from the aspect of rule reduction and cannot deal with high-dimensional data, where optimal feature selection is necessary for controlling the length of rules and achieving good interpretability. Besides, the strategy is indeed an exhaustive search that is computationally intensive and cannot select the rules adaptively in a way like the proposed method with SL. Other methods such as random feature selection and subspace clustering as discussed in Section II-B have also been used to reduce model complexity, but the construction of concise and compact TSK FS is usually not considered at the same time.

G. Statistical Analysis

To further evaluate the effectiveness of the proposed method, statistical tests are conducted to analyze the significance of the experimental results. The non-parametric Friedman test [43] is used to determine whether the results obtained by the six methods under comparison are significantly different. In the Friedman test, the value of α is set as 0.05 such that if the p-value is less than α, the null hypothesis that the performance of all the algorithms is the same is rejected. The post-hoc test is then used to further determine whether the performance of the best method, as identified by the Friedman test, is significantly different from that of the other methods.

TABLE VII
FRIEDMAN TEST ON THE RESULTS OF F-MEASURE

Algorithm	Ranking	*p*-value	Hypothesis
TSK-FC	4.55	0.002673	Reject
L2-TSK-FS	3.8		
TSK-IQP	3.35		
TSK-LSSLI	3.4		
TSK-FS-PCA	4.65		
ESSC-SL-TSK	1.25		

TABLE VIII
FRIEDMAN TEST ON THE RESULTS OF MODEL COMPLEXITY

Algorithm	Ranking	*p*-value	Hypothesis
TSK-FC	3.5	0.000001	Reject
L2-TSK-FS	6		
TSK-IQP	3.5		
TSK-LSSLI	3.5		
TSK-FS-PCA	3.5		
ESSC-SL-TSK	1		

Tables VII and VIII show the results of Friedman test on F-measure and model complexity, which indicate that the performance of the six methods is significantly different and that the proposed ESSC-SL-CTSK-FS is ranked first, superior to the other methods.

Based on the results of Friedman test, the post-hoc test is conducted to compare the best method, i.e., ESSC-SL-CTSK-FS, with each of the other methods regarding their F-measure performance and model complexity respectively. The results of the post-hoc test are shown in Tables IX and X. From the aspect of F-measure, the results in Table IX show that the proposed ESSC-SL-CTSK-FS is significantly superior to L2-TSK-FS and TSK-FS-PCA, but not so for the other three methods. Nevertheless, it can be seen from Table IV that ESSC-SL-CTSK-FS still outperforms these three methods to some extent. From the aspect of model complexity, Table X shows that ESSC-SL-CTSK-FS is significantly better than all the other algorithms.

The results of the statistical analysis presented above show that the purpose of the proposed method is met, i.e., to construct concise and interpretable FS while keeping the classification performance competitive with, or even better than conventional fuzzy models.

TABLE IX
POST-HOC TEST ON THE RESULTS OF F-MEASURE

i	Algorithm	$z=(R_0-R_i)/SE$	p	Holm= α/i	Hypothesis
5	L2-TSK-FS	3.34664	0.000818	0.01	Reject
4	TSK-FS-PCA	3. 34664	0. 000818	0.0125	Reject
3	TSK-FC	2.988072	0.002807	0.016667	Not Reject
2	TSK-LSSLI	2.031889	0.042165	0.025	Not Reject
1	TSK-IQP	1.195229	0.231998	0.05	Not Reject

TABLE X
POST-HOC TEST ON THE RESULTS OF MODEL COMPLEXITY

i	Algorithm	$z=(R_0-R_i)/SE$	p	Holm= α/i	Hypothesis
5	L2-TSK-FS	5.976143	0	0.01	Reject
4	TSK-FS-PCA	2.988072	0.002807	0.0125	Reject
3	TSK-FC	2.988072	0.002807	0.016667	Reject
2	TSK-LSSLI	2.988072	0.002807	0.025	Reject
1	TSK-IQP	2.988072	0.002807	0.05	Reject

VI. CONCLUSIONS

Interpretability of decision model is very important in many practical applications, such as medical diagnosis. To meet the requirement, this paper investigates the development of highly interpretable intelligent models based on concise zero-order TSK FS. Two techniques are used to improve the interpretability of zero-order TSK FS. First, ESSC is adopted to partition the input space of the training dataset. Elastic fuzzy rules can then be generated where each rule only contains a few important features and different rules are constructed with different feature subsets. Thus, the concise fuzzy rules not only remove the noisy features but also possess human-like inference mechanism that consider different views from different experts on the same task. Second, sparse learning is adopted to remove redundant rules by solving the sparse solution to the consequent parameters of the fuzzy rules. With the two techniques, the ESSC-SL-CTSK-FS method is proposed to construct concise and highly interpretable zero-order TSK FS, which have demonstrated promising performance with extensive experiments conducted on various medical datasets.

Further research of the project include the investigation of using other FS models, e.g. first order TSK FS and Mamdani-Larsen-type FS, to improve high-dimensional data-driven fuzzy modeling in terms of interpretability and conciseness. Another interesting work is to investigate compact subspace extraction methods for ESSC-SL-CTSK-FS when the input features in different rules are almost equally important.

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