Strain gradient differential quadrature finite element for moderately thick micro-plates

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Summary

In this study, we integrate the advantages of differential quadrature method (DQM) and finite element method (FEM) to construct a C^1-type four-node quadrilateral element with 48 degrees of freedom (DOF) for strain gradient Mindlin micro-plates. This element is free of shape functions and shear locking. The C^1-continuity requirements of deflection and rotation functions are accomplished by a fourth-order differential quadrature (DQ)-based geometric mapping scheme, which facilitates the conversion of the displacement parameters at Gauss-Lobatto quadrature (GLQ) points into those at element nodes. The appropriate application of DQ rule to non-rectangular domains is proceeded by the natural-to-Cartesian geometric mapping technique. Using GLQ and DQ rules, we discretize the total potential energy functional of a generic micro-plate element into a function of nodal displacement parameters. Then, we adopt the principle of minimum potential energy to determine element stiffness matrix, mass matrix, and load vector. The efficacy of the present element is validated through several examples associated with the static bending and free vibration problems of rectangular, annular sectorial, and elliptical micro-plates. Finally, the developed element is applied to study the behavior of freely vibrating moderately thick micro-plates with irregular shapes. It is shown that our element has better convergence and adaptability than that of Bogner-Fox-Schmit (BFS) one, and strain gradient effects can cause a significant increase in vibration frequencies and a certain change in vibration mode shapes.

KEYWORDS

Differential quadrature method, finite element method, C^1-type four-node quadrilateral element, strain gradient Mindlin micro-plates, geometric mapping technique.

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1. Introduction

Over the past century, classical continuum mechanics has played a crucial role in modeling and analyzing the mechanical behavior of macroscopic solids and structures. However, at micro-scale, its validity has been challenged by several experimental works, such as the torsion of ultra-thin copper wires[1-4] and the bending of ultra-thin beams[5-8]. This is because the leading dimensions (e.g., diameter or thickness) of micro-structures are comparable to the intrinsic length scales of constituent materials (e.g., grain size, void radius, or dislocation spacing). In other words, classical continuum mechanics does not admit the size-dependence in the elastic fields of defects. Consequently, there are discrepancies between experimental observations and classical theoretical predictions. In such a background, the framework of classical continuum mechanics must be updated by redefining the underlying propositions of Cauchy continuum while retaining its original homogenizing nature. To this end, researchers are now focusing on gradient elasticity theories[5,9-15], which incorporate the higher-order spatial derivatives of strain, stress, and/or inertia, and one or more material length scale parameters (MLSPs). In the current work, we primarily mention two well-accepted gradient elasticity theories, i.e., the modified strain gradient elasticity theory (MSGT)[5] and the modified couple stress theory (MCST)[13]. For more details on the application of MSGT and MCST, we can refer to a literature review of Thai et al[16].

Although gradient elasticity-based continuum modeling for micro-structural members has been quite successful, only limited model problems with special boundary conditions and simple geometries can be analytically solved, as pointed out in Ref.[16]. A predicament of developing solution schemes for gradient elastic structural models lies in that the inclusion of gradient effects may cause a remarkable rise in the order of equations of motion and boundary conditions. Naturally, basis kinematic functions require more demanding displacement continuity. For instance, gradient elastic Kirchhoff plate models[17,18] and Kirchhoff-Love shell models[19] are described by the sixth-order differential equations and involve the curvature boundary conditions; thus, the C^2-continuity of deflection function should be satisfied. In the context of MCST, the related Mindlin plate model leads to the twelfth-order governing equations[20]. For gradient elastic bar and plane strain/stress models[21], the fourth-order boundary value problems are involved. Accordingly, there are considerable challenges in solving the gradient-enhanced models by both analytical and numerical approaches[19,22,23].

During the past few years, a few researchers have made computational contributions to non-classical finite elements to solve gradient elasticity-related boundary value problems. For instance, Swaddiwudhipong et al[24] established several C^0-type solid elements in the context of

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a mechanism-based strain gradient plasticity theory. Papanicolopulos et al[25] developed a C^1-type hexahedral element to deal with three-dimensional (3D) gradient elasticity problems. Using the enhanced patch test theory, Chen[26] presented the patch test functions to assess the convergence of C^0- and C^1-continuous couple stress-based finite elements. Zhao et al[27] developed a 24-DOF quadrilateral element for couple stress/strain gradient elasticity using the refined Kirchhoff plate element RDKQ with weak C^1-continuity, as well as the nonconforming element CQ12 with weak C^0-continuity. Zhang et al[28,29] constructed a MSGT-based 8-DOF two-node Timoshenko beam element and a MCST-based 36-DOF four-node Mindlin plate element, which have weak C^1-continuity and are free of shear locking. Kim and Reddy[30] performed a nonlinear finite element analysis for thick FG micro-plates with couple stress and von Kármán nonlinearity. Kwon and Lee[31,32] analyzed the torsion behavior of micro-bars, bending behavior of micro-beams, and stress concentration behavior of circular micro-plates by constructing MCST-based hexahedral and quadrilateral elements with C^1-weak continuity. Choi and Lee[33] proposed a MCST-based 3-node C^0-continuous triangular element employing the node-based smoothed finite element method to approximately express the second-order derivatives of displacement. Torabi et al[34] developed a 4-node non-conforming strain gradient tetrahedron element that holds the translational displacements together with their first partial derivatives as nodal unknowns to investigate the free vibration behavior of circular/elliptical micro-plates. Babu and Patel[35] proposed a rectangular plate finite element for a single- parameter strain gradient Kirchhoff plate model to solve the static bending, free vibration, and buckling problems of rectangular nano-plates. Papanicolopulos et al[36] devised a new family of hybrid finite element for strain gradient-related boundary value problems, where the Lagrange multiplier method was used. Sze and Hu[37] developed three 24-DOF four-node gradient- enhanced quadrilateral elements by extending the discrete Kirchhoff method, the relaxed hybrid-stress method, and the hybrid-stress method adopted in macro-scale thin plate analysis. More recently, Sze and his co-workers[38] used the discrete Kirchhoff method and the relaxed hybrid-stress method to set up two 48-DOF four-node tetrahedral gradient-enhanced elements for three-dimensional gradient elasticity analysis. Within the framework of gradient elasticity theories, however, there are still many issues during the construction of complete conforming elements, especially for gradient elastic plate and shell models. Most of the published works inevitably applied higher-order, nonconforming, mixed, or other non-standard finite elements.

Given the deficiencies of standard FEM in dealing with higher-order continuity problems, Isogeometric Analysis (IGA), proposed by Hughes et al[39], has attracted considerable attention from researchers in the field of micro-structural mechanics. IGA method can facilitate the integration of standard FEM with non-uniform rational basis spline(NURBS)-based computer-aided design (CAD) tool to achieve arbitrary-order continuities of kinematic variables, while avoiding the exploitation of higher-order nonconforming or mixed elements. Some associated problems requiring global C^1-continuity or even C^2-continuity have been effectively solved within the IGA-based computational framework.To name a few, Niiranen et al[21] presented the variational formulation to address the fourth-order boundary value problems of gradient elastic bar and plane strain/stress models along with their relevant isogeometric C^1-continuous discretization schemes. Then, Niiranen et al[22] established a C^2-type IGA-based Galerkin discretization scheme in the context of an H^3-Sobolev space setting to handle the sixth-order boundary value problems of gradient elastic Kirchhoff plates. Balobanov and Niiranen[23] proposed two types of non-standard shear locking-free variational formulations for gradient elastic Timoshenko beams and the related C^2-continuous isogeometric discretization scheme. Nguyen et al[40] developed an IGA-based numerical model for MCST-based FG micro-plates, including both shear and normal deformation effects. In the context of MSGT, Thai and his co-workers[41,42] presented IGA-based simulation models for the linear/nonlinear static and dynamic analysis of FG Reddy micro-plates. Using the IGA approach, Makvandi et al[43] investigated the behavior of second- and third-order gradient materials under edge and point displacement boundary conditions, respectively. Kolo et al[44] extended adaptive IGA to Aifantis' gradient elasticity and Engelen' implicit gradient plasticity theories, applying the hierarchical refinement built upon truncated multi-level basis functions that interact through an inter-level subdivision operator in the Beziér element framework. More recently, Kolo and de Borst[45] presented a gradient plasticity-related IGA model, and confirmed that their model has distinct advantages in terms of the different order interpolations of displacement and plastic multiplier compared to FEM and meshless method. Although IGA approach can yield arbitrary-order continuous basis functions, there are still inadequacies in the integration of the weak-form and the imposition of essential boundary conditions in such method. Besides, the basis functions of an IGA model often have a larger support domain than those of the related finite element model, implying less sparse system matrices and higher computational expense.

From the literature research above, we can conclude that the existing solution methods for general boundary value problems of gradient elastic beams and plates primarily focus on the IGA approach and non-standard FEM. Each solution method has its merits and demerits in analyzing structural mechanics problem; therefore, it is necessary to develop alternatives. As is well-known, DQM initiated by Bellman et al[46,47]can flexibly address the higher-order smoothness requirement and yield highly accurate results with much less computational expense than standard FEM. The fundamental propositions of DQM and its engineering application to macroscopic structural members were summarized in academic books[48,49]. Although DQM has distinct convergence and accuracy advantages, it is still inconvenient to deal with structural mechanics problems with non-rectangular domains and sudden changes in thickness, distributed loads, or edge supports. Recently, a few researchers have drawn the advantages of DQM and domain decomposition technique to develop some gradient-enhanced quadrature elements (QEs) and differential quadrature finite elements (DQFEs) for gradient elastic beams and plates. For example, Wang[50] proposed a weak-form QEM to analyze the free vibration behavior of nonlocal strain gradient Euler-Bernoulli beams. Zhang et al[51] developed two types of DQFEs corresponding to MSGT-based Euler-Bernoulli and Timoshenko beam models, respectively. Based on Ref.[51], Zhang et al[52] proposed a strain gradient DQFE to examine the size-dependent static and dynamic behavior of Reddy-type micro-beams. Using a 2D fourth-order DQ-based geometric mapping scheme, Zhang et al[53] constructed a C^1-continuous rectangular element to analyze the vibration and buckling behavior of a couple stress-based moderately thick FG plate. Afterwards, these authors[54] proposed an incomplete C^2-continuous DQFE for the MSGT-based Kirchhoff plate model. Using the DQFEs proposed in Refs. [51,52], Zhang et al[55] analyzed the coupling effects of surface energy, strain gradient, and inertia gradient on the behavior of freely vibrating small-scale beams. Ishaquddin and Gopalakrishnan[56] constructed a novel weak-form QEM for a one-parameter gradient elastic Euler-Bernoulli beam model.

However, to the best of our knowledge, the numerical analysis of gradient elastic Mindlin micro-plates, with arbitrary shapes and sudden variations in edge support and thickness based on either IGA, FEM, QEM, or DQFEM, has never been reported. Our particular aim is to fill this research void. The organization of this paper is as follows. Section 2 recapitulates the geometric and constitutive equations of Lam's gradient elasticity theory[5] (i.e., MSGT). Section 3 presents a 48-DOF four-node quadrilateral element for the MSGT-based Mindlin plate model using a 2D DQ-based displacement mapping scheme and a natural-to-Cartesian geometric mapping technique. Section 4 demonstrates the applicability of our method through some typical examples related to the static and vibration problems of micro-plates with rectangular, annular sectorial, and elliptical shapes. Finally, the study is concluded in Section 5.

2. Lam's modified strain gradient elasticity theory (MSGT)

Within the framework of MSGT[5], the strain energy Π_s of an isotropic elastic deformable body occupying the space Ω is defined as

$$\Pi_s = \frac{1}{2}\int_\Omega \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^{(s)}\chi_{ij}^{(s)} \right) d\Omega \,, \tag{1}$$

where ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$, and $\chi_{ij}^{(s)}$ are the strain, dilatation gradient, deviatoric stretch gradient, and symmetrical part of rotation gradient tensors work-conjugated to the stress metrics σ_{ij}, p_i, $\tau_{ijk}^{(1)}$, and $m_{ij}^{(s)}$, respectively. Here, the elastic body is assumed to experience infinitesimal deformation. The above deformation and stress metrics are written as

$$\begin{aligned}
&\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right), \quad \gamma_i = \frac{\partial \varepsilon_{mm}}{\partial X_i}, \\
&\chi_{ij}^{(s)} = \frac{1}{4}\left(e_{imn}\frac{\partial^2 U_n}{\partial X_m \partial X_j} + e_{jmn}\frac{\partial^2 U_n}{\partial X_m \partial X_i} \right), \\
&\eta_{ijk}^{(1)} = \frac{1}{3}\left(\frac{\partial \varepsilon_{jk}}{\partial X_i} + \frac{\partial \varepsilon_{ki}}{\partial X_j} + \frac{\partial \varepsilon_{ij}}{\partial X_k} \right) - \frac{1}{15}\delta_{ij}\left(\frac{\partial \varepsilon_{mm}}{\partial X_k} + 2\frac{\partial \varepsilon_{mk}}{\partial X_m} \right) \\
&\qquad - \frac{1}{15}\delta_{jk}\left(\frac{\partial \varepsilon_{mm}}{\partial X_i} + 2\frac{\partial \varepsilon_{mi}}{\partial X_m} \right) - \frac{1}{15}\delta_{ki}\left(\frac{\partial \varepsilon_{mm}}{\partial X_j} + 2\frac{\partial \varepsilon_{mj}}{\partial X_m} \right),
\end{aligned} \tag{2}$$

$$\sigma_{ij} = \lambda\varepsilon_{mm}\delta_{ij} + 2G\varepsilon_{ij}, \quad p_i = 2Gl_0^2\gamma_i, \quad \tau_{ijk}^{(1)} = 2Gl_1^2\eta_{ijk}^{(1)}, \quad m_{ij}^{(s)} = 2Gl_2^2\chi_{ij}^{(s)}, \tag{3}$$

where U_i is the displacement vector; δ_{ij} and e_{ijk} are Kronecker delta and permutation symbol, respectively; l_0, l_1, and l_2 are three MLSPs. The Latin indices run from 1 to 3 unless otherwise indicated. λ and G are the first and second Lamé constants as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}, \tag{4}$$

where E and ν are Young's modulus and Poisson's ratio.

3. Strain gradient differential quadrature finite element

Fig. 1 displays an initially flat, isotropic, and moderately thick micro-plate of uniform thickness h, where the mid-plane A is located in the OXY plane, and the bold symbols $\boldsymbol{n}$ and $\boldsymbol{s}$ indicate the unit normal and tangent vectors at a point of boundary curve ∂A, respectively. It is noted that ∂A may consist of several smooth segments. The material properties include: mass density ρ, shear modulus G, Young's modulus E, and Poisson's ratio ν. When a transversely distributed load Q is applied on the upper flat surface, the plate has an deflection W and two transverse normal rotations Φ_X and Φ_Y about the Y- and X-axes, respectively.

Based on the Mindlin hypothesis, the displacement field for a moderately thick plate can be expressed as

$$U_1 = Z\Phi_X(X,\ Y,\ t),\ \ U_2 = Z\Phi_Y(X,\ Y,\ t),\ \ U_3 = W(X,\ Y,\ t), \tag{5}$$

where U_1, U_2, and U_3 are the displacement components along the X, Y, and Z directions, respectively.

Using Eqs. (2) and (5), we can obtain the following nonzero components of deformation metrics:

$$\varepsilon_{11}=Z\frac{\partial\Phi_X}{\partial X},\ \varepsilon_{22}=Z\frac{\partial\Phi_Y}{\partial Y},\ \ \varepsilon_{21}=\varepsilon_{12}=\frac{1}{2}Z\left(\frac{\partial\Phi_X}{\partial Y}+\frac{\partial\Phi_Y}{\partial X}\right),\ \ \varepsilon_{31}=\varepsilon_{13}=\frac{1}{2}\left(\Phi_X+\frac{\partial W}{\partial X}\right),$$
$$\varepsilon_{32}=\varepsilon_{23}=\frac{1}{2}\left(\Phi_Y+\frac{\partial W}{\partial Y}\right), \tag{6}$$

$$\gamma_1=Z\left(\frac{\partial^2\Phi_X}{\partial X^2}+\frac{\partial^2\Phi_Y}{\partial X\partial Y}\right),\ \ \gamma_2=Z\left(\frac{\partial^2\Phi_X}{\partial X\partial Y}+\frac{\partial^2\Phi_Y}{\partial Y^2}\right),\ \ \gamma_3=\left(\frac{\partial\Phi_X}{\partial X}+\frac{\partial\Phi_Y}{\partial Y}\right), \tag{7}$$

$$\eta_{111}^{(1)}=\frac{Z}{5}\left(2\frac{\partial^2\Phi_X}{\partial X^2}-\frac{\partial^2\Phi_X}{\partial Y^2}-2\frac{\partial^2\Phi_Y}{\partial X\partial Y}\right),\ \ \eta_{222}^{(1)}=\frac{Z}{5}\left(2\frac{\partial^2\Phi_Y}{\partial Y^2}-2\frac{\partial^2\Phi_X}{\partial X\partial Y}-\frac{\partial^2\Phi_Y}{\partial X^2}\right),$$
$$\eta_{333}^{(1)}=-\frac{1}{5}\left(2\frac{\partial\Phi_X}{\partial X}+\frac{\partial^2 W}{\partial X^2}+2\frac{\partial\Phi_Y}{\partial Y}+\frac{\partial^2 W}{\partial Y^2}\right),\ \ \eta_{211}^{(1)}=\eta_{112}^{(1)}=\eta_{121}^{(1)}=\frac{Z}{15}\left(8\frac{\partial^2\Phi_X}{\partial X\partial Y}+4\frac{\partial^2\Phi_Y}{\partial X^2}-3\frac{\partial^2\Phi_Y}{\partial Y^2}\right),$$
$$\eta_{113}^{(1)}=\eta_{311}^{(1)}=\eta_{131}^{(1)}=\frac{1}{15}\left(8\frac{\partial\Phi_X}{\partial X}+4\frac{\partial^2 W}{\partial X^2}-2\frac{\partial\Phi_Y}{\partial Y}-\frac{\partial^2 W}{\partial Y^2}\right),$$
$$\eta_{212}^{(1)}=\eta_{122}^{(1)}=\eta_{221}^{(1)}=\frac{Z}{15}\left(4\frac{\partial^2\Phi_X}{\partial Y^2}+8\frac{\partial^2\Phi_Y}{\partial X\partial Y}-3\frac{\partial^2\Phi_X}{\partial X^2}\right),$$
$$\eta_{132}^{(1)}=\eta_{321}^{(1)}=\eta_{213}^{(1)}=\eta_{312}^{(1)}=\eta_{123}^{(1)}=\eta_{231}^{(1)}=\frac{1}{3}\left(\frac{\partial\Phi_X}{\partial Y}+\frac{\partial\Phi_Y}{\partial X}+\frac{\partial^2 W}{\partial X\partial Y}\right),$$
$$\eta_{133}^{(1)}=\eta_{313}^{(1)}=\eta_{331}^{(1)}=-\frac{Z}{15}\left(3\frac{\partial^2\Phi_X}{\partial X^2}+\frac{\partial^2\Phi_X}{\partial Y^2}+2\frac{\partial^2\Phi_Y}{\partial X\partial Y}\right),$$
$$\eta_{322}^{(1)}=\eta_{223}^{(1)}=\eta_{232}^{(1)}=\frac{1}{15}\left(8\frac{\partial\Phi_Y}{\partial Y}+4\frac{\partial^2 W}{\partial Y^2}-2\frac{\partial\Phi_X}{\partial X}-\frac{\partial^2 W}{\partial X^2}\right),$$
$$\eta_{233}^{(1)}=\eta_{323}^{(1)}=\eta_{332}^{(1)}=-\frac{Z}{15}\left(2\frac{\partial^2\Phi_X}{\partial X\partial Y}+\frac{\partial^2\Phi_Y}{\partial X^2}+3\frac{\partial^2\Phi_Y}{\partial Y^2}\right), \tag{8}$$

$$\chi_{11}^{(s)}=\frac{1}{2}\left(\frac{\partial^2 W}{\partial X\partial Y}-\frac{\partial\Phi_Y}{\partial X}\right),\ \ \chi_{12}^{(s)}=\chi_{21}^{(s)}=\frac{1}{4}\left(\frac{\partial\Phi_X}{\partial X}-\frac{\partial^2 W}{\partial X^2}-\frac{\partial\Phi_Y}{\partial Y}+\frac{\partial^2 W}{\partial Y^2}\right),$$
$$\chi_{13}^{(s)}=\chi_{31}^{(s)}=\frac{Z}{4}\left(\frac{\partial^2\Phi_Y}{\partial X^2}-\frac{\partial^2\Phi_X}{\partial X\partial Y}\right),\ \ \chi_{22}^{(s)}=\frac{1}{2}\left(\frac{\partial\Phi_X}{\partial Y}-\frac{\partial^2 W}{\partial X\partial Y}\right),\ \ \chi_{32}^{(s)}=\chi_{23}^{(s)}=\frac{Z}{4}\left(\frac{\partial^2\Phi_Y}{\partial X\partial Y}-\frac{\partial^2\Phi_X}{\partial Y^2}\right),$$
$$\chi_{33}^{(ss)}=\frac{1}{2}\left(\frac{\partial\Phi_Y}{\partial X}-\frac{\partial\Phi_X}{\partial Y}\right). \tag{9}$$

According to Eq. (3) and Eqs. (6)-(9), we can obtain the related nonzero classical stress and non-classical stress components.

The total potential energy for a MSGT-based Mindlin plate is

$$
\begin{aligned}
\Pi = &\frac{1}{2}\int_A \left[\rho h \left(\frac{\partial W}{\partial t} \right)^2 + \frac{\rho h^3}{12}\left(\frac{\partial \Phi_X}{\partial t} \right)^2 + \frac{\rho h^3}{12}\left(\frac{\partial \Phi_Y}{\partial t} \right)^2 \right] dXdY \\
&-\frac{1}{2}\int_A \Bigg\{ \Sigma_1 \left[\left(\frac{\partial^2 W}{\partial X^2} \right)^2 + \left(\frac{\partial^2 W}{\partial Y^2} \right)^2 \right] + \Sigma_2 \frac{\partial^2 W}{\partial X^2}\frac{\partial^2 W}{\partial Y^2} + \Sigma_4 \left[\left(\frac{\partial \Phi_Y}{\partial Y} \right)^2 + \left(\frac{\partial \Phi_X}{\partial X} \right)^2 \right] \\
&+ \Sigma_3 \frac{\partial \Phi_X}{\partial X}\frac{\partial \Phi_Y}{\partial Y} + \Sigma_5 \left(\frac{\partial \Phi_Y}{\partial Y}\frac{\partial^2 W}{\partial X^2} + \frac{\partial \Phi_X}{\partial X}\frac{\partial^2 W}{\partial Y^2} \right) + \Sigma_8 \left(\frac{\partial \Phi_X}{\partial X}\frac{\partial^2 W}{\partial X^2} + \frac{\partial \Phi_Y}{\partial Y}\frac{\partial^2 W}{\partial Y^2} \right) \\
&+ \Sigma_6 \left(\frac{\partial \Phi_X}{\partial Y}\frac{\partial^2 W}{\partial Y \partial X} + \frac{\partial^2 W}{\partial Y \partial X}\frac{\partial \Phi_Y}{\partial X} + \frac{\partial \Phi_X}{\partial Y}\frac{\partial \Phi_Y}{\partial X} \right) + \Sigma_{10} \left[\left(\frac{\partial^2 \Phi_Y}{\partial X \partial Y} \right)^2 + \left(\frac{\partial^2 \Phi_X}{\partial X \partial Y} \right)^2 \right] \\
&+ \Sigma_7 \left[\left(\frac{\partial \Phi_Y}{\partial X} \right)^2 + \left(\frac{\partial^2 W}{\partial X \partial Y} \right)^2 + \left(\frac{\partial \Phi_X}{\partial Y} \right)^2 \right] + \Sigma_9 \left[\left(\Phi_X + \frac{\partial W}{\partial X} \right)^2 + \left(\Phi_Y + \frac{\partial W}{\partial Y} \right)^2 \right] \\
&+ \Sigma_{11} \left(\frac{\partial^2 \Phi_Y}{\partial X \partial Y}\frac{\partial^2 \Phi_X}{\partial X^2} + \frac{\partial^2 \Phi_X}{\partial X \partial Y}\frac{\partial^2 \Phi_Y}{\partial Y^2} \right) + \Sigma_{12} \left(\frac{\partial^2 \Phi_Y}{\partial X^2}\frac{\partial^2 \Phi_X}{\partial X \partial Y} + \frac{\partial^2 \Phi_Y}{\partial X \partial Y}\frac{\partial^2 \Phi_X}{\partial Y^2} \right) \\
&+ \Sigma_{13} \left[\left(\frac{\partial^2 \Phi_X}{\partial X^2} \right)^2 + \left(\frac{\partial^2 \Phi_Y}{\partial Y^2} \right)^2 \right] + \Sigma_{14} \left[\left(\frac{\partial^2 \Phi_Y}{\partial X^2} \right)^2 + \left(\frac{\partial^2 \Phi_X}{\partial Y^2} \right)^2 \right] \\
&+ \Sigma_{15} \left(\frac{\partial^2 \Phi_Y}{\partial X^2}\frac{\partial^2 \Phi_Y}{\partial Y^2} + \frac{\partial^2 \Phi_X}{\partial X^2}\frac{\partial^2 \Phi_X}{\partial Y^2} \right) + \Sigma_{16} \left[\left(\frac{\partial \Phi_Y}{\partial Y} \right)^2 + \left(\frac{\partial \Phi_X}{\partial X} \right)^2 \right] \\
&+ \Sigma_{17} \frac{\partial \Phi_X}{\partial X}\frac{\partial \Phi_Y}{\partial Y} + \Sigma_{18} \left(\frac{\partial \Phi_X}{\partial Y} + \frac{\partial \Phi_Y}{\partial X} \right)^2 \Bigg\} dXdY + \int_A QW dXdY
\end{aligned}
\quad , \qquad (10)
$$

where

$$
\Sigma_1 = Gh\left(\frac{4l_1^2}{15} + \frac{l_2^2}{8} \right),\ \Sigma_2 = -Gh\left(\frac{2l_1^2}{15} + \frac{l_2^2}{4} \right),\ \Sigma_3 = Gh\left(2l_0^2 - \frac{8l_1^2}{15} - \frac{l_2^2}{4} \right),
$$

$$
\Sigma_4 = Gh\left(l_0^2 + \frac{16l_1^2}{15} + \frac{l_2^2}{8} \right),\ \Sigma_5 = -Gh\left(\frac{4l_1^2}{15} - \frac{l_2^2}{4} \right),\ \Sigma_6 = Gh\left(\frac{4l_1^2}{3} - \frac{l_2^2}{2} \right),\ \Sigma_7 = Gh\left(\frac{2l_1^2}{3} + \frac{l_2^2}{2} \right),
$$

$$
\Sigma_8 = Gh\left(\frac{16l_1^2}{15} - \frac{l_2^2}{4} \right),\ \Sigma_9 = \frac{GhK_s}{2},\ \Sigma_{10} = \frac{Gh^3}{3}\left(\frac{l_0^2}{4} + \frac{4l_1^2}{15} + \frac{l_2^2}{32} \right),\ \Sigma_{11} = \frac{Gh^3}{3}\left(\frac{l_0^2}{2} - \frac{l_1^2}{5} \right),
$$

$$
\Sigma_{12} = \frac{Gh^3}{3}\left(\frac{4l_1^2}{15} - \frac{l_2^2}{16} \right),\ \Sigma_{13} = \frac{Gh^3}{6}\left(\frac{l_0^2}{2} + \frac{l_1^2}{5} \right),\ \Sigma_{14} = \frac{Gh^3}{3}\left(\frac{l_1^2}{15} + \frac{l_2^2}{32} \right),\ \Sigma_{15} = -\frac{Gh^3 l_1^2}{30},
$$

$$
\Sigma_{16} = D/2,\ \Sigma_{17} = D\nu,\ \Sigma_{18} = D(1-\nu)/4,\ D = Eh^3 \big/ \left[12\left(1-\nu^2\right) \right]. \qquad (11)
$$

By setting $l_0 = l_1 = 0$ and $Q = 0$, Eq. (10) degenerates into the MCST-based Mindlin plate model studied by Ke et al[20]. Similar to Ref. [20], we can use Hamilton principle or Euler-Lagrange equation to derive the equations of motion and related boundary conditions for the present model. Evidently, Eq. (10) involves the second-order partial derivatives of W, Φ_X, and Φ_Y, which indicates the C^1-continuity requirements of them. However, the classical Mindlin plate element has only C^0-continuity and does not include any size effect. In addition, the reduced integration scheme has to be used to address the shear locking problem.

It is well established that the major advantages of FEM lie in its rigorous mathematical foundation and excellent flexibility in handling irregular geometries and complex boundary conditions, but this method is inconvenient to handle higher-order continuity issues. On the contrary, DQM exhibits good performance in realizing higher-order continuity conditions and can obtain highly accurate results with much less computational expense as compared to that required in standard FEM. [48,49] However, DQM also suffers from shortcomings in dealing with structural mechanics problems with non-rectangular solution domains and sudden changes in thickness and/or edge supports. To this end, some researchers integrated DQ rule with domain decomposition technique to develop several mixed methods, including Fourier expansion- based DQM[57], strong-form DQFEM[58], QEM[59-61], and weak-form DQFEM[62-64]. It may be noted that these studies only focused on macroscopic structural components.

Here, we propose a 48-DOF four-node quadrilateral element for the present Mindlin plate model. Fig. 2 depicts the natural-to-Cartesian geometric mapping scheme that can convert an arbitrary quadrilateral element into a square parent element, and a 2D DQ-based geometric mapping scheme that is responsible for realizing the C^1-continuity conditions of W, Φ_X, and Φ_Y. XOY and $\bar{X}\bar{O}\bar{Y}$ indicate the global and local coordinate systems, respectively.

Each node has twelve displacement unknowns, i.e., W, $W_{,\bar{X}}$, $W_{,\bar{Y}}$, $W_{,\bar{X}\bar{Y}}$, Φ_X, $\Phi_{X,\bar{X}}$, $\Phi_{X,\bar{Y}}$, $\Phi_{X,\bar{X}\bar{Y}}$, Φ_Y, $\Phi_{Y,\bar{X}}$, $\Phi_{Y,\bar{Y}}$, and $\Phi_{Y,\bar{X}\bar{Y}}$, which are same as those of C^1-conforming Bogner-Fox-Schmidt (BFS) element[65] based on bi-cubic Hermitian interpolation. Motivated by

the above work, Petera and Pittman[66] developed triangular and quadrilateral isoparametric elements with C^1-continuity and demonstrated their efficacy in dealing with irregular solution domains.

In Fig. 2, the partial derivatives of a generic kinematic variable in the global and local coordinate systems are associated with the following expressions:

$$\begin{bmatrix} \dfrac{\partial}{\partial X} \\ \dfrac{\partial}{\partial Y} \end{bmatrix} = \boldsymbol{J}_{\bar{X}\bar{Y}}^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \bar{X}} \\ \dfrac{\partial}{\partial \bar{Y}} \end{bmatrix}, \quad \begin{bmatrix} \dfrac{\partial^2}{\partial X^2} \\ \dfrac{\partial^2}{\partial Y^2} \\ \dfrac{\partial^2}{\partial X \partial Y} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial \bar{X}}{\partial X}\right)^2 & \left(\dfrac{\partial \bar{Y}}{\partial X}\right)^2 & 2\dfrac{\partial \bar{X}}{\partial X}\dfrac{\partial \bar{Y}}{\partial X} \\ \left(\dfrac{\partial \bar{X}}{\partial Y}\right)^2 & \left(\dfrac{\partial \bar{Y}}{\partial Y}\right)^2 & 2\dfrac{\partial \bar{X}}{\partial Y}\dfrac{\partial \bar{Y}}{\partial Y} \\ \dfrac{\partial \bar{X}}{\partial X}\dfrac{\partial \bar{X}}{\partial Y} & \dfrac{\partial \bar{Y}}{\partial X}\dfrac{\partial \bar{Y}}{\partial Y} & \dfrac{\partial \bar{Y}}{\partial X}\dfrac{\partial \bar{X}}{\partial Y} + \dfrac{\partial \bar{X}}{\partial X}\dfrac{\partial \bar{Y}}{\partial Y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2}{\partial \bar{X}^2} \\ \dfrac{\partial^2}{\partial \bar{Y}^2} \\ \dfrac{\partial^2}{\partial \bar{X} \partial \bar{Y}} \end{bmatrix}, \tag{13}$$

where

$$\boldsymbol{J}_{\bar{X}\bar{Y}} = \begin{bmatrix} \dfrac{\partial X}{\partial \bar{X}} & \dfrac{\partial Y}{\partial \bar{X}} \\ \dfrac{\partial X}{\partial \bar{Y}} & \dfrac{\partial Y}{\partial \bar{Y}} \end{bmatrix}, \quad \boldsymbol{J}_{\bar{X}\bar{Y}}^{-1} = \begin{bmatrix} \dfrac{\partial \bar{X}}{\partial X} & \dfrac{\partial \bar{Y}}{\partial X} \\ \dfrac{\partial \bar{X}}{\partial Y} & \dfrac{\partial \bar{Y}}{\partial Y} \end{bmatrix}. \tag{14}$$

According to Eqs. (10), (13), and (14), the total potential energy of a strain gradient quadrilateral Mindlin plate element can be transformed as

$$\Pi_e = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\left[\rho h\left(\frac{\partial W}{\partial t}\right)^2 + \frac{\rho h^3}{12}\left(\frac{\partial \Phi_X}{\partial t}\right)^2 + \frac{\rho h^3}{12}\left(\frac{\partial \Phi_Y}{\partial t}\right)^2\right]\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right| d\bar{X}d\bar{Y} + \int_{-1}^{1}\int_{-1}^{1} QW\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right| d\bar{X}d\bar{Y}$$
$$-\int_{-1}^{1}\int_{-1}^{1}\left[\beta_1\left(\frac{\partial^2 W}{\partial \bar{X}^2}\right)^2 + \beta_2\left(\frac{\partial^2 W}{\partial \bar{Y}^2}\right)^2 + \beta_3\left(\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\right)^2 + \beta_4\frac{\partial^2 W}{\partial \bar{X}^2}\frac{\partial^2 W}{\partial \bar{Y}^2} + \beta_5\frac{\partial^2 W}{\partial \bar{X}^2}\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}} + \beta_6\frac{\partial^2 W}{\partial \bar{Y}^2}\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\right.$$
$$+\beta_7\left(\frac{\partial^2 \Phi_X}{\partial \bar{X}^2}\right)^2 + \beta_8\left(\frac{\partial^2 \Phi_X}{\partial \bar{Y}^2}\right)^2 + \beta_9\left(\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}}\right)^2 + \beta_{10}\frac{\partial^2 \Phi_X}{\partial \bar{X}^2}\frac{\partial^2 \Phi_X}{\partial \bar{Y}^2} + \beta_{11}\frac{\partial^2 \Phi_X}{\partial \bar{X}^2}\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}} + \beta_{12}\frac{\partial^2 \Phi_X}{\partial \bar{Y}^2}\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}}$$
$$+\beta_{13}\left(\frac{\partial^2 \Phi_Y}{\partial \bar{X}^2}\right)^2 + \beta_{14}\left(\frac{\partial^2 \Phi_Y}{\partial \bar{Y}^2}\right)^2 + \beta_{15}\left(\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}}\right)^2 + \beta_{16}\frac{\partial^2 \Phi_Y}{\partial \bar{X}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{Y}^2} + \beta_{17}\frac{\partial^2 \Phi_Y}{\partial \bar{X}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}} + \beta_{18}\frac{\partial^2 \Phi_Y}{\partial \bar{Y}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}}$$
$$+\beta_{19}\frac{\partial^2 \Phi_X}{\partial \bar{X}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{X}^2} + \beta_{20}\frac{\partial^2 \Phi_X}{\partial \bar{Y}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{Y}^2} + \beta_{21}\frac{\partial^2 \Phi_X}{\partial \bar{X}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{Y}^2} + \beta_{22}\frac{\partial^2 \Phi_Y}{\partial \bar{X}^2}\frac{\partial^2 \Phi_X}{\partial \bar{Y}^2} + \beta_{23}\frac{\partial^2 \Phi_X}{\partial \bar{X}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}} +$$
$$\beta_{24}\frac{\partial^2 \Phi_X}{\partial \bar{Y}^2}\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}} + \beta_{25}\frac{\partial^2 \Phi_Y}{\partial \bar{Y}^2}\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}} + \beta_{26}\frac{\partial^2 \Phi_Y}{\partial \bar{X}^2}\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}} + \beta_{27}\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}}\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}} + \beta_{28}\frac{\partial^2 W}{\partial \bar{Y}^2}\frac{\partial \Phi_X}{\partial \bar{Y}} + \beta_{29}\frac{\partial^2 W}{\partial \bar{Y}^2}\frac{\partial \Phi_X}{\partial \bar{X}}$$
$$+\beta_{30}\frac{\partial^2 W}{\partial \bar{Y}^2}\frac{\partial \Phi_Y}{\partial \bar{Y}} + \beta_{31}\frac{\partial^2 W}{\partial \bar{Y}^2}\frac{\partial \Phi_Y}{\partial \bar{X}} + \beta_{32}\frac{\partial^2 W}{\partial \bar{X}^2}\frac{\partial \Phi_X}{\partial \bar{Y}} + \beta_{33}\frac{\partial^2 W}{\partial \bar{X}^2}\frac{\partial \Phi_X}{\partial \bar{X}} + \beta_{34}\frac{\partial^2 W}{\partial \bar{X}^2}\frac{\partial \Phi_Y}{\partial \bar{Y}} + \beta_{35}\frac{\partial^2 W}{\partial \bar{X}^2}\frac{\partial \Phi_Y}{\partial \bar{X}} +$$
$$\beta_{36}\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\frac{\partial \Phi_X}{\partial \bar{Y}} + \beta_{37}\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\frac{\partial \Phi_X}{\partial \bar{X}} + \beta_{38}\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\frac{\partial \Phi_Y}{\partial \bar{Y}} + \beta_{39}\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\frac{\partial \Phi_Y}{\partial \bar{X}} + \beta_{40}\frac{\partial \Phi_X}{\partial \bar{Y}}\frac{\partial \Phi_Y}{\partial \bar{Y}} + \beta_{41}\frac{\partial \Phi_Y}{\partial \bar{X}}\frac{\partial \Phi_X}{\partial \bar{Y}}$$
$$+\beta_{42}\frac{\partial \Phi_X}{\partial \bar{X}}\frac{\partial \Phi_X}{\partial \bar{Y}} + \beta_{43}\frac{\partial \Phi_X}{\partial \bar{X}}\frac{\partial \Phi_Y}{\partial \bar{Y}} + \beta_{44}\frac{\partial \Phi_X}{\partial \bar{X}}\frac{\partial \Phi_Y}{\partial \bar{X}} + \beta_{45}\frac{\partial \Phi_Y}{\partial \bar{X}}\frac{\partial \Phi_Y}{\partial \bar{Y}} + \beta_{46}\left(\frac{\partial W}{\partial \bar{X}}\right)^2 + \beta_{47}\left(\frac{\partial W}{\partial \bar{Y}}\right)^2 +$$
$$\beta_{48}\left(\frac{\partial \Phi_X}{\partial \bar{X}}\right)^2 + \beta_{49}\left(\frac{\partial \Phi_X}{\partial \bar{Y}}\right)^2 + \beta_{50}\left(\frac{\partial \Phi_Y}{\partial \bar{X}}\right)^2 + \beta_{51}\left(\frac{\partial \Phi_Y}{\partial \bar{Y}}\right)^2 + \beta_{52}\frac{\partial W}{\partial \bar{X}}\frac{\partial W}{\partial \bar{Y}} + \beta_{53}\Phi_X\frac{\partial W}{\partial \bar{X}} + \beta_{54}\Phi_X\frac{\partial W}{\partial \bar{Y}} +$$
$$\left.\beta_{55}\Phi_Y\frac{\partial W}{\partial \bar{X}} + \beta_{56}\Phi_Y\frac{\partial W}{\partial \bar{Y}} + \beta_{57}\left(\Phi_X^2 + \Phi_Y^2\right)\right]\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right| d\bar{X}d\bar{Y} \quad , (15)$$

where the coordinate transformation coefficients β_{m} are listed in Appendix A. Next, we use the GLQ and DQ rules to discretize Eq. (15).

In a standard interval (i.e., $\bar{X} \in [-1,\ 1]$), the integral points and weighting coefficients for the fourth-order Gauss-Lobatto quadrature rule are as follows[51]:

$$\bar{X}_1 = -1,\ \ \bar{X}_2 = -1/\sqrt{5},\ \ \bar{X}_3 = 1/\sqrt{5},\ \ \bar{X}_4 = 1, \quad (16a)$$

$$C_1 = C_4 = 1/6,\ \ C_2 = C_3 = 5/6\,. \quad (16b)$$

Based on Eq. (16a), we can obtain the following horizontal and longitudinal coordinate vectors formed at all Gauss-Lobatto quadrature points in the domain $[-1,\ 1]\times[-1,\ 1]$:

$$\bar{\boldsymbol{X}}_{(GL)} = \begin{bmatrix} -1 & -\frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 1 \end{bmatrix},\ \ \bar{\boldsymbol{Y}}_{(GL)} = \begin{bmatrix} -1 & -\frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 1 \end{bmatrix}. \quad (17)$$

where the subscript “(GL)” is used to label the quantities at Gauss-Lobatto quadrature points.

Applying the Lagrange interpolation technique, the trial functions of W, Φ_X, and Φ_Y are expressed as

$$\Delta = \sum_{i=1}^{4}\sum_{j=1}^{4} l_{\bar{X}i}\left(\bar{X}\right) l_{\bar{Y}j}\left(\bar{Y}\right)\Delta_{ij}\ , \quad (18)$$

where Δ_{ij} denotes the function value of W, Φ_X, or Φ_Y at the ij^{th} GLQ point; $l_{\bar{X}i}\left(\bar{X}\right)$ and $l_{\bar{Y}j}\left(\bar{Y}\right)$ are the Lagrange interpolation polynomials along the $\bar{X}$ and $\bar{Y}$ directions, respectively.

Using the 2D DQ rule, the first- and second-order partial derivatives of W, Φ_X, and Φ_Y at all quadrature points can be written in matrix form as follows:

$$\begin{bmatrix} \boldsymbol{D}_{\bar{X}}^{(1)}\boldsymbol{W}_{(GL)} \\ \boldsymbol{D}_{\bar{X}}^{(1)}\boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{D}_{\bar{X}}^{(1)}\boldsymbol{\Phi}_{Y(GL)} \end{bmatrix} = \boldsymbol{A}_{\bar{X}}^{(1)} \begin{bmatrix} \boldsymbol{W}_{(GL)} \\ \boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{\Phi}_{Y(GL)} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{D}_{\bar{Y}}^{(1)}\boldsymbol{W}_{(GL)} \\ \boldsymbol{D}_{\bar{Y}}^{(1)}\boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{D}_{\bar{Y}}^{(1)}\boldsymbol{\Phi}_{Y(GL)} \end{bmatrix} = \boldsymbol{A}_{\bar{Y}}^{(1)} \begin{bmatrix} \boldsymbol{W}_{(GL)} \\ \boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{\Phi}_{Y(GL)} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{D}_{\bar{X}}^{(2)}\boldsymbol{W}_{(GL)} \\ \boldsymbol{D}_{\bar{X}}^{(2)}\boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{D}_{\bar{X}}^{(2)}\boldsymbol{\Phi}_{Y(GL)} \end{bmatrix} = \boldsymbol{A}_{\bar{X}}^{(2)} \begin{bmatrix} \boldsymbol{W}_{(GL)} \\ \boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{\Phi}_{Y(GL)} \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{D}_{\bar{X}\bar{Y}}^{(1\oplus 1)}\boldsymbol{W}_{(GL)} \\ \boldsymbol{D}_{\bar{X}\bar{Y}}^{(1\oplus 1)}\boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{D}_{\bar{X}\bar{Y}}^{(1\oplus 1)}\boldsymbol{\Phi}_{Y(GL)} \end{bmatrix} = \boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)} \begin{bmatrix} \boldsymbol{W}_{(GL)} \\ \boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{\Phi}_{Y(GL)} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{D}_{\bar{Y}}^{(2)}\boldsymbol{W}_{(GL)} \\ \boldsymbol{D}_{\bar{Y}}^{(2)}\boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{D}_{\bar{Y}}^{(2)}\boldsymbol{\Phi}_{Y(GL)} \end{bmatrix} = \boldsymbol{A}_{\bar{Y}}^{(2)} \begin{bmatrix} \boldsymbol{W}_{(GL)} \\ \boldsymbol{\Phi}_{X(GL)} \\ \boldsymbol{\Phi}_{Y(GL)} \end{bmatrix}, \tag{19}$$

where $\boldsymbol{D}_{\bar{X}}^{(p)}\boldsymbol{F}_{(GL)}$, $\boldsymbol{D}_{\bar{Y}}^{(q)}\boldsymbol{F}_{(GL)}$, and $\boldsymbol{D}_{\bar{X}\bar{Y}}^{(p\oplus q)}\boldsymbol{F}_{(GL)}$ are the following partial derivative matrices:

$$\boldsymbol{D}_{\bar{X}}^{(p)}\boldsymbol{F}_{(GL)} = \left[\left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{11}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{21}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{31}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{41}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{12}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{22}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{32}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{42}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{13}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{23}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{33}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{43}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{14}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{24}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{34}, \left(\frac{\partial^p F}{\partial \bar{X}^p}\right)_{44} \right]^{\mathrm{T}}, \tag{20a}$$

$$\boldsymbol{D}_{\bar{Y}}^{(q)}\boldsymbol{F}_{(GL)} = \left[\left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{11}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{21}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{31}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{41}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{12}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{22}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{32}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{42}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{13}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{23}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{33}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{43}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{14}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{24}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{34}, \left(\frac{\partial^q F}{\partial \bar{Y}^q}\right)_{44} \right]^{\mathrm{T}}, \tag{20b}$$

$$\boldsymbol{D}_{\bar{X}\bar{Y}}^{(p\oplus q)}\boldsymbol{F}_{(GL)} = \left[\left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{11}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{21}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{31}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{41}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{12}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{22}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{32}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{42}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{13}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{23}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{33}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{43}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{14}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{24}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{34}, \left(\frac{\partial^{p+q} F}{\partial \bar{X}^p \partial \bar{Y}^q}\right)_{44} \right]^{\mathrm{T}}, \tag{20c}$$

where $(f)_{ij}$ indicates the function value of f at the ijth GLQ point.

$\boldsymbol{W}_{(GL)}$, $\boldsymbol{\Phi}_{X(GL)}$, and $\boldsymbol{\Phi}_{Y(GL)}$ can be uniformly written in the following column vector:

$$\boldsymbol{\Delta}_{(GL)} = \left[\Delta_{11}, \Delta_{21}, \Delta_{31}, \Delta_{41}, \Delta_{12}, \Delta_{22}, \Delta_{32}, \Delta_{42}, \Delta_{13}, \Delta_{23}, \Delta_{33}, \Delta_{43}, \Delta_{14}, \Delta_{24}, \Delta_{34}, \Delta_{44}\right]^{\mathrm{T}}. \tag{21}$$

$\boldsymbol{A}_{\bar{X}}^{(1)}$, $\boldsymbol{A}_{\bar{Y}}^{(1)}$, $\boldsymbol{A}_{\bar{X}}^{(2)}$, $\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}$, and $\boldsymbol{A}_{\bar{Y}}^{(2)}$ are the following 16×16 weighting coefficient matrices:

$$\boldsymbol{A}_{\bar{X}}^{(1)} = \frac{1}{4}\begin{bmatrix} \boldsymbol{\Lambda}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Lambda}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Lambda}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Lambda}_1 \end{bmatrix}, \quad \boldsymbol{\Lambda}_1 = \begin{bmatrix} -6 & \frac{5}{2}\left(1+\sqrt{5}\right) & \frac{5}{2}\left(1-\sqrt{5}\right) & 1 \\ -\frac{1}{2}\left(1+\sqrt{5}\right) & 0 & \sqrt{5} & \frac{1}{2}\left(1-\sqrt{5}\right) \\ -\frac{1}{2}\left(1-\sqrt{5}\right) & -\sqrt{5} & 0 & \frac{1}{2}\left(1+\sqrt{5}\right) \\ -1 & -\frac{5}{2}\left(1-\sqrt{5}\right) & -\frac{5}{2}\left(1+\sqrt{5}\right) & 6 \end{bmatrix},$$

$$\boldsymbol{A}_{\bar{X}}^{(2)}=\frac{1}{4}\begin{bmatrix}\boldsymbol{\Lambda}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Lambda}_2 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_2\end{bmatrix},\quad \boldsymbol{\Lambda}_2=\begin{bmatrix}20 & -5\left(1+3\sqrt{5}\right) & -5\left(1-3\sqrt{5}\right) & -10\\ 5+3\sqrt{5} & -20 & 10 & 5-3\sqrt{5}\\ 5-3\sqrt{5} & 10 & -20 & 5+3\sqrt{5}\\ -10 & -5\left(1-3\sqrt{5}\right) & -5\left(1+3\sqrt{5}\right) & 20\end{bmatrix},$$

$$\boldsymbol{A}_{\bar{Y}}^{(1)}=\frac{1}{2}\begin{bmatrix}\langle -6\rangle & \left\langle \frac{5}{2}\left(1+\sqrt{5}\right)\right\rangle & \left\langle \frac{5}{2}\left(1-\sqrt{5}\right)\right\rangle & \langle 1\rangle\\ \left\langle -\frac{1}{2}\left(1+\sqrt{5}\right)\right\rangle & \langle 0\rangle & \left\langle \sqrt{5}\right\rangle & \left\langle \frac{1}{2}\left(1-\sqrt{5}\right)\right\rangle\\ \left\langle -\frac{1}{2}\left(1-\sqrt{5}\right)\right\rangle & \left\langle -\sqrt{5}\right\rangle & \langle 0\rangle & \left\langle \frac{1}{2}\left(1+\sqrt{5}\right)\right\rangle\\ \langle -1\rangle & \left\langle -\frac{5}{2}\left(1-\sqrt{5}\right)\right\rangle & \left\langle -\frac{5}{2}\left(1+\sqrt{5}\right)\right\rangle & \langle 6\rangle\end{bmatrix},$$

$$\boldsymbol{A}_{\bar{X}}^{(2)}=\frac{1}{4}\begin{bmatrix}\langle 20\rangle & \left\langle -5\left(1+3\sqrt{5}\right)\right\rangle & \left\langle -5\left(1-3\sqrt{5}\right)\right\rangle & \langle -10\rangle\\ \left\langle 5+3\sqrt{5}\right\rangle & \langle -20\rangle & \langle 10\rangle & \left\langle 5-3\sqrt{5}\right\rangle\\ \left\langle 5-3\sqrt{5}\right\rangle & \langle 10\rangle & \langle -20\rangle & \left\langle 5+3\sqrt{5}\right\rangle\\ \langle -10\rangle & \left\langle -5\left(1-3\sqrt{5}\right)\right\rangle & \left\langle -5\left(1+3\sqrt{5}\right)\right\rangle & \langle 20\rangle\end{bmatrix}, \tag{22}$$

where $\langle f\rangle$ represents a 4×4 diagonal matrix with identical element f ; $\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}$ can be obtained by multiplying $\boldsymbol{A}_{\bar{X}}^{(1)}$ and $\boldsymbol{A}_{\bar{Y}}^{(1)}$.

The integral weighting coefficient matrix $\boldsymbol{C}_{(GL)}$ formed at all quadrature points is

$$\boldsymbol{C}_{(GL)}=diag\left(\left[1,\ 5,\ 5,\ 1,\ 5,\ 25,\ 25,\ 5,\ 5,\ 25,\ 25,\ 5,\ 1,\ 5,\ 5,\ 1\right]\right)/36, \tag{23}$$

In view of Eq. (19), we can discretize Eq. (15) as follows

$$\begin{aligned}\Pi_e=&\frac{1}{2}\left(\dot{\boldsymbol{W}}_{(GL)}^{\mathrm{T}}\bar{\boldsymbol{M}}_{(WW)}^{(e)}\dot{\boldsymbol{W}}_{(GL)}+\dot{\boldsymbol{\Phi}}_{X(GL)}^{\mathrm{T}}\bar{\boldsymbol{M}}_{(\Phi_X\Phi_X)}^{(e)}\dot{\boldsymbol{\Phi}}_{X(GL)}+\dot{\boldsymbol{\Phi}}_{Y(GL)}^{\mathrm{T}}\bar{\boldsymbol{M}}_{(\Phi_Y\Phi_Y)}^{(e)}\dot{\boldsymbol{\Phi}}_{Y(GL)}\right)\\ &-\frac{1}{2}\begin{pmatrix}\boldsymbol{W}_{(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(WW)}^{(e)}\boldsymbol{W}_{(GL)}+\boldsymbol{\Phi}_{X(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(\Phi_X\Phi_X)}^{(e)}\boldsymbol{\Phi}_{X(GL)}+\boldsymbol{\Phi}_{Y(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(\Phi_Y\Phi_Y)}^{(e)}\boldsymbol{\Phi}_{Y(GL)}+\\ \boldsymbol{W}_{(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(W\Phi_X)}^{(e)}\boldsymbol{\Phi}_{X(GL)}+\boldsymbol{\Phi}_{X(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(\Phi_X W)}^{(e)}\boldsymbol{W}_{(GL)}+\boldsymbol{W}_{(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(W\Phi_Y)}^{(e)}\boldsymbol{\Phi}_{Y(GL)}+\\ \boldsymbol{\Phi}_{Y(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(\Phi_Y W)}^{(e)}\boldsymbol{W}_{(GL)}+\boldsymbol{\Phi}_{X(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(\Phi_X\Phi_Y)}^{(e)}\boldsymbol{\Phi}_{Y(GL)}+\boldsymbol{\Phi}_{Y(GL)}^{\mathrm{T}}\bar{\boldsymbol{K}}_{(\Phi_Y\Phi_X)}^{(e)}\boldsymbol{\Phi}_{X(GL)}\end{pmatrix},\\ &+\boldsymbol{W}_{(GL)}^{\mathrm{T}}\bar{\boldsymbol{Q}}_{(W)}^{(e)}\end{aligned} \tag{24}$$

where

$$\bar{\boldsymbol{K}}_{(WW)}^{(e)}=\boldsymbol{J}\begin{bmatrix}2\boldsymbol{\beta}_1\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+2\boldsymbol{\beta}_2\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+2\boldsymbol{\beta}_3\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}\\ +\boldsymbol{\beta}_4\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+\boldsymbol{\beta}_4\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_5\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}+\\ \boldsymbol{\beta}_5\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_6\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}+\boldsymbol{\beta}_6\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus 1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+\\ 2\boldsymbol{\beta}_{46}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+2\boldsymbol{\beta}_{47}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{52}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\\ \boldsymbol{\beta}_{52}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}\end{bmatrix},$$

$$\bar{\boldsymbol{K}}_{(\Phi_X\Phi_X)}^{(e)}=\boldsymbol{J}\begin{bmatrix} 2\boldsymbol{\beta}_7\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+2\boldsymbol{\beta}_8\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+2\boldsymbol{\beta}_9\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)} \\ +\boldsymbol{\beta}_{10}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+\boldsymbol{\beta}_{10}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_{11}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}+ \\ \boldsymbol{\beta}_{11}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_{12}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}+\boldsymbol{\beta}_{12}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)} \\ +\boldsymbol{\beta}_{42}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{42}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+2\boldsymbol{\beta}_{48}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+ \\ 2\boldsymbol{\beta}_{49}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+2\boldsymbol{\beta}_{57}\boldsymbol{C}_{(GL)} \end{bmatrix},$$

$$\bar{\boldsymbol{K}}_{(\Phi_Y\Phi_Y)}^{(e)}=\boldsymbol{J}\begin{bmatrix} 2\boldsymbol{\beta}_{13}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+2\boldsymbol{\beta}_{14}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+2\boldsymbol{\beta}_{15}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)} \\ +\boldsymbol{\beta}_{16}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+\boldsymbol{\beta}_{16}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_{17}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}+ \\ \boldsymbol{\beta}_{17}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_{18}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1\oplus1)}+\boldsymbol{\beta}_{18}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+ \\ 2\boldsymbol{\beta}_{50}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+2\boldsymbol{\beta}_{51}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{45}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+ \\ \boldsymbol{\beta}_{45}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+2\boldsymbol{\beta}_{57}\boldsymbol{C}_{(GL)} \end{bmatrix},$$

$$\left(\bar{\boldsymbol{K}}_{(\Phi_X W)}^{(e)}\right)^{\mathrm{T}}=\bar{\boldsymbol{K}}_{(W\Phi_X)}^{(e)}=\boldsymbol{J}\begin{bmatrix} \boldsymbol{\beta}_{28}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{29}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+\boldsymbol{\beta}_{32}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+ \\ \boldsymbol{\beta}_{33}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+\boldsymbol{\beta}_{36}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{37}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)} \\ +\boldsymbol{\beta}_{53}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}+\boldsymbol{\beta}_{54}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)} \end{bmatrix},$$

$$\left(\bar{\boldsymbol{K}}_{(\Phi_Y\Phi_X)}^{(e)}\right)^{\mathrm{T}}=\bar{\boldsymbol{K}}_{(\Phi_X\Phi_Y)}^{(e)}=\boldsymbol{J}\begin{bmatrix} \boldsymbol{\beta}_{19}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_{20}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+\boldsymbol{\beta}_{21}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)} \\ +\boldsymbol{\beta}_{22}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_{23}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}+\boldsymbol{\beta}_{24}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)} \\ +\boldsymbol{\beta}_{25}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(2)}+\boldsymbol{\beta}_{26}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(2)}+\boldsymbol{\beta}_{27}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)} \\ +\boldsymbol{\beta}_{40}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{41}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+\boldsymbol{\beta}_{43}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+ \\ \boldsymbol{\beta}_{44}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{T}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)} \end{bmatrix},$$

$$\left(\bar{\boldsymbol{K}}_{(\Phi_Y W)}^{(e)}\right)^{\mathrm{T}}=\bar{\boldsymbol{K}}_{(W\Phi_Y)}^{(e)}=\boldsymbol{J}\begin{bmatrix} \boldsymbol{\beta}_{30}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{31}\left(\boldsymbol{A}_{\bar{Y}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+\boldsymbol{\beta}_{34}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+ \\ \boldsymbol{\beta}_{35}\left(\boldsymbol{A}_{\bar{X}}^{(2)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)}+\boldsymbol{\beta}_{38}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{Y}}^{(1)}+\boldsymbol{\beta}_{39}\left(\boldsymbol{A}_{\bar{X}\bar{Y}}^{(1\oplus1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}\boldsymbol{A}_{\bar{X}}^{(1)} \\ +\boldsymbol{\beta}_{55}\left(\boldsymbol{A}_{\bar{X}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)}+\boldsymbol{\beta}_{56}\left(\boldsymbol{A}_{\bar{Y}}^{(1)}\right)^{\mathrm{T}}\boldsymbol{C}_{(GL)} \end{bmatrix}, \tag{25}$$

$$\bar{\boldsymbol{M}}_{(WW)}^{(e)}=\rho h\boldsymbol{J}\boldsymbol{C}_{(GL)},\quad \bar{\boldsymbol{M}}_{(\Phi_X\Phi_X)}^{(e)}=\bar{\boldsymbol{M}}_{(\Phi_Y\Phi_Y)}^{(e)}=\frac{\rho h^3\boldsymbol{J}\boldsymbol{C}_{(GL)}}{12}, \tag{26}$$

$$\bar{\boldsymbol{Q}}_{(W)}^{(e)}=\boldsymbol{J}\boldsymbol{C}_{(GL)}\boldsymbol{Q}_{(GL)}. \tag{27}$$

Unless otherwise specified, the superscript "T" denotes the transpose of a matrix throughout this paper. The matrices $\boldsymbol{J}$, $\boldsymbol{\beta}_n$, and $\boldsymbol{Q}_{(GL)}$ are defined as follows:

$$\boldsymbol{J}=diag\left(\begin{matrix}\left[\left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{11},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{21},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{31},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{41},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{12},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{22},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{32},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{42},\right. \\ \left.\left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{13},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{23},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{33},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{43},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{14},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{24},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{34},\ \left(\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|\right)_{44}\right]\end{matrix}\right), \tag{28}$$

$$\boldsymbol{\beta}_n = diag\left(\begin{matrix}\left[(\beta_n)_{11},\ (\beta_n)_{21},\ (\beta_n)_{31},\ (\beta_n)_{41},\ (\beta_n)_{12},\ (\beta_n)_{22},\ (\beta_n)_{32},\ (\beta_n)_{42},\right.\\ \left.(\beta_n)_{13},\ (\beta_n)_{23},\ (\beta_n)_{33},\ (\beta_n)_{43},\ (\beta_n)_{14},\ (\beta_n)_{24},\ (\beta_n)_{34},\ (\beta_n)_{44}\right]\end{matrix}\right), \tag{29}$$

$$\boldsymbol{Q}_{(GL)} = \left[Q_{11},\ Q_{21},\ Q_{31},\ Q_{41},\ Q_{12},\ Q_{22},\ Q_{32},\ Q_{42},\right.$$
$$\left. Q_{13},\ Q_{23},\ Q_{33},\ Q_{43},\ Q_{14},\ Q_{24},\ Q_{34},\ Q_{44}\right]^{\mathrm{T}}. \tag{30}$$

where $\left|\boldsymbol{J}_{\bar{X}\bar{Y}}\right|_{pq}$ denotes the determinant of Jacobian matrix at the pq^{th} quadrature point.

To realize the C^1-continuity conditions, the following local nodal displacement vector is defined:

$$\begin{aligned}\boldsymbol{\Theta}_N^{(Local)} = \Bigg[& (W)_{11},\ \left(\frac{\partial W}{\partial \bar{X}}\right)_{11},\ \left(\frac{\partial W}{\partial \bar{Y}}\right)_{11},\ \left(\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\right)_{11},\ (\Phi_X)_{11},\ \left(\frac{\partial \Phi_X}{\partial \bar{X}}\right)_{11},\ \left(\frac{\partial \Phi_X}{\partial \bar{Y}}\right)_{11},\ \left(\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}}\right)_{11},\\ & (\Phi_Y)_{11},\ \left(\frac{\partial \Phi_Y}{\partial \bar{X}}\right)_{11},\ \left(\frac{\partial \Phi_Y}{\partial \bar{Y}}\right)_{11},\ \left(\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}}\right)_{11},\ (W)_{41},\ \left(\frac{\partial W}{\partial \bar{X}}\right)_{41},\ \left(\frac{\partial W}{\partial \bar{Y}}\right)_{41},\ \left(\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\right)_{41},\\ & (\Phi_X)_{41},\ \left(\frac{\partial \Phi_X}{\partial \bar{X}}\right)_{41},\ \left(\frac{\partial \Phi_X}{\partial \bar{Y}}\right)_{41},\ \left(\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}}\right)_{41},\ (\Phi_Y)_{41},\ \left(\frac{\partial \Phi_Y}{\partial \bar{X}}\right)_{41},\ \left(\frac{\partial \Phi_Y}{\partial \bar{Y}}\right)_{41},\ \left(\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}}\right)_{41},\\ & (W)_{44},\ \left(\frac{\partial W}{\partial \bar{X}}\right)_{44},\ \left(\frac{\partial W}{\partial \bar{Y}}\right)_{44},\ \left(\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\right)_{44},\ (\Phi_X)_{44},\ \left(\frac{\partial \Phi_X}{\partial \bar{X}}\right)_{44},\ \left(\frac{\partial \Phi_X}{\partial \bar{Y}}\right)_{44},\ \left(\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}}\right)_{44},\\ & (\Phi_Y)_{44},\ \left(\frac{\partial \Phi_Y}{\partial \bar{X}}\right)_{44},\ \left(\frac{\partial \Phi_Y}{\partial \bar{Y}}\right)_{44},\ \left(\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}}\right)_{44},\ (W)_{14},\ \left(\frac{\partial W}{\partial \bar{X}}\right)_{14},\ \left(\frac{\partial W}{\partial \bar{Y}}\right)_{14},\ \left(\frac{\partial^2 W}{\partial \bar{X}\partial \bar{Y}}\right)_{14},\\ & (\Phi_X)_{14},\ \left(\frac{\partial \Phi_X}{\partial \bar{X}}\right)_{14},\ \left(\frac{\partial \Phi_X}{\partial \bar{Y}}\right)_{14},\ \left(\frac{\partial^2 \Phi_X}{\partial \bar{X}\partial \bar{Y}}\right)_{14},\ (\Phi_Y)_{14},\ \left(\frac{\partial \Phi_Y}{\partial \bar{X}}\right)_{14},\ \left(\frac{\partial \Phi_Y}{\partial \bar{Y}}\right)_{14},\ \left(\frac{\partial^2 \Phi_Y}{\partial \bar{X}\partial \bar{Y}}\right)_{14}\Bigg]^{\mathrm{T}}\end{aligned} \tag{31}$$

Using DQ rule, $\boldsymbol{W}_{(GL)}$, $\boldsymbol{\Phi}_{X(GL)}$, and $\boldsymbol{\Phi}_{Y(GL)}$ are written in terms of $\boldsymbol{\Theta}_N^{(Local)}$ as follows:

$$\boldsymbol{W}_{(GL)} = \boldsymbol{B}^{-1}\boldsymbol{T}_W\boldsymbol{\Theta}_N^{(Local)},\quad \boldsymbol{\Phi}_{X(GL)} = \boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}\boldsymbol{\Theta}_N^{(Local)},\quad \boldsymbol{\Phi}_{Y(GL)} = \boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}\boldsymbol{\Theta}_N^{(Local)}, \tag{32}$$

where

$$\boldsymbol{T}_W = \begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} \end{bmatrix}, \tag{33a}$$

$$\boldsymbol{T}_{\Phi_X} = \begin{bmatrix} \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \end{bmatrix}, \tag{33b}$$

$$\boldsymbol{T}_{\Phi_Y} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{bmatrix}, \tag{33c}$$

where $\mathbf{I}$ and $\mathbf{O}$ are 4×4 identity and null matrices, respectively, and

$$\boldsymbol{B}=\begin{bmatrix} \boldsymbol{B}_{(11)(11)} & \boldsymbol{B}_{(11)(41)} & \boldsymbol{B}_{(11)(44)} & \boldsymbol{B}_{(11)(14)} \\ \boldsymbol{B}_{(41)(11)} & \boldsymbol{B}_{(41)(41)} & \boldsymbol{B}_{(41)(44)} & \boldsymbol{B}_{(41)(14)} \\ \boldsymbol{B}_{(44)(11)} & \boldsymbol{B}_{(44)(41)} & \boldsymbol{B}_{(44)(44)} & \boldsymbol{B}_{(44)(14)} \\ \boldsymbol{B}_{(14)(11)} & \boldsymbol{B}_{(14)(41)} & \boldsymbol{B}_{(14)(44)} & \boldsymbol{B}_{(14)(14)} \end{bmatrix}. \tag{34}$$

The sub-block matrices $\boldsymbol{B}_{(ij)(kl)}$ are listed in Appendix B.

Substituting Eq. (32) into Eq. (24), we can obtain the local stiffness and mass matrices and the local load vector of the element as follows:

$$\bar{\boldsymbol{M}}^{(e)}_{(Local)}=\left(\boldsymbol{B}^{-1}\boldsymbol{T}_W\right)^{\mathrm{T}}\bar{\boldsymbol{M}}^{(e)}_{(WW)}\boldsymbol{B}^{-1}\boldsymbol{T}_W+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}\right)^{\mathrm{T}}\bar{\boldsymbol{M}}^{(e)}_{(\Phi_X\Phi_X)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}\right)^{\mathrm{T}}\bar{\boldsymbol{M}}^{(e)}_{(\Phi_Y\Phi_Y)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}, \tag{35}$$

$$\begin{aligned}\bar{\boldsymbol{K}}^{(e)}_{(Local)}=&\left(\boldsymbol{B}^{-1}\boldsymbol{T}_W\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(WW)}\boldsymbol{B}^{-1}\boldsymbol{T}_W+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(\Phi_X\Phi_X)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(\Phi_Y\Phi_Y)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}+\\&\left(\boldsymbol{B}^{-1}\boldsymbol{T}_W\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(W\Phi_X)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(\Phi_XW)}\boldsymbol{B}^{-1}\boldsymbol{T}_W+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_W\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(W\Phi_Y)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}+\\&\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(\Phi_YW)}\boldsymbol{B}^{-1}\boldsymbol{T}_W+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(\Phi_X\Phi_Y)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}+\left(\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_Y}\right)^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(\Phi_Y\Phi_X)}\boldsymbol{B}^{-1}\boldsymbol{T}_{\Phi_X}\end{aligned}, \tag{36}$$

$$\bar{\boldsymbol{Q}}^{(e)}_{(Local)}=\left(\boldsymbol{B}^{-1}\boldsymbol{T}_W\right)^{\mathrm{T}}\bar{\boldsymbol{Q}}^{(e)}_{(W)}. \tag{37}$$

Since the element stiffness and mass matrices as well as the load vector have to be assembled in the global coordinate system, $\boldsymbol{\Theta}_N^{(Local)}$ should be converted into its global counterpart as follows:

$$\boldsymbol{\Theta}_N^{(\mathrm{Local})}=\boldsymbol{R}\begin{bmatrix}\boldsymbol{\Theta}_{N(1)}^{(Global)}\\ \boldsymbol{\Theta}_{N(2)}^{(Global)}\end{bmatrix}, \tag{38}$$

where

$$\begin{aligned}\boldsymbol{\Theta}_{N(1)}^{(Global)}=\Bigg[&(W)_{11},\ \left(\frac{\partial W}{\partial X}\right)_{11},\ \left(\frac{\partial W}{\partial Y}\right)_{11},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{11},\ (\Phi_X)_{11},\ \left(\frac{\partial \Phi_X}{\partial X}\right)_{11},\ \left(\frac{\partial \Phi_X}{\partial Y}\right)_{11},\ \left(\frac{\partial^2 \Phi_X}{\partial X\,\partial Y}\right)_{11},\\&(\Phi_Y)_{11},\ \left(\frac{\partial \Phi_Y}{\partial X}\right)_{11},\ \left(\frac{\partial \Phi_Y}{\partial Y}\right)_{11},\ \left(\frac{\partial^2 \Phi_Y}{\partial X\,\partial Y}\right)_{11},\ (W)_{41},\ \left(\frac{\partial W}{\partial X}\right)_{41},\ \left(\frac{\partial W}{\partial Y}\right)_{41},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{41},\\&(\Phi_X)_{41},\ \left(\frac{\partial \Phi_X}{\partial X}\right)_{41},\ \left(\frac{\partial \Phi_X}{\partial Y}\right)_{41},\ \left(\frac{\partial^2 \Phi_X}{\partial X\,\partial Y}\right)_{41},\ (\Phi_Y)_{41},\ \left(\frac{\partial \Phi_Y}{\partial X}\right)_{41},\ \left(\frac{\partial \Phi_Y}{\partial Y}\right)_{41},\ \left(\frac{\partial^2 \Phi_Y}{\partial X\,\partial Y}\right)_{41},\\&(W)_{44},\ \left(\frac{\partial W}{\partial X}\right)_{44},\ \left(\frac{\partial W}{\partial Y}\right)_{44},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{44},\ (\Phi_X)_{44},\ \left(\frac{\partial \Phi_X}{\partial X}\right)_{44},\ \left(\frac{\partial \Phi_X}{\partial Y}\right)_{44},\ \left(\frac{\partial^2 \Phi_X}{\partial X\,\partial Y}\right)_{44},\\&(\Phi_Y)_{44},\ \left(\frac{\partial \Phi_Y}{\partial X}\right)_{44},\ \left(\frac{\partial \Phi_Y}{\partial Y}\right)_{44},\ \left(\frac{\partial^2 \Phi_Y}{\partial X\,\partial Y}\right)_{44},\ (W)_{14},\ \left(\frac{\partial W}{\partial X}\right)_{14},\ \left(\frac{\partial W}{\partial Y}\right)_{14},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{14},\\&(\Phi_X)_{14},\ \left(\frac{\partial \Phi_X}{\partial X}\right)_{14},\ \left(\frac{\partial \Phi_X}{\partial Y}\right)_{14},\ \left(\frac{\partial^2 \Phi_X}{\partial X\,\partial Y}\right)_{14},\ (\Phi_Y)_{14},\ \left(\frac{\partial \Phi_Y}{\partial X}\right)_{14},\ \left(\frac{\partial \Phi_Y}{\partial Y}\right)_{14},\ \left(\frac{\partial^2 \Phi_Y}{\partial X\,\partial Y}\right)_{14}\Bigg]^{\mathrm{T}}\end{aligned}, \tag{39a}$$

$$\mathbf{\Theta}_{N(2)}^{(Global)}=\left[\left(\frac{\partial^2 W}{\partial X^2}\right)_{11},\ \left(\frac{\partial^2 W}{\partial Y^2}\right)_{11},\ \left(\frac{\partial^2 \Phi_X}{\partial X^2}\right)_{11},\ \left(\frac{\partial^2 \Phi_X}{\partial Y^2}\right)_{11},\ \left(\frac{\partial^2 \Phi_Y}{\partial X^2}\right)_{11},\ \left(\frac{\partial^2 \Phi_Y}{\partial Y^2}\right)_{11},\right.$$
$$\left(\frac{\partial^2 W}{\partial X^2}\right)_{41},\ \left(\frac{\partial^2 W}{\partial Y^2}\right)_{41},\ \left(\frac{\partial^2 \Phi_X}{\partial X^2}\right)_{41},\ \left(\frac{\partial^2 \Phi_X}{\partial Y^2}\right)_{41},\ \left(\frac{\partial^2 \Phi_Y}{\partial X^2}\right)_{41},\ \left(\frac{\partial^2 \Phi_Y}{\partial Y^2}\right)_{41},$$
$$\left(\frac{\partial^2 W}{\partial X^2}\right)_{44},\ \left(\frac{\partial^2 W}{\partial Y^2}\right)_{44},\ \left(\frac{\partial^2 \Phi_X}{\partial X^2}\right)_{44},\ \left(\frac{\partial^2 \Phi_X}{\partial Y^2}\right)_{44},\ \left(\frac{\partial^2 \Phi_Y}{\partial X^2}\right)_{44},\ \left(\frac{\partial^2 \Phi_Y}{\partial Y^2}\right)_{44},$$
$$\left.\left(\frac{\partial^2 W}{\partial X^2}\right)_{14},\ \left(\frac{\partial^2 W}{\partial Y^2}\right)_{14},\ \left(\frac{\partial^2 \Phi_X}{\partial X^2}\right)_{14},\ \left(\frac{\partial^2 \Phi_X}{\partial Y^2}\right)_{14},\ \left(\frac{\partial^2 \Phi_Y}{\partial X^2}\right)_{14},\ \left(\frac{\partial^2 \Phi_Y}{\partial Y^2}\right)_{14}\right]^{\mathrm{T}}. \quad (39b)$$

$\boldsymbol{R}$ is a 48×72 matrix with the following nonzero entries:

$$\boldsymbol{R}(1,\ 1)=\boldsymbol{R}(9,\ 9)=\boldsymbol{R}(45,\ 45)=\boldsymbol{R}(21,\ 21)=\boldsymbol{R}(13,\ 13)=\boldsymbol{R}(5,\ 5)=\boldsymbol{R}(37,\ 37)$$
$$=\boldsymbol{R}(25,\ 25)=\boldsymbol{R}(17,\ 17)=\boldsymbol{R}(29,\ 29)=\boldsymbol{R}(41,\ 41)=\boldsymbol{R}(33,\ 33)=d_1,$$
$$\boldsymbol{R}(3,\ 2)=\boldsymbol{R}(7,\ 6)=\boldsymbol{R}(11,\ 10)=d_2,\quad \boldsymbol{R}(10,\ 10)=\boldsymbol{R}(2,\ 2)=\boldsymbol{R}(6,\ 6)=d_3,$$
$$\boldsymbol{R}(3,\ 3)=\boldsymbol{R}(7,\ 7)=\boldsymbol{R}(11,\ 11)=d_4,\quad \boldsymbol{R}(10,\ 11)=\boldsymbol{R}(6,\ 7)=\boldsymbol{R}(2,\ 3)=d_5,$$
$$\boldsymbol{R}(4,\ 49)=\boldsymbol{R}(8,\ 51)=\boldsymbol{R}(12,\ 51)=d_6,\ \boldsymbol{R}(4,\ 50)=\boldsymbol{R}(8,\ 52)=\boldsymbol{R}(12,\ 54)=d_7,$$
$$\boldsymbol{R}(8,\ 8)=\boldsymbol{R}(4,\ 4)=\boldsymbol{R}(12,\ 12)=d_8,\quad \boldsymbol{R}(19,\ 18)=\boldsymbol{R}(15,\ 14)=\boldsymbol{R}(23,\ 22)=d_9,$$
$$\boldsymbol{R}(22,\ 22)=\boldsymbol{R}(18,\ 18)=\boldsymbol{R}(14,\ 14)=d_{10},\quad \boldsymbol{R}(19,\ 19)=\boldsymbol{R}(23,\ 23)=\boldsymbol{R}(15,\ 15)=d_{11},$$
$$\boldsymbol{R}(14,\ 15)=\boldsymbol{R}(18,\ 19)=\boldsymbol{R}(22,\ 23)=d_{12},\quad \boldsymbol{R}(16,\ 55)=\boldsymbol{R}(20,\ 57)=\boldsymbol{R}(24,\ 57)=d_{13},$$
$$\boldsymbol{R}(20,\ 58)=\boldsymbol{R}(24,\ 60)=\boldsymbol{R}(16,\ 56)=d_{14},\quad \boldsymbol{R}(16,\ 16)=\boldsymbol{R}(24,\ 24)=\boldsymbol{R}(20,\ 20)=d_{15},$$
$$\boldsymbol{R}(27,\ 26)=\boldsymbol{R}(31,\ 30)=\boldsymbol{R}(35,\ 34)=d_{16},\quad \boldsymbol{R}(30,\ 30)=\boldsymbol{R}(34,\ 34)=\boldsymbol{R}(26,\ 26)=d_{17},$$
$$\boldsymbol{R}(27,\ 27)=\boldsymbol{R}(35,\ 35)=\boldsymbol{R}(31,\ 31)=d_{18},\quad \boldsymbol{R}(26,\ 27)=\boldsymbol{R}(34,\ 35)=\boldsymbol{R}(30,\ 31)=d_{19},$$
$$\boldsymbol{R}(32,\ 63)=\boldsymbol{R}(36,\ 63)=\boldsymbol{R}(28,\ 61)=d_{20},\quad \boldsymbol{R}(28,\ 62)=\boldsymbol{R}(36,\ 66)=\boldsymbol{R}(32,\ 64)=d_{21},$$
$$\boldsymbol{R}(32,\ 32)=\boldsymbol{R}(36,\ 36)=\boldsymbol{R}(28,\ 28)=d_{22},\quad \boldsymbol{R}(39,\ 38)=\boldsymbol{R}(43,\ 42)=\boldsymbol{R}(47,\ 46)=d_{23},$$
$$\boldsymbol{R}(42,\ 42)=\boldsymbol{R}(46,\ 46)=\boldsymbol{R}(38,\ 38)=d_{24},\quad \boldsymbol{R}(39,\ 39)=\boldsymbol{R}(43,\ 43)=\boldsymbol{R}(47,\ 47)=d_{25},$$
$$\boldsymbol{R}(38,\ 39)=\boldsymbol{R}(42,\ 43)=\boldsymbol{R}(46,\ 47)=d_{26},\quad \boldsymbol{R}(40,\ 67)=\boldsymbol{R}(44,\ 69)=\boldsymbol{R}(48,\ 69)=d_{27},$$
$$\boldsymbol{R}(40,\ 68)=\boldsymbol{R}(44,\ 70)=\boldsymbol{R}(48,\ 72)=d_{28},\quad \boldsymbol{R}(40,\ 40)=\boldsymbol{R}(44,\ 44)=\boldsymbol{R}(48,\ 48)=d_{29}, \quad (40)$$

where

$$d_1=1,\quad d_2=\left(\frac{\partial X}{\partial \bar{Y}}\right)_{11},\quad d_3=\left(\frac{\partial X}{\partial \bar{X}}\right)_{11},\quad d_4=\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{11},\quad d_5=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{11},\quad d_6=\left(\frac{\partial X}{\partial \bar{X}}\right)_{11}\left(\frac{\partial X}{\partial \bar{Y}}\right)_{11},$$
$$d_7=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{11}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{11},\quad d_8=\left(\frac{\partial X}{\partial \bar{Y}}\right)_{11}\left(\frac{\partial Y}{\partial \bar{X}}\right)_{11}+\left(\frac{\partial X}{\partial \bar{X}}\right)_{11}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{11},\quad d_9=\left(\frac{\partial X}{\partial \bar{Y}}\right)_{41},\quad d_{10}=\left(\frac{\partial X}{\partial \bar{X}}\right)_{41},$$
$$d_{11}=\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{41},\quad d_{12}=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{41},\quad d_{13}=\left(\frac{\partial X}{\partial \bar{X}}\right)_{41}\left(\frac{\partial X}{\partial \bar{Y}}\right)_{41},\quad d_{14}=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{41}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{41},\quad d_{16}=\left(\frac{\partial X}{\partial \bar{X}}\right)_{44},$$
$$d_{15}=\left(\frac{\partial X}{\partial \bar{Y}}\right)_{41}\left(\frac{\partial Y}{\partial \bar{X}}\right)_{41}+\left(\frac{\partial X}{\partial \bar{X}}\right)_{41}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{41},\quad d_{17}=\left(\frac{\partial X}{\partial \bar{X}}\right)_{44},\quad d_{18}=\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{44},\quad d_{19}=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{44},$$
$$d_{20}=\left(\frac{\partial X}{\partial \bar{X}}\right)_{44}\left(\frac{\partial X}{\partial \bar{Y}}\right)_{44},\quad d_{21}=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{44}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{44},\quad d_{22}=\left(\frac{\partial X}{\partial \bar{Y}}\right)_{44}\left(\frac{\partial Y}{\partial \bar{X}}\right)_{44}+\left(\frac{\partial X}{\partial \bar{X}}\right)_{44}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{44},$$

$$d_{23}=\left(\frac{\partial X}{\partial \bar{Y}}\right)_{14},\ d_{24}=\left(\frac{\partial X}{\partial \bar{X}}\right)_{14},\ d_{25}=\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{14},\ d_{26}=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{14},\ d_{27}=\left(\frac{\partial X}{\partial \bar{X}}\right)_{14}\left(\frac{\partial X}{\partial \bar{Y}}\right)_{14},$$
$$d_{28}=\left(\frac{\partial Y}{\partial \bar{X}}\right)_{14}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{14},\ d_{29}=\left(\frac{\partial X}{\partial \bar{Y}}\right)_{14}\left(\frac{\partial Y}{\partial \bar{X}}\right)_{14}+\left(\frac{\partial X}{\partial \bar{X}}\right)_{14}\left(\frac{\partial Y}{\partial \bar{Y}}\right)_{14}. \tag{41}$$

To ensure the same number of local and nodal displacement parameters, $\boldsymbol{\Theta}_{N(2)}^{(Global)}$ has to be eliminated. Similar to the work of Petera and Pittman[66], we use the bi-cubic Hermitian interpolation technique to express $\boldsymbol{\Theta}_{N(2)}^{(Global)}$ in terms of $\boldsymbol{\Theta}_{N(1)}^{(Global)}$ as follows:

$$\boldsymbol{\Theta}_{N(2)}^{(Global)}=\boldsymbol{\Lambda}\boldsymbol{\Theta}_{N(1)}^{(Global)}, \tag{42}$$

where $\boldsymbol{\Lambda}$ is a 24×48 matrix with the following entries:

$$\begin{aligned}
&\boldsymbol{\Lambda}(1,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(1,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(2,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(2,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(3,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(1,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(4,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(2,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(5,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(1,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(6,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(2,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(7,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(3,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(8,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(4,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(9,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(3,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(10,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(4,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(11,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(3,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(12,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(4,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(13,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(5,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(14,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(6,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(15,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(5,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(16,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(6,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(17,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(5,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(18,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(6,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(19,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(7,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(20,\ [1\sim4,\ 13\sim16,\ 25\sim28,\ 37\sim40])=\boldsymbol{\Gamma}(8,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(21,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(7,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(22,\ [5\sim8,\ 17\sim20,\ 29\sim32,\ 41\sim44])=\boldsymbol{\Gamma}(8,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(23,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(7,\ [1\sim16]),\\
&\boldsymbol{\Lambda}(24,\ [9\sim12,\ 21\sim24,\ 33\sim36,\ 45\sim48])=\boldsymbol{\Gamma}(8,\ [1\sim16]),\\
&\text{others } \boldsymbol{\Lambda}(i,\ j)=0.
\end{aligned} \tag{43}$$

$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_1\cdot\boldsymbol{\Gamma}_2$ is an 8×16 matrix with

$$\boldsymbol{\Gamma}_1=\begin{bmatrix}
0&0&0&2&0&0&6X_1&2Y_1&0&0&6X_1Y_1&2Y_1^2&0&6X_1Y_1^2&2Y_1^3&6X_1Y_1^3\\
0&0&0&0&0&2&0&0&2X_1&6Y_1&0&2X_1^2&6X_1Y_1&2X_1^3&6X_1^2Y_1&6X_1^3Y_1\\
0&0&0&2&0&0&6X_2&2Y_2&0&0&6X_2Y_2&2Y_2^2&0&6X_2Y_2^2&2Y_2^3&6X_2Y_2^3\\
0&0&0&0&0&2&0&0&2X_2&6Y_2&0&2X_2^2&6X_2Y_2&2X_2^3&6X_2^2Y_2&6X_2^3Y_2\\
0&0&0&2&0&0&6X_3&2Y_3&0&0&6X_3Y_3&2Y_3^2&0&6X_3Y_3^2&2Y_3^3&6X_3Y_3^3\\
0&0&0&0&0&2&0&0&2X_3&6Y_3&0&2X_3^2&6X_3Y_3&2X_3^3&6X_3^2Y_3&6X_3^3Y_3\\
0&0&0&2&0&0&6X_4&2Y_4&0&0&6X_4Y_4&2Y_4^2&0&6X_4Y_4^2&2Y_4^3&6X_4Y_4^3\\
0&0&0&0&0&2&0&0&2X_4&6Y_4&0&2X_4^2&6X_4Y_4&2X_4^3&6X_4^2Y_4&6X_4^3Y_4
\end{bmatrix}, \tag{44}$$

$$\boldsymbol{\Gamma}_2=\begin{bmatrix}
1&X_1&Y_1&X_1^2&X_1Y_1&Y_1^2&X_1^3&X_1^2Y_1&X_1Y_1^2&Y_1^3&X_1^3Y_1&X_1^2Y_1^2&X_1Y_1^3&X_1^3Y_1^2&X_1^2Y_1^3&X_1^3Y_1^3\\
0&1&0&2X_1&Y_1&0&3X_1^2&2X_1Y_1&Y_1^2&0&3X_1^2Y_1&2X_1Y_1^2&Y_1^3&3X_1^2Y_1^2&2X_1Y_1^3&3X_1^2Y_1^3\\
0&0&1&0&X_1&2Y_1&0&X_1^2&2X_1Y_1&3Y_1^2&X_1^3&2X_1^2Y_1&3X_1Y_1^2&2X_1^3Y_1&3X_1^2Y_1^2&3X_1^3Y_1^2\\
0&0&0&0&1&0&0&2X_1&2Y_1&0&3X_1^2&4X_1Y_1&3Y_1^2&6X_1^2Y_1&6X_1Y_1^2&9X_1^2Y_1^2\\
1&X_2&Y_2&X_2^2&X_2Y_2&Y_2^2&X_2^3&X_2^2Y_2&X_2Y_2^2&Y_2^3&X_2^3Y_2&X_2^2Y_2^2&X_2Y_2^3&X_2^3Y_2^2&X_2^2Y_2^3&X_2^3Y_2^3\\
0&1&0&2X_2&Y_2&0&3X_2^2&2X_2Y_2&Y_2^2&0&3X_2^2Y_2,&2X_2Y_2^2&Y_2^3&3X_2^2Y_2^2&2X_2Y_2^3&3X_2^2Y_2^3\\
0&0&1&0&X_2&2Y_2&0&X_2^2&2X_2Y_2&3Y_2^2&X_2^3&2X_2^2Y_2&3X_2Y_2^2&2X_2^3Y_2&3X_2^2Y_2^2&3X_2^3Y_2^2\\
0&0&0&0&1&0&0&2X_2&2Y_2&0&3X_2^2&4X_2Y_2&3Y_2^2&6X_2^2Y_2&6X_2Y_2^2&9X_2^2Y_2^2\\
1&X_3&Y_3&X_3^2&X_3Y_3&Y_3^2&X_3^3&X_3^2Y_3&X_3Y_3^2&Y_3^3&X_3^3Y_3&X_3^2Y_3^2&X_3Y_3^3&X_3^3Y_2^2&X_3^2Y_3^3&X_3^3Y_3^3\\
0&1&0&2X_3&Y_3&0&3X_3^2&2X_3Y_3&Y_3^2&0&3X_3^2Y_3&2X_3Y_3^2&Y_3^3&3X_3^2Y_3^2&2X_3Y_3^3&3X_2^2Y_2^3\\
0&0&1&0&X_3&2Y_3&0&X_3^2&2X_3Y_3&3Y_3^2&X_3^3&2X_3^2Y_3&3X_3Y_3^2&2X_3^3Y_3&3X_3^2Y_3^2&3X_3^3Y_3^2\\
0&0&0&0&1&0&0&2X_3&2Y_3&0&3X_3^2&4X_3Y_3&3Y_3^2&6X_3^2Y_3&6X_3Y_3^2&9X_3^2Y_3^2\\
1&X_4&Y_4&X_4^2&X_4Y_4&Y_4^2&X_4^3&X_4^2Y_4&X_4Y_4^2&Y_4^3&X_4^3Y_4&X_4^2Y_4^2&X_4Y_4^3&X_4^3Y_4^2&X_4^2Y_4^3&X_4^3Y_4^3\\
0&1&0&2X_4&Y_4&0&3X_4^2&2X_4Y_4&Y_4^2&0&3X_4^2Y_4&2X_4Y_4^2&Y_4^3&3X_4^2Y_4^2&2X_4Y_4^3&3X_4^2Y_4^3\\
0&0&1&0&X_4&2Y_4&0&X_4^2&2X_4Y_4&3Y_4^2&X_4^3&2X_4^2Y_4&3X_4Y_4^2&2X_4^3Y_4&3X_4^2Y_4^2&3X_4^3Y_4^2\\
0&0&0&0&1&0&0&2X_4&2Y_4&0&3X_4^2&4X_4Y_4&3Y_4^2&6X_4^2Y_4&6X_4Y_4^2&9X_4^2Y_4^2
\end{bmatrix}^{-1}. \tag{45}$$

According to Eqs. (35), (36), (37), and (42), we obtain the global element stiffness and mass matrices as well as the global load vector as follows:

$$\begin{aligned}\boldsymbol{M}^{(e)}_{(Global)}=&\bar{\boldsymbol{M}}^{(e)}_{(Global)}\left([1\sim 48],\ [1\sim 48]\right)+\bar{\boldsymbol{M}}^{(e)}_{(Global)}\left([1\sim 48],\ [49\sim 72]\right)\boldsymbol{\Lambda}+\\ &\boldsymbol{\Lambda}^{\mathrm{T}}\bar{\boldsymbol{M}}^{(e)}_{(Global)}\left([49\sim 72],\ [1\sim 48]\right)+\boldsymbol{\Lambda}^{\mathrm{T}}\bar{\boldsymbol{M}}^{(e)}_{(Global)}\left([49\sim 72],\ [49\sim 72]\right)\boldsymbol{\Lambda}\end{aligned}, \tag{46}$$

$$\begin{aligned}\boldsymbol{K}^{(e)}_{(Global)}=&\bar{\boldsymbol{K}}^{(e)}_{(Global)}\left([1\sim 48],\ [1\sim 48]\right)+\bar{\boldsymbol{K}}^{(e)}_{(Global)}\left([1\sim 48],\ [49\sim 72]\right)\boldsymbol{\Lambda}+\\ &\boldsymbol{\Lambda}^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(Global)}\left([49\sim 72],\ [1\sim 48]\right)+\boldsymbol{\Lambda}^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(Global)}\left([49\sim 72],\ [49\sim 72]\right)\boldsymbol{\Lambda}\end{aligned}, \tag{47}$$

$$\boldsymbol{Q}^{(e)}_{(Global)}=\bar{\boldsymbol{Q}}^{(e)}_{(Global)}\left([1\sim 48],\ [1\sim 48]\right)+\boldsymbol{\Lambda}^{\mathrm{T}}\bar{\boldsymbol{Q}}^{(e)}_{(Global)}\left([49\sim 72],\ [1\sim 48]\right), \tag{48}$$

where $\left[i\sim j\right]$ indicates that the the row/column of a matrix ranges from i to j, and

$$\bar{\boldsymbol{M}}^{(e)}_{(Global)}=\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{M}}^{(e)}_{(Local)}\boldsymbol{R},\quad \bar{\boldsymbol{K}}^{(e)}_{(Global)}=\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{K}}^{(e)}_{(Local)}\boldsymbol{R},\quad \bar{\boldsymbol{Q}}^{(e)}_{(Global)}=\boldsymbol{R}^{\mathrm{T}}\bar{\boldsymbol{Q}}^{(e)}_{(Local)}. \tag{49}$$

From Eqs. (46)-(48), the equation of motion for a genetic plate element can be presented as:

$$\boldsymbol{M}^{(e)}_{(Global)}\ddot{\boldsymbol{\Theta}}^{(Global)}_{N(1)}+\boldsymbol{K}^{(e)}_{(Global)}\boldsymbol{\Theta}^{(Global)}_{N(1)}=\boldsymbol{Q}^{(e)}_{(Global)}. \tag{50}$$

For the rectangular case, we have used MAPLE symbolic computation software to give explicit algebraic expressions of $\boldsymbol{M}_e$ and $\boldsymbol{K}_e$, as listed in Appendices C and D. The overall DQ-based finite element equation for a moderately thick micro-plate is formed by assembling all element stiffness and mass matrices and element load vectors. Based on the earlier studies[20], we consider the following displacement boundary conditions:

Simply supported edge (**S**):

$$W=\Phi_s=0\text{, where }\ \Phi_s=-\Phi_X n_Y+\Phi_Y n_X\ . \tag{51}$$

Clamped edge (**C**):

$$W=\Phi_s=\Phi_n=0\ \ \text{(without strain gradient effects), where }\ \Phi_n=\Phi_X n_X+\Phi_Y n_Y\ . \tag{52a}$$

$$W=\Phi_s=\Phi_n=\frac{\partial W}{\partial s}=\frac{\partial W}{\partial n}=0\ \text{(with strain gradient effects), where}$$

$$\frac{\partial W}{\partial s}=-\frac{\partial W}{\partial X}n_Y+\frac{\partial W}{\partial Y}n_X\ \ \text{and}\ \ \frac{\partial W}{\partial n}=\frac{\partial W}{\partial X}n_X+\frac{\partial W}{\partial Y}n_Y\ . \tag{52b}$$

Free edge (**F**): No constraint.

The essential boundary conditions of the present micro-plate can be written as

$$\boldsymbol{C}_{(\partial A)}\boldsymbol{\Theta}_{N(1)}^{(Global)}=0\ , \tag{53}$$

where $\boldsymbol{C}_{(\partial A)}$ is called constraint matrix of size $m\times n$ with m and n denoting the numbers of boundary constraints and nodal displacements, respectively. To apply the coupling displacement boundary conditions, we conduct the following singular value decomposition of $\boldsymbol{C}_{(\partial A)}$:

$$\boldsymbol{C}_{(\partial A)}=\boldsymbol{U}_{(\partial A)}\boldsymbol{\Sigma}_{(\partial A)}\boldsymbol{V}_{(\partial A)}^{\mathrm{T}}\ , \tag{54}$$

where $\boldsymbol{U}_{(\partial A)}$ is a $m\times m$ orthogonal matrix, $\boldsymbol{V}_{(\partial A)}$ is a $n\times n$ orthogonal matrix, and $\boldsymbol{\Sigma}_{(\partial A)}$ has the special diagonal form

$$\boldsymbol{\Sigma}_{(\partial A)}=\begin{bmatrix}\Psi_{r\times r} & \mathbf{0}_{r\times(n-r)}\\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)}\end{bmatrix},\ \ \boldsymbol{V}_{(\partial A)}^{\mathrm{T}}=\begin{bmatrix}\boldsymbol{V}_{n\times r} & \boldsymbol{V}_{n\times(n-r)}\end{bmatrix}^{\mathrm{T}}. \tag{55}$$

Here, r is the number of independent boundary constraints.

Applying coordinate transformation, we get

$$\boldsymbol{\Theta}_{N(1)}^{(Global)}=\boldsymbol{V}_{n\times(n-r)}\boldsymbol{\phi}\ . \tag{56}$$

According to Eqs. (50) and (56), the dimensionality-reduced stiffness and mass matrices and load vector can be obtained

$$\hat{\boldsymbol{M}}_{(Global)}^{(e)}=\left(\boldsymbol{V}_{n\times(n-r)}\right)^{\mathrm{T}}\boldsymbol{M}_{(Global)}^{(e)}\boldsymbol{V}_{n\times(n-r)}\ ,\quad \hat{\boldsymbol{K}}_{(Global)}^{(e)}=\left(\boldsymbol{V}_{n\times(n-r)}\right)^{\mathrm{T}}\boldsymbol{K}_{(Global)}^{(e)}\boldsymbol{V}_{n\times(n-r)}\ ,$$

$$\hat{\boldsymbol{Q}}_{(Global)}^{(e)}=\left(\boldsymbol{V}_{n\times(n-r)}\right)^{T}\boldsymbol{Q}_{(Global)}^{(e)}\ . \tag{57}$$

4. Numerical results and discussion

4.1 Convergence of static responses

In this subsection, we conduct patch test to check the convergence of the predicted static bending responses. In comparison to the conventional plate finite elements, the convergence criteria for gradient-enhanced plate finite elements are still inadequate. Here, we employ the enhanced patch test theory[26,67] for the constrained couple stress plane element and the classical Mindlin plate element to assess the convergence of our element. In our present analysis, we select an element patch with six quadrilateral elements, as shown in Fig. 3. Unless stated otherwise, we assume that the micro-plates are composed of epoxy resin with the material properties as follows[5]: $\mathrm{E=1.44\ GPa}$, $\rho=1220\ \mathrm{kg/m^3}$, and $\nu=0.38$.

Patch test should use the partial differential equations within a generic domain consisting of several elements that are set up in such a manner in which exact solutions have been given. Based on Eq. (10) and Euler-Lagrange equations, we obtain the homogeneous equilibrium equations as follows:

$$
\begin{aligned}
&2\Sigma_9\left(\frac{\partial\Phi_X}{\partial X}+\frac{\partial^2 W}{\partial X^2}+\frac{\partial\Phi_Y}{\partial Y}+\frac{\partial^2 W}{\partial Y^2}\right)-2\Sigma_1\left(\frac{\partial^4 W}{\partial X^4}+\frac{\partial^4 W}{\partial Y^4}\right)-\Sigma_8\left(\frac{\partial^3\Phi_X}{\partial X^3}+\frac{\partial^3\Phi_Y}{\partial Y^3}\right)\\
&-\Sigma_5\left(\frac{\partial^3\Phi_Y}{\partial Y\partial X^2}+\frac{\partial^3\Phi_X}{\partial Y^2\partial X}\right)-\Sigma_6\left(\frac{\partial^3\Phi_X}{\partial Y^2\partial X}+\frac{\partial^3\Phi_Y}{\partial Y\partial X^2}\right)-2\Sigma_7\frac{\partial^4 W}{\partial Y^2\partial X^2}-2\Sigma_2\frac{\partial^4 W}{\partial X^2\partial Y^2}=0
\end{aligned}
\quad , \qquad (58a)
$$

$$
\begin{aligned}
&-2\Sigma_9\left(\Phi_X+\frac{\partial W}{\partial X}\right)+\Sigma_8\frac{\partial^3 W}{\partial X^3}+\left(\Sigma_5+\Sigma_6\right)\frac{\partial^3 W}{\partial Y^2\partial X}+\left(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18}\right)\frac{\partial^2\Phi_Y}{\partial Y\partial X}\\
&+2\left(\Sigma_4+\Sigma_{16}\right)\frac{\partial^2\Phi_X}{\partial X^2}-2\left(\Sigma_{13}\frac{\partial^4\Phi_X}{\partial X^4}+\Sigma_{14}\frac{\partial^4\Phi_X}{\partial Y^4}\right)-\left(\Sigma_{12}+\Sigma_{11}\right)\left(\frac{\partial^4\Phi_Y}{\partial Y\partial X^3}+\frac{\partial^4\Phi_Y}{\partial Y^3\partial X}\right),\\
&-2\left(\Sigma_{15}+\Sigma_{10}\right)\frac{\partial^4\Phi_X}{\partial Y^2\partial X^2}+2\left(\Sigma_7+\Sigma_{18}\right)\frac{\partial^2\Phi_X}{\partial Y^2}=0
\end{aligned}
\qquad (58b)
$$

$$
\begin{aligned}
&-2\Sigma_9\left(\Phi_Y+\frac{\partial W}{\partial Y}\right)+\Sigma_8\frac{\partial^3 W}{\partial Y^3}+\left(\Sigma_5+\Sigma_6\right)\frac{\partial^3 W}{\partial Y\partial X^2}+\left(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18}\right)\frac{\partial^2\Phi_X}{\partial Y\partial X}\\
&+2\left(\Sigma_4+\Sigma_{16}\right)\frac{\partial^2\Phi_Y}{\partial Y^2}-2\left(\Sigma_{14}\frac{\partial^4\Phi_Y}{\partial X^4}+\Sigma_{13}\frac{\partial^4\Phi_Y}{\partial Y^4}\right)-\left(\Sigma_{12}+\Sigma_{11}\right)\left(\frac{\partial^4\Phi_X}{\partial Y\partial X^3}+\frac{\partial^4\Phi_X}{\partial Y^3\partial X}\right).\\
&-2\left(\Sigma_{15}+\Sigma_{10}\right)\frac{\partial^4\Phi_Y}{\partial Y^2\partial X^2}+2\left(\Sigma_7+\Sigma_{18}\right)\frac{\partial^2\Phi_Y}{\partial X^2}=0
\end{aligned}
\qquad (58c)
$$

Based on Refs. [26,67] and Eqs. (58a)-(58c), three test functions are assumed as

$$W=S_7X^3+S_8X^2Y+S_9XY^2+S_{10}Y^3+S_4X^2+S_5XY+S_6Y^2+S_2X+S_3Y+S_1, \qquad (59a)$$

$$\Phi_X=S_{17}X^3+S_{18}X^2Y+S_{19}XY^2+S_{20}Y^3+S_{14}X^2+S_{15}XY+S_{16}Y^2+S_{12}X+S_{13}Y+S_{11}, \qquad (59b)$$

$$\Phi_Y=S_{27}X^3+S_{28}X^2Y+S_{29}XY^2+S_{30}Y^3+S_{24}X^2+S_{25}XY+S_{26}Y^2+S_{22}X+S_{23}Y+S_{21}. \qquad (59c)$$

where S_n are undetermined coefficients.

Substituting Eqs. (59a)-(59c) into Eqs. (58a)-(58c), we have

$$
\begin{aligned}
&S_{17}=S_{18}=S_{19}=S_{20}=S_{27}=S_{28}=S_{29}=S_{30}=0,\ \ S_{14}=-3S_7,\ \ S_{16}=-S_9,\ \ S_{15}=-2S_8,\\
&S_{12}=-2S_4,\ \ S_{13}=-S_5,\ \ S_{24}=-S_8,\ \ S_{26}=-3S_{10},\ \ S_{25}=-2S_9,\ \ S_{22}=-S_5,\ \ S_{23}=-2S_6,\\
&S_{11}=-S_2-\frac{3S_7\left(2\Sigma_4-\Sigma_8+2\Sigma_{16}\right)+S_9\left(\Sigma_3-\Sigma_5+2\Sigma_7+\Sigma_{17}+4\Sigma_{18}\right)}{\Sigma_9},\\
&S_{21}=-S_3-\frac{S_8\left(\Sigma_3-\Sigma_5+2\Sigma_7+\Sigma_{17}+4\Sigma_{18}\right)+3S_{10}\left(2\Sigma_4-\Sigma_8+2\Sigma_{16}\right)}{\Sigma_9}.
\end{aligned}
\qquad (60)
$$

Therefore, we express W, Φ_X, and Φ_Y as

$$W = S_7 X^3 + S_8 X^2 Y + S_9 XY^2 + S_{10} Y^3 + S_4 X^2 + S_5 XY + S_6 Y^2 + S_2 X + S_3 Y + S_1, \tag{61a}$$

$$\begin{aligned}\Phi_X = &-\frac{1}{\Sigma_9}\left[3S_7\left(2\Sigma_4 - \Sigma_8 + 2\Sigma_{16}\right) + S_9\left(\Sigma_3 - \Sigma_5 + 2\Sigma_7 + \Sigma_{17} + 4\Sigma_{18}\right)\right] \\ &-\left(3S_7 X^2 + 2S_8 XY + S_9 Y^2\right) - \left(2S_4 X + S_5 Y\right) - S_2\end{aligned}, \tag{61b}$$

$$\begin{aligned}\Phi_Y = &-\frac{1}{\Sigma_9}\left[S_8\left(\Sigma_3 - \Sigma_5 + 2\Sigma_7 + \Sigma_{17} + 4\Sigma_{18}\right) + 3S_{10}\left(2\Sigma_4 - \Sigma_8 + 2\Sigma_{16}\right)\right] \\ &-\left(S_8 X^2 + 2S_9 XY + 3S_{10} Y^2\right) - \left(S_5 X + 2S_6 Y\right) - S_3\end{aligned}. \tag{61c}$$

Without loss of generality, we adopt $S_n = n/10$ ($1 \le n \le 10$), $l_0 = h/3$, $l_1 = h/2$, $l_2 = h$, $h = 0.1$, and $K_s = 1$. Thus, we have

$$W = \frac{7X^3}{10} + \frac{4X^2Y}{5} + \frac{9XY^2}{10} + Y^3 + \frac{2X^2}{5} + \frac{XY}{2} + \frac{3Y^2}{5} + \frac{X}{5} + \frac{3Y}{10} + \frac{1}{10}, \tag{62a}$$

$$\Phi_X = -\left(\frac{21X^2}{10} + \frac{8XY}{5} + \frac{9Y^2}{10} + \frac{4X}{5} + \frac{Y}{2} + 0.2754623656\right), \tag{62b}$$

$$\Phi_Y = -\left(\frac{4X^2}{5} + \frac{9XY}{5} + 3Y^2 + \frac{X}{2} + \frac{6Y}{5} + 0.3955856631\right). \tag{62c}$$

Table 1 presents the related analytical and numerical static bending responses. It is clear that our element can pass the enhanced patch test by regenerating the displacement solutions.

Table 1 Responses of enhanced patch test.

Node	Nodal parameter								
	W	$W_{,X}$	$W_{,Y}$	Φ_X	$\Phi_{X,X}$	$\Phi_{X,Y}$	Φ_Y	$\Phi_{Y,X}$	$\Phi_{Y,Y}$
(0.4, 0.2)	0.4608 (a)	1.1200	1.1320	-1.1955	-2.8000	-1.5000	-1.2276	-1.5000	-3.1200
	0.4608 (2)	1.1200	1.1320	-1.1955	-2.8000	-1.5000	-1.2276	-1.5000	-3.1200
(1.8, 0.3)	7.2028 (a)	9.5390	5.3940	-9.6145	-8.8400	-3.9200	-5.4896	-3.9200	-6.2400
	7.2028 (b)	9.5390	5.3940	-9.6145	-8.8400	-3.9200	-5.4896	-3.9200	-6.2400
(1.6, 0.8)	8.6472 (a)	9.8800	8.3320	-9.9555	-8.8000	-4.5000	-8.4276	-4.5000	-8.8800
	8.6472 (b)	9.8800	8.3320	-9.9555	-8.8000	-4.5000	-8.4276	-4.5000	-8.8800
(0.8, 0.8)	3.2008 (a)	4.1840	5.2440	-4.2595	-5.4400	-3.2200	-5.3396	-3.2200	-7.4400
	3.2008 (b)	4.1840	5.2440	-4.2595	-5.4400	-3.2200	-5.3396	-3.2200	-7.4400

(a) Exact; (b) Numerical

Tables 2 and 3 display the effects of strain gradient and transverse shear deformation on the convergence of static responses of epoxy micro-plates with an uniformly distributed load, respectively. Here, $K_s = 5/6$, $L_X = L_Y = 1$, $l_0 = l_1 = l_2 = l$, $\log\left(\mathrm{Cond}\left(\boldsymbol{K}, 2\right)\right)$ is the common logarithm of the condition number of the reduced global stiffness matrix (after introducing essential boundary conditions), and $\bar{W}_c = 10 W_c E h^3 / L_X^4$ is the dimensionless center deflection. From the two tables, it is evident that the center deflection results can always converge to stable values. Moreover, the increase of l/h or h/L_X leads to an increase of the convergence speed of static responses and a decrease of $\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$. As is widely known, the smaller the condition number of a matrix, the better the convergence of the solution of the related linear algebraic equation system is.

Table 2 Size effects on the convergence of static response. Here, $h = 0.1$.

Plate type	l/h	Parameter	Mesh				
			4×4	8×8	12×12	16×16	20×20
SFSF	0	$\bar{W}_c$	1.7244	1.7242	1.7230	1.7225	1.7223
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	9.4845	10.7348	11.4481	11.9505	12.3389
	0.5	$\bar{W}_c$	0.3971	0.3966	0.3965	0.3965	0.3965
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.7666	9.9261	10.6177	11.1125	11.4978
	1	$\bar{W}_c$	0.1429	0.1428	0.1427	0.1427	0.1427
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.5124	9.8223	10.5650	11.0814	11.4776
SSSS	0	$\bar{W}_c$	0.4546	0.4470	0.4443	0.4431	0.4426
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	9.3562	10.6553	11.3863	11.8977	12.2914
	0.5	$\bar{W}_c$	0.1231	0.1209	0.1205	0.1205	0.1204
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.6918	9.8878	10.5892	11.0880	11.4755
	1	$\bar{W}_c$	0.05149	0.05099	0.05092	0.05091	0.05091
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.4537	9.7891	10.5386	11.0582	11.4562
CCCC	0	$\bar{W}_c$	0.1660	0.1587	0.1574	0.1571	0.1571
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	9.3479	10.6475	11.3809	11.8940	12.2891
	0.5	$\bar{W}_c$	0.05184	0.05027	0.05015	0.05014	0.05013
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.6888	9.8868	10.5889	11.0878	11.4752
	1	$\bar{W}_c$	0.02441	0.02388	0.02384	0.02383	0.02382
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.45293	9.7886	10.5380	11.0575	11.4553

Table 3 Transverse shear deformation effect on the convergence of static response. Here, $l/h = 0.5$.

Plate type	h/L_X	Parameter	Mesh				
			4×4	8×8	12×12	16×16	20×20
SFSF	0.2	$\bar{W}_c$	0.4997	0.4994	0.4994	0.4994	0.4994
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.1545	9.3247	10.0198	10.5152	10.9004
	0.1	$\bar{W}_c$	0.3971	0.3966	0.3965	0.3965	0.3965
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.7666	9.9261	10.6177	11.1125	11.4978
	0.05	$\bar{W}_c$	0.3671	0.3669	0.3668	0.3668	0.3668
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	9.3423	10.5388	11.2250	11.7153	12.0979
SSSS	0.2	$\bar{W}_c$	0.1820	0.1808	0.1807	0.1807	0.1807
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.1131	9.2991	9.9981	10.4955	10.8820
	0.1	$\bar{W}_c$	0.1231	0.1209	0.1205	0.1205	0.1204
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.6918	9.8878	10.5892	11.0880	11.4755

	0.05	$\bar{W}_c$	0.1058	0.1040	0.1035	0.1034	0.1033
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	9.2178	10.4674	11.1771	11.6781	12.0667
	0.2	$\bar{W}_c$	0.0884	0.08697	0.08686	0.08684	0.08683
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.1126	9.2990	9.9976	10.4950	10.8815
CCCC	0.1	$\bar{W}_c$	0.05184	0.05027	0.05015	0.05014	0.05013
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	8.68882	9.88683	10.5889	11.0878	11.4752
	0.05	$\bar{W}_c$	0.03808	0.03651	0.03628	0.03624	0.03622
		$\log(\mathrm{Cond}(\boldsymbol{K},\ 2))$	9.2121	10.4632	11.1752	11.6772	12.0662

4.2 Convergence of vibration frequencies

This subsection examines the convergence of the new element in analyzing the free vibration problem of rectangular, annular sectorial, and elliptical plates. For simplicity, we assume $l_0 = l_1 = l_2 = l$.

Table 4 gives the first five dimensionless frequencies $\bar{\omega}_n = \omega_n L_X^2 \sqrt{\rho h / D} / \pi^2$ with varying mesh numbers for a square macro-plate that was studied by Liew et al[68] using the *p*-version Ritz method. Here, $L_X / h = 10$, $\nu = 0.3$, and $K_s = \pi^2/12$. Note that the repeated eigenvalues are treated as the same. It is evident that the frequency predictions always converge to stable values with the mesh number. When a 24×24 mesh is used, convergent frequency results are obtained. As evinced below, one can see that our results agree well with the reported ones.

Table 4 Convergence of the first five dimensionless frequencies for a moderately thick square macro-plate.

Plate type	Mode	Ref. [68]	Mesh						
			4×4	8×8	12×12	16×16	20×20	24×24	28×28
	$\bar{\omega}_1$	0.333	0.3304	0.3318	0.3324	0.3326	0.3327	0.3328	0.3328
	$\bar{\omega}_2$	1.677	1.6724	1.6749	1.6763	1.6768	1.6771	1.6772	1.6773
SSFF	$\bar{\omega}_3$	1.874	1.8709	1.8719	1.8730	1.8736	1.8738	1.8740	1.8741
	$\bar{\omega}_4$	3.557	3.5365	3.5453	3.5514	3.5538	3.5550	3.5556	3.5560
	$\bar{\omega}_5$	4.718	4.7290	4.7186	4.7183	4.7183	4.7184	4.7184	4.7184
	$\bar{\omega}_1$	1.931	1.8972	1.9159	1.9233	1.9264	1.9280	1.9289	1.9294
	$\bar{\omega}_2$	4.605	4.5615	4.5818	4.5930	4.5978	4.6002	4.6016	4.6024
SSSS	$\bar{\omega}_3$	7.064	6.9656	7.0085	7.0349	7.0468	7.0527	7.0561	7.0581
	$\bar{\omega}_4$	8.605	8.6062	8.5931	8.6000	8.6026	8.6036	8.6041	8.6041
	$\bar{\omega}_5$	10.792	10.695	10.721	10.754	10.770	10.778	10.782	10.785
	$\bar{\omega}_1$	2.700	2.6392	2.6871	2.6980	2.7008	2.7016	2.7017	2.7017
	$\bar{\omega}_2$	4.971	4.8942	4.9419	4.9625	4.9694	4.9719	4.9728	4.9732
SCSC	$\bar{\omega}_3$	5.990	5.9478	5.9921	6.0049	6.0063	6.0052	6.0038	6.0024
	$\bar{\omega}_4$	7.972	7.8253	7.9164	7.9630	7.9783	7.9830	7.9843	7.9842
	$\bar{\omega}_5$	8.787	8.7637	8.7612	8.7773	8.7843	8.7872	8.7884	8.7888
	$\bar{\omega}_1$	0.348	0.3298	0.3423	0.3452	0.3462	0.3467	0.3469	0.3471
	$\bar{\omega}_2$	0.816	0.7724	0.7937	0.8041	0.8089	0.8114	0.8128	0.8137
CFFF	$\bar{\omega}_3$	2.034	1.9316	1.9891	2.0105	2.0200	2.0249	2.0277	2.0294
	$\bar{\omega}_4$	2.582	2.5829	2.5795	2.5808	2.5816	2.5821	2.5823	2.5824
	$\bar{\omega}_5$	2.860	2.7239	2.7828	2.8172	2.8338	2.8425	2.8475	2.8507

CCFF	$\bar{\omega}_1$	0.676	0.6256	0.6542	0.6646	0.6691	0.6714	0.6727	0.6735
	$\bar{\omega}_2$	2.242	2.0874	2.1585	2.1955	2.2134	2.2230	2.2285	2.2320
	$\bar{\omega}_3$	2.503	2.4372	2.4824	2.4948	2.4991	2.5009	2.5019	2.5024
	$\bar{\omega}_4$	4.251	4.1184	4.1552	4.1964	4.2184	4.2302	4.2370	4.2412
	$\bar{\omega}_5$	5.557	5.4138	5.4866	5.5240	5.5397	5.5472	5.5511	5.5533
CCCC	$\bar{\omega}_1$	3.292	3.2088	3.2780	3.2919	3.2952	3.2959	3.2959	3.2957
	$\bar{\omega}_2$	6.276	6.2034	6.2688	6.2892	6.2930	6.2927	6.2914	6.2900
	$\bar{\omega}_3$	8.792	8.6152	8.7293	8.7928	8.8119	8.8168	8.8172	8.8161
	$\bar{\omega}_4$	10.356	10.550	10.426	10.422	10.415	10.408	10.401	10.396
	$\bar{\omega}_5$	10.445	10.619	10.522	10.520	10.512	10.505	10.499	10.493

Next, we apply the new element to calculate the vibration frequencies of an annular sectorial macro-plate (inner radius $R_{\text{in}}=0.5$, outer radius $R_{\text{out}}=1.0$, and sectorial angle $\alpha=45^\circ$) and an elliptical macro-plate (semi-major axis $L_X=1$ and semiminor axis $L_Y=0.5$). Here, $K_s=0.86667$. Figs. 4 and 5 display three meshing schemes for annular sectorial and elliptical cases, respectively. For verification purposes, frequency results reported by Liew et al[69] for nonrectangular moderately thick macro-plates are adopted as benchmarks.

Tables 5 and 6 present the first six dimensionless frequencies $\bar{\omega}_n=\omega_n R_{\text{out}}^2\sqrt{\rho h/D}$ for CCCC and SSSS annular sectorial macro-plates, respectively. It is clear that, as the number of meshes increase, the calculated frequency results gradually converge to stable values, and the convergent solutions are slightly larger than those available. Moreover, the convergence trend and speed are affected by h/R_{out} and the order of vibration mode. The thinner the plate thickness or the higher the order of vibration mode, the slower the convergent speed is, implying that the present element is more suitable for analyzing the free vibration of thick plates. The reason for this is that $\log\left(\text{Cond}\left(\boldsymbol{K},2\right)\right)$ increases as the decrease of the plate thickness, and more iterations are required in solving higher vibration frequencies. To show the efficacy of our method, Figs. 6 and 7 reproduce the first six vibration mode shapes for two types of macro-plates with $h/R_{\text{out}}=0.1$, respectively.

Table 5 Convergence of the first six dimensionless frequencies for an annular sectorial macro-plate with CCCC boundary conditions.

h/R_{out}	Source	Mode					
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
0.001	Ref. [69]	125.5	227.4	280.2	378.3	387.2	517.3
	Mesh I	165.3338	292.2252	353.9860	482.9315	523.1557	669.4078
	Mesh II	129.8238	232.0112	284.6804	384.5185	395.3162	524.5558
	Mesh III	127.1476	228.8035	282.1036	379.5005	388.7205	518.3038
0.1	Ref. [69]	95.10	153.8	179.1	226.9	230.1	281.0
	Mesh I	93.5207	151.2895	177.0323	222.9385	226.8761	278.5019
	Mesh II	95.7227	155.0461	180.7708	229.0871	232.1881	284.2072
	Mesh III	95.7850	155.1778	180.8985	229.2941	232.5605	284.3705

0.2	Ref. [69]	64.50	97.10	109.5	136.9	138.3	164.4
	Mesh I	64.1567	96.5479	109.3252	135.4655	137.5767	163.7575
	Mesh II	65.3313	98.4207	111.0663	138.8192	140.2543	166.8580
	Mesh III	65.3610	98.4770	111.1144	138.9060	140.4041	166.9238

Table 6 Convergence of the first six dimensionless frequencies for an annular sectorial macro-plate with SSSS boundary conditions.

h/R_{out}	Source	Mode					
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
0.001	Ref. [69]	68.34	150.8	189.7	278.2	283.5	387.8
	Mesh I	106.2108	187.5356	250.4228	308.9057	375.0737	473.2842
	Mesh II	75.0577	155.5681	196.4829	281.7796	293.7869	397.9409
	Mesh III	71.1306	152.7452	191.9595	279.6327	286.8946	391.0256
0.1	Ref. [69]	61.25	122.3	147.6	200.2	203.1	256.9
	Mesh I	60.8749	121.8455	147.0971	199.2506	201.5419	256.3791
	Mesh II	61.4139	122.9647	148.3905	201.6777	204.4149	259.0707
	Mesh III	61.4387	123.0248	148.4315	201.7827	204.5717	259.1352
0.2	Ref. [69]	49.29	88.18	102.7	132.0	133.5	161.6
	Mesh I	49.1262	88.0102	102.6758	131.4542	132.9335	161.4503
	Mesh II	49.6583	89.0938	103.8620	133.7442	135.1513	163.9958
	Mesh III	49.6765	89.1348	103.8929	133.8144	135.2656	164.0401

Tables 7 and 8 list the first six dimensionless frequencies $\bar{\omega}_n = \omega_n L_Y^2 \sqrt{\rho h/D}$ for CCCC and SSSS elliptical macro-plates, respectively. As can be seen, increasing the mesh number can yield good convergent frequency results in consistent with the reported ones. Figs. 8 and 9 plot the first six vibration mode shapes for two types of elliptical macro-plates with $h/L_X = 0.1$, respectively.

Table 7 Convergence of the first six dimensionless frequencies for an elliptical macro-plate with clamped boundary conditions.

h/L_X	Source	Mode					
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
0.05	Ref. [69]	26.81	38.42	54.00	66.72	73.60	83.36
	Mesh I	26.7306	38.1318	53.5053	66.4974	72.9236	82.6263
	Mesh II	26.8049	38.3696	53.9076	66.7142	73.4337	83.2499
	Mesh III	26.8090	38.4096	53.9984	66.7389	73.5682	83.3293
0.1	Ref. [69]	25.31	35.72	49.32	59.56	65.92	73.12
	Mesh I	25.1988	35.4030	48.8433	59.3108	65.2884	72.4464
	Mesh II	25.2962	35.6580	49.2427	59.5444	65.7667	73.0302
	Mesh III	25.3084	35.7021	49.3287	59.5708	65.8810	73.1014
0.15	Ref. [69]	23.33	32.35	43.88	51.64	57.48	62.48
	Mesh I	23.2252	32.0883	43.4945	51.4346	57.0177	61.9719
	Mesh II	23.3126	32.2998	43.8068	51.6184	57.3777	62.4093
	Mesh III	23.3245	32.3370	43.8736	51.6392	57.4608	62.4622

Table 8 Convergence of the first six dimensionless frequencies for an elliptical macro-plate with simply supported boundary conditions.

h/L_X	Source	Mode					
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
0.05	Ref. [69]	13.10	23.25	37.43	45.08	55.68	60.68
	Mesh I	13.2059	23.4712	37.7644	45.2656	56.1084	61.0845
	Mesh II	13.1647	23.4196	37.7055	45.2248	56.0642	61.0585
	Mesh III	13.1509	23.4093	37.6965	45.2165	56.0615	61.0527
0.1	Ref. [69]	12.84	22.43	35.51	42.56	51.80	56.12
	Mesh I	12.9294	22.7154	35.9554	42.7666	52.3769	56.7265
	Mesh II	12.9218	22.7115	35.9620	42.7677	52.4092	56.7647
	Mesh III	12.9191	22.7108	35.9635	42.7672	52.4157	56.7692
0.15	Ref. [69]	12.46	21.34	33.06	39.30	47.12	50.72
	Mesh I	12.5791	21.7029	33.6019	39.5438	47.7881	51.4269
	Mesh II	12.5784	21.7099	33.6246	39.5566	47.8391	51.4872
	Mesh III	12.5779	21.7114	33.6289	39.5579	47.8481	51.4946

Comparing Table 4 with Tables 5-8, we can see that the convergence for the rectangular case is markedly better than that for two non-rectangular cases. This is because the coordinate transformation from an arbitrary quadrilateral element to a standard element will bring about inevitable numerical errors.

The next example shows the convergence of our element in analyzing the free vibration problem of rectangular and annular sectorial micro-plates made of epoxy resin. The related frequency results against the mesh number are provided in Tables 9 and 10, respectively. Here, $\bar{\omega}_n = \omega_n L_X^2 \sqrt{\rho h/D} / \pi^2$, $L_X / h = 10$, and $l/h = 1$ for the rectangular case, and $\bar{\omega}_n = \omega_n R_{\text{out}}^2 \sqrt{\rho h/D}$ and $h/R_{\text{out}} = 0.1$ for the annular sectorial one.

The results listed in Table 9 show that the calculated frequencies converge to stable values gradually with the mesh number. Under the 24×24 mesh, good convergent frequencies are obtained. Through comparing the two tables, it is clear that the convergence of our element in the annular sectorial case is not as good as that of the rectangular case, due to the inevitable calculation errors during the process of coordinate transformation. For the annular case, it also can be seen that the convergence trends do not accord with each other. Such inconsistency is attributed to that $\log(\text{Cond}(\boldsymbol{K}, 2))$ and $\log(\text{Cond}(\boldsymbol{M}, 2))$ vary with mesh number and l/h monotonically, as shown in Figs.10 and 11. Consequently, the analysis of irregular micro-plates requires a greater number of meshes and higher calculation cost as compared to the rectangular case.

Table 9 Convergence of the first six dimensionless frequencies for a square micro-plate made of epoxy resin.

Plate type	Mode	Mesh						
		4×4	8×8	12×12	16×16	20×20	24×24	28×28
SSFF	$\bar{\omega}_1$	1.2163	1.2239	1.2248	1.2250	1.2251	1.2251	1.2251

	$\bar{\omega}_2$	5.4189	5.4292	5.4306	5.4309	5.4310	5.4310	5.4310
	$\bar{\omega}_3$	5.7312	5.7324	5.7327	5.7328	5.7328	5.7328	5.7328
	$\bar{\omega}_4$	10.1637	10.1799	10.1822	10.1827	10.1828	10.1828	10.1829
	$\bar{\omega}_5$	12.8184	12.7746	12.7711	12.7704	12.7702	12.7701	12.7701
	$\bar{\omega}_1$	5.6212	5.6477	5.6513	5.6521	5.6523	5.6524	5.6524
	$\bar{\omega}_2$	11.7705	11.7683	11.7689	11.7691	11.7691	11.7691	11.7691
SSSS	$\bar{\omega}_3$	17.1544	17.1457	17.1461	17.1461	17.1461	17.1461	17.1461
	$\bar{\omega}_4$	20.7440	20.5748	20.5637	20.5615	20.5609	20.5606	20.5605
	$\bar{\omega}_5$	25.7294	25.5851	25.5757	25.5739	25.5733	25.5731	25.5730
	$\bar{\omega}_1$	7.0535	7.1060	7.1113	7.1124	7.1128	7.1129	7.1130
	$\bar{\omega}_2$	12.4501	12.5204	12.5308	12.5333	12.5341	12.5345	12.5347
SCSC	$\bar{\omega}_3$	14.6196	14.6384	14.6438	14.6453	14.6458	14.6461	14.6462
	$\bar{\omega}_4$	19.1466	19.3420	19.3736	19.3818	19.3847	19.3861	19.3868
	$\bar{\omega}_5$	21.1386	21.0563	21.0618	21.0642	21.0652	21.0657	21.0660
	$\bar{\omega}_1$	1.1679	1.2069	1.2120	1.2132	1.2135	1.2137	1.2137
	$\bar{\omega}_2$	2.6412	2.6861	2.6930	2.6946	2.6950	2.6952	2.6952
CFFF	$\bar{\omega}_3$	5.8334	5.8870	5.8946	5.8963	5.8968	5.8970	5.8971
	$\bar{\omega}_4$	7.9215	7.9197	7.9210	7.9214	7.9216	7.9216	7.9217
	$\bar{\omega}_5$	8.2105	8.3045	8.3193	8.3228	8.3240	8.3244	8.3246
	$\bar{\omega}_1$	2.1511	2.2166	2.2257	2.2278	2.2284	2.2286	2.2287
	$\bar{\omega}_2$	6.4659	6.5629	6.5789	6.5827	6.5839	6.5844	6.5846
CCFF	$\bar{\omega}_3$	6.9202	6.9607	6.9649	6.9658	6.9661	6.9662	6.9663
	$\bar{\omega}_4$	11.5220	11.6496	11.6719	11.6776	11.6796	11.6804	11.6808
	$\bar{\omega}_5$	14.4943	14.5429	14.5526	14.5552	14.5561	14.5565	14.5567
	$\bar{\omega}_1$	8.2545	8.3536	8.3625	8.3644	8.3650	8.3653	8.3654
	$\bar{\omega}_2$	15.1597	15.2642	15.2825	15.2870	15.2887	15.2894	15.2898
CCCC	$\bar{\omega}_3$	20.9630	21.3267	21.3992	21.4175	21.4241	21.4270	21.4286
	$\bar{\omega}_4$	25.5568	25.3455	25.3512	25.3556	25.3576	25.3587	25.3593
	$\bar{\omega}_5$	25.6497	25.4508	25.4561	25.4593	25.4608	25.4616	25.4620

Table 10 Convergence of the first six dimensionless frequencies for an annular sectorial micro-plate made of epoxy resin.

Plate type	l/h	Mesh	Mode					
			$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
CCCC	0.1	Mesh I	97.1580	157.4126	183.4259	234.1531	237.2928	290.8727
		Mesh II	98.3432	158.3574	184.3887	234.0605	237.1294	291.1177
		Mesh III	98.4007	158.5079	184.5827	234.2262	237.3068	291.3135
	0.5	Mesh I	152.6664	247.5716	292.4992	384.5919	393.0635	498.6688
		Mesh II	155.0588	249.4766	292.8201	385.1208	387.9228	498.9786
		Mesh III	155.1634	250.7414	294.2454	385.7977	387.7611	497.9399
	1.0	Mesh I	229.9975	385.0091	465.7339	629.7734	650.1887	845.4804
		Mesh II	235.4577	391.1775	465.5301	633.6635	636.1833	844.0642
		Mesh III	235.4851	394.6116	469.4749	634.6716	636.2966	841.5375
SSSS	0.1	Mesh I	63.8594	125.6830	151.0549	203.4523	205.7115	260.2139
		Mesh II	64.2366	126.3456	151.6353	204.7885	207.4166	261.5618
		Mesh III	64.2553	126.3778	151.6240	204.7778	207.3521	261.3891
	0.5	Mesh I	104.0389	192.1592	228.6844	313.6300	316.4009	408.8149
		Mesh II	105.5801	193.2220	228.0397	314.1392	315.4739	408.8319
		Mesh III	106.0433	194.1517	228.6948	313.7921	315.2343	407.1835
	1.0	Mesh I	144.5363	274.8118	331.1315	477.8997	491.6618	653.1602
		Mesh II	149.7154	278.9491	331.9462	480.0729	488.6299	653.5332

Mesh III	151.4868	282.1210	333.4310	479.3540	488.4752	649.6737

4.3 Convergence comparison of the present and available elements

To demonstrate the good convergence of our element, we compare it with some available ones with the Bogner-Fox-Schmit (BFS) element[65] of C^1-continuity or the classical Kirchhoff plate element of C^1-weak continuity. Without loss of generality, we apply the present element and the Bogner-Fox-Schmit (BFS) element[65] to solve the free vibration of a moderately thick rectangular macro-plate with $L_X/L_Y = 0.4$ and $h/L_Y = 0.1$ and a thin rectangular macro-plate with $L_X/L_Y = 1$ and $h/L_Y = 0.001$.

The related dimensionless frequencies for two cases are presented Tables 11 and 12, respectively. It should be noted that two types of elements both introduce the mixed partial derivatives of kinematic variables as nodal displacement parameters, and possess the same structures in terms of their corresponding stiffness (mass) matricies but have different matrix elements. From Tables 11 and 12, it can be seen that the present element exhibits better convergence than the BFS one during the mesh refinement. To explain this point, Figs. 12-14 present the condition numbers of the reduced global mass and stiffness matrices in a histogram
form. The three figures demonstrate that the values of $\log(\mathrm{Cond}(\boldsymbol{M}, 2))$ and $\log(\mathrm{Cond}(\boldsymbol{K}, 2))$ for the present case are smaller than their counterparts for the BFS case, respectively. Through comparing Figs. 12 and 13 with Fig. 14, it is seen that the value of $\log(\mathrm{Cond}(\boldsymbol{K}, 2))$ for the present case and that for the BFS case can be negligible as the plate becomes very thin. The underlying reason is that the construction of basis functions in our element is distinctly different from that in the BFS one. The present method adopts non-uniform GLQ points to construct basis functions, which not only reduces the extra interpolation calculation but also improves the numerical characteristics of stiffness and mass matrices.

Table 12 also reveals that the present element and the available one are both free of shear locking, which can be proved by comparing the magnitude of the shear element stiffness matrix with that of the bending one. The underlying cause is that the present element uses the bi-cubic Hermitian interpolation technique to express the trial functions of kinematic variables, as seen from Eq. (18). For its one-dimensional reduced model (i.e., Timoshenko beam model), Zhang et al[28,51] have validated such shear locking-free performance explicitly.

Table 11 The first eight dimensionless frequencies of a moderately thick rectangular macro-plate.

Plate type	Mode	2×5		4×10		8×20		Ref. [69]
		BFS[65]	Present	BFS[65]	Present	BFS[65]	Present	
SSSS	$\bar{\omega}_1$	6.3829	6.3908	6.4274	6.4315	6.4544	6.4558	6.467

	$\bar{\omega}_2$	8.5744	8.5898	8.6841	8.6947	8.7590	8.7629	8.793
	$\bar{\omega}_3$	12.0560	12.0737	12.1911	12.2055	12.2992	12.3050	12.348
	$\bar{\omega}_4$	16.5301	16.5425	16.6316	16.6463	16.7522	16.7588	16.808
	$\bar{\omega}_5$	19.2274	19.3344	18.9819	18.9843	18.9947	18.9958	19.004
	$\bar{\omega}_6$	20.7828	20.8836	20.5853	20.5935	20.6468	20.6505	20.680
	$\bar{\omega}_7$	22.0125	22.1029	21.7202	21.7326	21.8359	21.8426	21.894
	$\bar{\omega}_8$	23.2933	23.3828	23.1493	23.1635	23.2716	23.2786	23.334
CCCC	$\bar{\omega}_1$	10.0232	10.1410	10.5254	10.5495	10.6465	10.6502	10.670
	$\bar{\omega}_2$	10.9149	11.0661	11.8954	11.9555	12.2399	12.2517	12.316
	$\bar{\omega}_3$	13.1776	13.3630	14.5844	14.6605	15.0818	15.1009	15.211
	$\bar{\omega}_4$	15.9224	16.2107	18.3825	18.4660	18.9657	18.9894	19.133
	$\bar{\omega}_5$	17.6633	18.2038	22.3160	22.3541	22.4999	22.5058	22.537
	$\bar{\omega}_6$	17.6709	18.2218	22.9215	23.0172	23.5854	23.6115	23.779
	$\bar{\omega}_7$	18.3697	18.6999	23.1297	23.2374	23.7427	23.7633	23.877
	$\bar{\omega}_8$	21.6197	21.8415	24.7735	24.9260	25.8328	25.8710	26.095

Table 12 The first ten dimensionless frequencies of a thin rectangular macro-plate with SSSS boundary conditions.

Mode	2×2		4×4		6×6		8×8		10×10	
	BFS[65]	Present	BFS[65]	Present	BFS[65]	Present	BFS[65]	Present	BFS[65]	Present
$\bar{\omega}_1$	2.0043	2.0040	2.0003	2.0003	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
$\bar{\omega}_2$	5.9486	5.9307	5.0193	5.0188	5.0032	5.0032	5.0009	5.0009	5.0003	5.0003
$\bar{\omega}_3$	5.9486	5.9307	5.0193	5.0188	5.0032	5.0032	5.0009	5.0009	5.0003	5.0003
$\bar{\omega}_4$	9.3777	9.3370	8.0270	8.0257	8.0041	8.0040	8.0011	8.0010	8.0003	8.0003
$\bar{\omega}_5$	12.0265	11.4797	10.2650	10.2526	10.0514	10.0504	10.0139	10.0137	10.0051	10.0051
$\bar{\omega}_6$	12.0265	11.4797	10.2650	10.2526	10.0514	10.0504	10.0139	10.0138	10.0051	10.0051
$\bar{\omega}_7$	15.2201	14.5817	13.2334	13.2184	13.0430	13.0417	13.0112	13.0110	13.0039	13.0039
$\bar{\omega}_8$	15.2201	14.5817	13.2334	13.2184	13.0430	13.0417	13.0112	13.0110	13.0039	13.0039
$\bar{\omega}_9$	20.7705	19.4410	18.3662	18.3343	17.3649	17.3539	17.0982	17.0964	17.0349	17.0345
$\bar{\omega}_{10}$	953.0456	1120.2897	25.8241	25.6062	17.3649	17.3539	17.0982	17.0964	17.0349	17.0345

Table 13 further demonstrates the shear locking-free behavior of our element for solving the free vibration problem of a simply supported plate with L_X/h ranging from 10 to 50000, where a 24×24 mesh is chosen and the Navier solution[17] based on the Kirchhoff plate model is provided for the sake of comparison. Here, the superscripts "(1)" and "(0)" indicate the size-dependent ($h/l=1$) and size-independent ($h/l=0$) cases, respectively. It is clear from Table 13 that the present frequency results are in excellent agreement with the analytical ones even when L_X/h reaches up to 50000, which means that our element can effectively address the problem of shear locking.

Table 13 Shear locking-free performance of the new element in analyzing the free vibration of a simply supported square micro-plate.

Mode	Ref. [17]	L_X/h							
		10	50	100	500	1000	5000	10000	50000
$\bar{\omega}_1$	$7.6818^{(1)}$	5.7049	7.5330	7.6390	7.6769	7.6791	7.6800	7.6800	7.6800
	$2.5311^{(0)}$	2.3927	2.5244	2.5292	2.5310	2.5311	2.5311	2.5311	2.5311
$\bar{\omega}_2$	$19.2112^{(1)}$	11.8313	18.3412	18.9654	19.1880	19.1968	19.1999	19.2000	19.2001

	6.3278$^{(0)}$	5.5675	6.2892	6.3177	6.3274	6.3278	6.3279	6.3279	6.3279
$\bar{\omega}_3$	30.7486$^{(1)}$	17.2142	28.6272	30.1273	30.6889	30.7115	30.7198	30.7200	30.7201
	10.1246$^{(0)}$	8.3750	10.0260	10.0984	10.1234	10.1243	10.1246	10.1246	10.1246
$\bar{\omega}_4$	38.4447$^{(1)}$	20.6340	35.2383	37.4946	38.3573	38.3892	38.4001	38.4005	38.4006
	12.6557$^{(0)}$	10.0923	12.5047	12.6165	12.6542	12.6555	12.6559	12.6559	12.6559
$\bar{\omega}_5$	49.9955$^{(1)}$	25.6540	44.8116	48.4084	49.8465	49.9009	49.9197	49.9203	49.9205
	16.4524$^{(0)}$	12.4792	16.1982	16.3859	16.4497	16.4518	16.4525	16.4525	16.4525

It may appear that the proposed DQ-based geometric mapping scheme is similar to the classical shape function-based scheme for the C^1-weak continuous Kirchhoff plate element. Therefore, we now compare the convergences of these two schemes. For the macroscopic Kirchhoff plate model, the corresponding DQ-based finite element formulation should be derived using the same procedure shown in Section 3. The nodal displacement vector is defined as

$$\Theta_{\mathrm{N}}=\left[(W)_{11},\ \left(\frac{\partial W}{\partial X}\right)_{11},\ \left(\frac{\partial W}{\partial Y}\right)_{11},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{11},\ (W)_{11},\ \left(\frac{\partial W}{\partial X}\right)_{11},\ \left(\frac{\partial W}{\partial Y}\right)_{41},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{41},\right.$$
$$\left.(W)_{44},\ \left(\frac{\partial W}{\partial X}\right)_{44},\ \left(\frac{\partial W}{\partial Y}\right)_{44},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{44},\ (W)_{44},\ \left(\frac{\partial W}{\partial X}\right)_{44},\ \left(\frac{\partial W}{\partial Y}\right)_{44},\ \left(\frac{\partial^2 W}{\partial X\,\partial Y}\right)_{44}\right]^{\mathrm{T}}. \quad (63)$$

The resulting explicit algebraic expressions of element matrices are given in Appendix E.

Table 14 compares the convergence of DQ-based Kirchhoff plate element (4 DOF per node) with that of C^1-weak continuous Kirchhoff plate element (3 DOF per node) when they are used to analyze the free vibration problem of a simply supported square macro-plate. For a reliable comparison, two numerical models should adopt the same total number of DOF. Here, 10×10, 20×20, 30×30, and 40×40 meshes are utilized in the classical model. Consequently, 8×8, 17×17, 25×25, and 34×34 meshes are considered in the DQ-based model. It is emphasized that the total number of DOF in the classical model are slightly higher than that in the present model. It is shown that the present element has much faster convergence speed than the classical one. Specifically, the proposed model with 8×8 mesh (256 DOF) can provide the same accuracy as that exhibited by the classical one with 40×40 mesh (4719 DOF). Similar conclusions have been drawn in the available literature[62,63]. Here, the dimensionless frequencies are defined as $\bar{\varpi}_n=\omega_n L_X^2\sqrt{\rho h/D}\big/\pi^2$. For a simply supported square macro-plate, the exact solution of $\bar{\varpi}_n$ is $\bar{\varpi}_n=n_1^2+n_2^2$, where the positive integers n_1 and n_2 denote the half-wave numbers along the X and Y directions, respectively.

Table 14 Comparison between the convergence of DQ-based Kirchhoff plate element and that of its classical counterpart in analyzing the free vibration of a simply supported macro-plate.

Mode	Classical				Present				Exact
	10×10	20×20	30×30	40×40	8×8	17×17	25×25	34×34	

$\bar{\omega}_1$	1.9895	1.9973	1.9988	1.9993	2.0000	2.0000	2.0000	2.0000	2.0000
$\bar{\omega}_2$	4.9593	4.9894	4.9953	4.9973	5.0008	5.0000	5.0000	5.0000	5.0000
$\bar{\omega}_3$	7.8409	7.9580	7.9812	7.9894	8.0011	8.0000	8.0000	8.0000	8.0000
$\bar{\omega}_4$	9.9155	9.9767	9.9895	9.9940	10.0103	10.0005	10.0001	10.0000	10.0000
$\bar{\omega}_5$	12.6586	12.9067	12.9578	12.9761	13.0084	13.0004	13.0001	13.0000	13.0000
$\bar{\omega}_6$	16.8746	16.9603	16.9816	16.9895	17.0583	17.0031	17.0007	17.0002	17.0000
$\bar{\omega}_7$	17.2623	17.7922	17.9055	17.9464	18.0118	18.0006	18.0001	18.0000	18.0000
$\bar{\omega}_8$	19.4386	19.8373	19.9257	19.9578	20.0502	20.0026	20.0006	20.0002	20.0000
$\bar{\omega}_9$	23.7682	24.6366	24.8333	24.9052	25.0450	25.0023	25.0005	25.0001	25.0000
$\bar{\omega}_{10}$	25.8735	25.9427	25.9722	25.9839	26.2156	26.0118	26.0026	26.0008	26.0000
$\bar{\omega}_{11}$	28.2274	28.7537	28.8854	28.9345	29.1951	29.0106	29.0023	29.0007	29.0000
$\bar{\omega}_{12}$	29.9301	31.3636	31.7056	31.8320	32.0648	32.0033	32.0007	32.0002	32.0000
$\bar{\omega}_{13}$	32.2375	33.4452	33.7422	33.8527	34.1717	34.0092	34.0020	34.0006	34.0000
$\bar{\omega}_{14}$	36.9796	36.9291	36.9623	36.9776	37.6079	37.0351	37.0077	37.0023	37.0000
$\bar{\omega}_{15}$	38.0063	39.6618	39.8384	39.9068	40.5664	40.0326	40.0071	40.0021	40.0000

4.4 Model validation

In this subsection, some illustrative examples are presented to verify the correctness of the frequency predictions. For the macroscopic case, three MLSPs should be excluded, namely, $l_0 = l_1 = l_2 = 0$.

Table 15 lists the first eight dimensionless frequencies $\bar{\omega}_n = \omega_n L_X^2 \sqrt{\rho/D}/\pi^2$ obtained by the present method and the *p*-version Ritz method[68] for a moderately thick macro-plate with $L_X/L_Y = 0.4$, $h/L_Y = 0.1$, $\nu = 0.3$ and $K_s = \pi^2/12$. Here, the mesh number of 20×50 is used and eight types of boundary conditions are considered. One can see that the predicted frequency parameters of the proposed element agree well with the available ones due to the maximum relative error being less than 1.5%.

Table 15 Comparison of the first eight dimensionless frequency parameters for a moderately thick macro-plate.

Plate type	Source	Mode							
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$
SSSF	Ref. [68]	0.792	2.766	5.769	8.572	9.890	10.449	13.348	14.937
	Present	0.7921	2.7657	5.7679	8.5720	9.8887	10.4486	13.3458	14.9367
SFSF	Ref. [68]	5.579	6.202	8.139	11.132	15.058	18.297	18.710	19.729
	Present	5.5789	6.2020	8.1382	11.1311	15.0560	18.2965	18.7100	19.7268
SSSS	Ref. [68]	6.467	8.793	12.348	16.808	19.004	20.680	21.894	23.334
	Present	6.4648	8.7876	12.3410	16.8008	19.0024	20.6748	21.8867	23.3254
SCSC	Ref. [68]	6.722	9.497	13.366	17.949	19.060	20.872	23.013	23.679
	Present	6.7236	9.5045	13.3875	17.9919	19.0618	20.8804	23.0841	23.6983
CFFF	Ref. [68]	2.099	2.727	4.387	7.253	10.399	11.130	11.424	13.148
	Present	2.0967	2.7197	4.3813	7.2504	10.4105	11.1396	11.4326	13.1648
CCFF	Ref. [68]	2.337	3.922	6.832	10.529	11.065	12.190	14.824	15.868
	Present	2.3332	3.9157	6.8266	10.5343	11.0775	12.2127	14.8557	15.8781
CFCF	Ref. [68]	10.132	10.353	11.275	13.193	16.310	20.476	22.011	22.288
	Present	10.1501	10.3484	11.2889	13.2222	16.3491	20.5179	22.1856	22.4338
CCCC	Ref. [68]	10.670	12.316	15.211	19.133	22.537	23.779	23.877	26.095

Present	10.7225	12.3730	15.2771	19.2177	22.7518	23.8901	24.0904	26.3086

Figs. 15 and 16 regenerate the first eight vibration mode shapes in terms of deflection contour plots for a moderately thick square plate with two sets of boundary conditions, namely, at least two edges simply supported and at least one edge clamped, respectively. Here, the mesh number of 100×100 is used and the dimensionless frequencies $\bar{\omega}_n = \omega_n L_X^2 \sqrt{\rho h / D} / \pi^2$ are provided. For most cases, the present vibration mode shapes are in agreement with those of *p*-version Ritz method[68] (see Figs. 2 and 3 therein). However, for the cases of SSSS and CCCC, there are some distinct differences between the present 2nd, 3rd, 7th and 8th mode shapes as well as their respective counterparts.

Table 16 exhibits the validity of the present element in calculating the first three vibration frequencies of an isotropic homogeneous micro-plate with couple stress effect (i.e., $l_0 = l_1 = 0$ and $l_2 = l$). The available results given by *p*-version Ritz method[20] and C^1-weak continuous finite element[29] are served as comparison benchmarks. Here, the frequency parameters are presented as $\hat{\omega}_i = \omega_i / (2\pi)$ and the mesh number of 32×32 is used. Evidently, three groups of frequency predictions coincide well with each other for most cases; however, for the case of $L_Y/L_X = 0.5$, the third vibration frequencies available in Ref. [20] are actually the present fourth vibration frequencies.

Table 16 Comparison of the first three dimensionless frequencies (Unit: MHZ) for SSSS and CCCC micro-plates. Here, $h/L_X = 0.1$.

Plate type	h/l	Source	$L_Y/L_X = 1$			$L_Y/L_X = 0.5$			
			$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	$\hat{\omega}_4$
SSSS	1	Ref. [20]	1.2431	2.8989	4.3947	2.9223	4.4222	8.2063	—
		Ref. [29]	1.2368	2.8865	4.3583	2.8896	4.3654	6.5749	—
		Present	1.2403	2.8926	4.3708	2.8925	4.3705	6.5785	8.1841
	1.5	Ref. [20]	0.6295	1.4889	2.2773	1.4997	2.2919	4.3419	—
		Ref. [29]	—	—	—	—	—	—	—
		Present	0.6281	1.4843	2.2643	1.4843	2.2641	3.4463	4.3160
	2	Ref. [20]	0.4042	0.9603	1.4720	0.9662	1.4799	2.8214	—
		Ref. [29]	0.4028	0.9559	1.4614	9.5655	1.4630	2.2334	—
		Present	0.4034	0.9570	1.4636	0.9569	1.4635	2.2332	2.7998
CCCC	1	Ref. [20]	2.0733	3.8810	5.4493	4.7581	6.0524	10.4102	—
		Ref. [29]	2.0670	3.8713	5.4167	4.7432	6.0273	8.1026	—
		Present	2.0647	3.8706	5.4158	4.7429	6.0144	8.0661	10.2984
	1.5	Ref. [20]	1.0769	2.0433	2.8758	2.5244	3.2115	5.5668	—
		Ref. [29]	—	—	—	—	—	—	—
		Present	1.0645	2.0222	2.8443	2.5130	3.1838	4.2890	5.5500
	2	Ref. [20]	0.6997	1.3356	1.8753	1.6462	2.0966	3.6196	—
		Ref. [29]	0.6901	1.3116	1.8462	1.6349	2.0746	3.6037	—
		Present	0.6870	1.3095	1.8428	1.6328	2.0656	2.7831	3.6104

4.5 Parametric study

In this section, we use the developed method to examine the size-dependent vibration behavior of moderately thick annular sectorial and elliptical micro-plates composed of epoxy resin. Here, $K_s = 5/6$ and $l_0 = l_1 = l_2 = l$. For the annular sectorial case, we set h/R_{out} =0.1 and $\bar{\omega}_n = \omega_n R_{\text{out}}^2 \sqrt{\rho h/D}$, and for the elliptical case,we set h/L_Y =0.1 and $\bar{\omega}_n = \omega_n L_Y^2 \sqrt{\rho h/D}$.

Tables 17 and 18 present the first nine dimensionless frequencies for two types of micro-plates, respectively. From these tabulated results, one can observe the size-dependent vibration behavior of micro-plates; the larger the l/h, the more increase in the dimensionless frequencies is. This is because the consideration of strain gradient can result in an enhancement of structural bending rigidity. It can also seen that the ratio of the frequency with strain gradient effects to that without them is notably affected by the boundary conditions, the order of vibration mode, and the value of l/h.

Figs. 17 and 18 present the first six vibration mode shapes in the form of contour plots for annular sectorial and elliptical micro-plates, respectively, where five values of l/h are considered. The two figures show that introducing strain gradient effects changes the vibration mode shapes, not the frequency values alone. The reason for this may be that strain gradient terms reflect the internal deformation constraints in structures.

As far as the authors are aware, little work has been reported on strain gradient Mindlin micro-plates with annular sectorial and elliptical shapes, whether applying IGA, FEM, QEM or DQ-based FEM. Thus, predictions on the vibration frequencies and their mode shapes can serve as reference values for future research.

Table 17 The first nine dimensionless frequency parameters for a moderately thick annular sectorial micro-plate made of epoxy resin.

Plate type	l/h	Mode								
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$	$\bar{\omega}_9$
SSSS	0	60.6894	120.3986	144.6748	195.5893	198.2003	249.6831	273.2125	275.4438	292.1619
	0.25	78.4409	149.5581	178.0540	240.3196	242.1211	306.7163	336.7095	340.7969	361.1640
	0.50	106.0433	194.1517	228.6948	313.7921	315.2343	407.1835	451.6975	461.4020	487.6864
	0.75	129.8976	237.4818	279.6613	393.2327	398.4617	523.4623	586.3757	602.7760	636.6312
CCCC	0	92.9892	149.6810	173.8650	220.0836	223.0511	271.8477	292.6607	294.8952	311.7445
	0.25	117.3710	188.2648	219.5435	281.3578	284.1325	354.5453	381.7134	386.8583	410.8212
	0.50	155.1634	250.7414	294.2454	385.7977	387.7611	497.9399	538.7884	550.8875	586.1657
	0.75	194.3097	320.1500	378.6958	505.9284	507.4002	664.0848	721.7276	741.4546	789.8210
CFCC	0	70.3626	99.3343	156.2947	159.0608	183.5790	231.6671	236.9777	257.6073	281.1181
	0.25	91.8731	129.4870	202.4773	203.5484	238.7831	296.7310	309.4466	342.9110	374.2707
	0.50	123.1199	176.7302	273.0730	278.0460	327.6297	409.5919	433.0416	484.5199	533.3159
	0.75	154.1744	225.6322	351.2926	360.0445	426.9272	538.1343	573.7768	647.6293	716.2162

Table 18 The first nine dimensionless frequency parameters for a moderately thick elliptical micro-plate made of epoxy resin.

Plate type	l/h	Mode								
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$	$\bar{\omega}_7$	$\bar{\omega}_8$	$\bar{\omega}_9$

S	0	13.0200	22.8010	35.9569	42.5485	52.1985	56.3504	71.0841	72.3218	84.1427
	0.25	19.1715	32.4162	49.8223	55.5708	70.5074	72.9586	92.9411	94.2855	106.8504
	0.50	29.7613	48.6860	72.6634	77.0662	99.5794	99.9874	125.2273	130.9733	142.1265
	0.75	40.2845	64.1366	93.6764	95.9244	123.1666	126.9045	154.3247	165.2579	174.5324
C	0	25.0937	35.3337	48.7249	58.6621	64.9346	71.8571	83.5465	87.0769	101.2945
	0.25	33.4727	46.5626	63.5047	73.3383	83.6185	89.3514	106.5211	107.8518	123.3468
	0.50	46.3737	63.5494	85.6250	93.2806	111.3503	113.5608	137.1082	140.7889	154.1379
	0.75	56.7940	77.5460	104.4605	108.5990	133.7451	135.9164	163.1897	172.7179	182.8995

5. Conclusions

In this work, we drew on the advantages of DQM and FEM to develop a C^1-type 48-DOF four-node quadrilateral plate element for a three-parameter strain gradient Mindlin micro-plates. To address the C^1-continuity requirements of W, Φ_X, and Φ_Y, we constructed a fourth-order DQ-based geometric mapping scheme that was responsible for converting the displacement parameters at GLQ points into those at element nodes. The proper exploitation of DQ rule to non-rectangular domains was realized by the natural-to-Cartesian geometric mapping technique. The GLQ and DQ rules were combined to discretize the total potential energy functional of a generic plate element.Then, the stiffness and mass matrices, as well as the load vector of the element, were obtained by the minimum total potential energy principle. Numerical results showed that the convergence and adaptability of our element is better than the available C^1-continuous BFS element. The boundary conditions, thickness-to-MLSP ratio, length-to-thickness ratio, and the order of vibration model have certain influences on the convergence and accuracy of predictions. Parametric studies reveal that the present element can predict the size-dependent vibration frequencies and mode shapes of irregular micro-plates, and strain gradient effects can result in an enhancement of vibration frequencies and a certain change in vibration mode shapes.

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Conflict of interest statement

The authors declare that they have no conflict of interest.

Appendix A

The coordinate transformation-related coefficients β_m are as follows:

$$\beta_1=(\Sigma_7+\Sigma_2)\left(\frac{\partial\bar{X}}{\partial X}\right)^2\left(\frac{\partial\bar{X}}{\partial Y}\right)^2+\Sigma_1\left(\frac{\partial\bar{X}}{\partial X}\right)^4+\Sigma_1\left(\frac{\partial\bar{X}}{\partial Y}\right)^4,\quad \beta_2=(\Sigma_7+\Sigma_2)\left(\frac{\partial\bar{Y}}{\partial X}\right)^2\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2+\Sigma_1\left(\frac{\partial\bar{Y}}{\partial X}\right)^4+\Sigma_1\left(\frac{\partial\bar{Y}}{\partial Y}\right)^4,$$

$$\beta_3 = \left(\frac{\partial \bar{Y}}{\partial X}\right)^2 \left[\Sigma_7 \left(\frac{\partial \bar{X}}{\partial Y}\right)^2 + 4\Sigma_1 \left(\frac{\partial \bar{X}}{\partial X}\right)^2\right] + \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 \left[\Sigma_7 \left(\frac{\partial \bar{X}}{\partial X}\right)^2 + 4\Sigma_1 \left(\frac{\partial \bar{X}}{\partial Y}\right)^2\right] + 2\left(\Sigma_7 + 2\Sigma_2\right) \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial X}\frac{\partial \bar{Y}}{\partial Y},$$

$$\beta_4 = \left(\frac{\partial \bar{Y}}{\partial X}\right)^2 \left[\Sigma_2 \left(\frac{\partial \bar{X}}{\partial Y}\right)^2 + 2\Sigma_1 \left(\frac{\partial \bar{X}}{\partial X}\right)^2\right] + \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 \left[\Sigma_2 \left(\frac{\partial \bar{X}}{\partial X}\right)^2 + 2\Sigma_1 \left(\frac{\partial \bar{X}}{\partial Y}\right)^2\right] + 2\Sigma_7 \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial X}\frac{\partial \bar{Y}}{\partial Y},$$

$$\beta_5 = 4\Sigma_1 \left[\left(\frac{\partial \bar{X}}{\partial X}\right)^3 \frac{\partial \bar{Y}}{\partial X} + \left(\frac{\partial \bar{X}}{\partial Y}\right)^3 \frac{\partial \bar{Y}}{\partial Y}\right] + 2\left(\Sigma_2 + \Sigma_7\right)\left[\frac{\partial \bar{X}}{\partial Y}\left(\frac{\partial \bar{X}}{\partial X}\right)^2 \frac{\partial \bar{Y}}{\partial Y} + \frac{\partial \bar{X}}{\partial X}\left(\frac{\partial \bar{X}}{\partial Y}\right)^2 \frac{\partial \bar{Y}}{\partial X}\right],$$

$$\beta_6 = 4\Sigma_1 \left[\frac{\partial \bar{X}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial X}\right)^3 + \frac{\partial \bar{X}}{\partial Y}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^3\right] + 2\left(\Sigma_2 + \Sigma_7\right)\left[\frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial Y}\left(\frac{\partial \bar{Y}}{\partial X}\right)^2 + \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{Y}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2\right],$$

$$\beta_7 = \left(\Sigma_{10} + \Sigma_{15}\right)\left(\frac{\partial \bar{X}}{\partial X}\right)^2\left(\frac{\partial \bar{X}}{\partial Y}\right)^2 + \Sigma_{13}\left(\frac{\partial \bar{X}}{\partial X}\right)^4 + \Sigma_{14}\left(\frac{\partial \bar{X}}{\partial Y}\right)^4, \ \beta_8 = \left(\Sigma_{10} + \Sigma_{15}\right)\left(\frac{\partial \bar{Y}}{\partial X}\right)^2\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 + \Sigma_{13}\left(\frac{\partial \bar{Y}}{\partial X}\right)^4 + \Sigma_{14}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^4,$$

$$\beta_9 = \left(\frac{\partial \bar{Y}}{\partial X}\right)^2 \left[\Sigma_{10} \left(\frac{\partial \bar{X}}{\partial Y}\right)^2 + 4\Sigma_{13} \left(\frac{\partial \bar{X}}{\partial X}\right)^2\right] + \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 \left[\Sigma_{10} \left(\frac{\partial \bar{X}}{\partial X}\right)^2 + 4\Sigma_{14} \left(\frac{\partial \bar{X}}{\partial Y}\right)^2\right] + 2\left(\Sigma_{10} + 2\Sigma_{15}\right) \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial X}\frac{\partial \bar{Y}}{\partial Y},$$

$$\beta_{10} = \left(\frac{\partial \bar{X}}{\partial X}\right)^2 \left[\Sigma_{15} \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 + 2\Sigma_{13} \left(\frac{\partial \bar{Y}}{\partial X}\right)^2\right] + \left(\frac{\partial \bar{X}}{\partial Y}\right)^2 \left[\Sigma_{15} \left(\frac{\partial \bar{Y}}{\partial X}\right)^2 + 2\Sigma_{14} \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2\right] + 2\Sigma_{10} \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial X}\frac{\partial \bar{Y}}{\partial Y},$$

$$\beta_{11} = 4\Sigma_{13}\left(\frac{\partial \bar{X}}{\partial X}\right)^3 \frac{\partial \bar{Y}}{\partial X} + 4\Sigma_{14}\left(\frac{\partial \bar{X}}{\partial Y}\right)^3 \frac{\partial \bar{Y}}{\partial Y} + 2\left(\Sigma_{10} + \Sigma_{15}\right)\left[\frac{\partial \bar{X}}{\partial X}\left(\frac{\partial \bar{X}}{\partial Y}\right)^2 \frac{\partial \bar{Y}}{\partial X} + \left(\frac{\partial \bar{X}}{\partial X}\right)^2 \frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial Y}\right],$$

$$\beta_{12} = 4\Sigma_{13}\left(\frac{\partial \bar{Y}}{\partial X}\right)^3 \frac{\partial \bar{X}}{\partial X} + 4\Sigma_{14}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^3 \frac{\partial \bar{X}}{\partial Y} + 2\left(\Sigma_{10} + \Sigma_{15}\right)\left[\left(\frac{\partial \bar{Y}}{\partial X}\right)^2 \frac{\partial \bar{Y}}{\partial Y}\frac{\partial \bar{X}}{\partial Y} + \frac{\partial \bar{Y}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 \frac{\partial \bar{X}}{\partial X}\right],$$

$$\beta_{13} = \left(\Sigma_{10} + \Sigma_{15}\right)\left(\frac{\partial \bar{X}}{\partial X}\right)^2\left(\frac{\partial \bar{X}}{\partial Y}\right)^2 + \Sigma_{14}\left(\frac{\partial \bar{X}}{\partial X}\right)^4 + \Sigma_{13}\left(\frac{\partial \bar{X}}{\partial Y}\right)^4, \quad \beta_{14} = \left(\Sigma_{10} + \Sigma_{15}\right)\left(\frac{\partial \bar{Y}}{\partial X}\right)^2\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 + \Sigma_{14}\left(\frac{\partial \bar{Y}}{\partial X}\right)^4 + \Sigma_{13}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^4,$$

$$\beta_{15} = \left(\frac{\partial \bar{X}}{\partial Y}\right)^2 \left[\Sigma_{10} \left(\frac{\partial \bar{Y}}{\partial X}\right)^2 + 4\Sigma_{13} \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2\right] + \left(\frac{\partial \bar{X}}{\partial X}\right)^2 \left[\Sigma_{10} \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 + 4\Sigma_{14} \left(\frac{\partial \bar{Y}}{\partial X}\right)^2\right] + 2\left(\Sigma_{10} + 2\Sigma_{15}\right) \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial X}\frac{\partial \bar{Y}}{\partial Y},$$

$$\beta_{16} = \left(\frac{\partial \bar{X}}{\partial X}\right)^2 \left[2\Sigma_{14} \left(\frac{\partial \bar{Y}}{\partial X}\right)^2 + \Sigma_{15} \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2\right] + \left(\frac{\partial \bar{X}}{\partial Y}\right)^2 \left[\Sigma_{15} \left(\frac{\partial \bar{Y}}{\partial X}\right)^2 + 2\Sigma_{13} \left(\frac{\partial \bar{Y}}{\partial Y}\right)^2\right] + 2\Sigma_{10} \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial Y}\frac{\partial \bar{Y}}{\partial X},$$

$$\beta_{17} = 4\Sigma_{14}\left(\frac{\partial \bar{X}}{\partial X}\right)^3 \frac{\partial \bar{Y}}{\partial X} + 4\Sigma_{13}\left(\frac{\partial \bar{X}}{\partial Y}\right)^3 \frac{\partial \bar{Y}}{\partial Y} + 2\left(\Sigma_{10} + \Sigma_{15}\right)\left[\frac{\partial \bar{X}}{\partial X}\left(\frac{\partial \bar{X}}{\partial Y}\right)^2 \frac{\partial \bar{Y}}{\partial X} + \left(\frac{\partial \bar{X}}{\partial X}\right)^2 \frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial Y}\right],$$

$$\beta_{18} = 4\Sigma_{13}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^3 \frac{\partial \bar{X}}{\partial Y} + 4\Sigma_{14}\left(\frac{\partial \bar{Y}}{\partial X}\right)^3 \frac{\partial \bar{Y}}{\partial X} + 2\left(\Sigma_{10} + \Sigma_{15}\right)\left[\left(\frac{\partial \bar{Y}}{\partial X}\right)^2 \frac{\partial \bar{Y}}{\partial Y}\frac{\partial \bar{X}}{\partial Y} + \frac{\partial \bar{Y}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 \frac{\partial \bar{X}}{\partial X}\right],$$

$$\beta_{19} = \left(\Sigma_{11} + \Sigma_{12}\right)\left[\left(\frac{\partial \bar{X}}{\partial X}\right)^3 \frac{\partial \bar{X}}{\partial Y} + \frac{\partial \bar{X}}{\partial X}\left(\frac{\partial \bar{X}}{\partial Y}\right)^3\right], \quad \beta_{20} = \left(\Sigma_{11} + \Sigma_{12}\right)\left[\left(\frac{\partial \bar{Y}}{\partial X}\right)^3 \frac{\partial \bar{Y}}{\partial Y} + \frac{\partial \bar{Y}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^3\right],$$

$$\beta_{21} = \frac{\partial \bar{Y}}{\partial X}\frac{\partial \bar{Y}}{\partial Y}\left[\Sigma_{11}\left(\frac{\partial \bar{X}}{\partial X}\right)^2 + \Sigma_{12}\left(\frac{\partial \bar{X}}{\partial Y}\right)^2\right] + \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\left[\Sigma_{11}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 + \Sigma_{12}\left(\frac{\partial \bar{Y}}{\partial X}\right)^2\right],$$

$$\beta_{22} = \frac{\partial \bar{Y}}{\partial X}\frac{\partial \bar{Y}}{\partial Y}\left[\Sigma_{11}\left(\frac{\partial \bar{X}}{\partial Y}\right)^2 + \Sigma_{12}\left(\frac{\partial \bar{X}}{\partial X}\right)^2\right] + \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{X}}{\partial Y}\left[\Sigma_{11}\left(\frac{\partial \bar{Y}}{\partial X}\right)^2 + \Sigma_{12}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2\right],$$

$$\beta_{23} = \Sigma_{11}\left(\frac{\partial \bar{X}}{\partial X}\right)^3 \frac{\partial \bar{Y}}{\partial Y} + \Sigma_{12}\left(\frac{\partial \bar{X}}{\partial Y}\right)^3 \frac{\partial \bar{Y}}{\partial X} + \left(\Sigma_{11} + 2\Sigma_{12}\right)\left(\frac{\partial \bar{X}}{\partial X}\right)^2 \frac{\partial \bar{X}}{\partial Y}\frac{\partial \bar{Y}}{\partial X} + \left(2\Sigma_{11} + \Sigma_{12}\right)\frac{\partial \bar{X}}{\partial X}\left(\frac{\partial \bar{X}}{\partial Y}\right)^2 \frac{\partial \bar{Y}}{\partial Y},$$

$$\beta_{24} = \Sigma_{11}\left(\frac{\partial \bar{Y}}{\partial X}\right)^3 \frac{\partial \bar{X}}{\partial Y} + \Sigma_{12}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^3 \frac{\partial \bar{X}}{\partial X} + \left(\Sigma_{11} + 2\Sigma_{12}\right)\left(\frac{\partial \bar{Y}}{\partial X}\right)^2 \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{Y}}{\partial Y} + \left(2\Sigma_{11} + \Sigma_{12}\right)\frac{\partial \bar{Y}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 \frac{\partial \bar{X}}{\partial Y},$$

$$\beta_{25} = \Sigma_{12}\left(\frac{\partial \bar{Y}}{\partial X}\right)^3 \frac{\partial \bar{X}}{\partial Y} + \Sigma_{11}\frac{\partial \bar{X}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^3 + \left(2\Sigma_{11} + \Sigma_{12}\right)\left(\frac{\partial \bar{Y}}{\partial X}\right)^2 \frac{\partial \bar{X}}{\partial X}\frac{\partial \bar{Y}}{\partial Y} + \left(\Sigma_{11} + 2\Sigma_{12}\right)\frac{\partial \bar{Y}}{\partial X}\left(\frac{\partial \bar{Y}}{\partial Y}\right)^2 \frac{\partial \bar{X}}{\partial Y},$$

$$\beta_{26}=\Sigma_{11}\frac{\partial\bar{Y}}{\partial X}\left(\frac{\partial\bar{X}}{\partial Y}\right)^3+\Sigma_{12}\left(\frac{\partial\bar{X}}{\partial X}\right)^3\frac{\partial\bar{Y}}{\partial Y}+\left(\Sigma_{12}+2\Sigma_{11}\right)\left(\frac{\partial\bar{X}}{\partial X}\right)^2\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{X}}{\partial Y}+\left(\Sigma_{11}+2\Sigma_{12}\right)\frac{\partial\bar{X}}{\partial X}\left(\frac{\partial\bar{X}}{\partial Y}\right)^2\frac{\partial\bar{Y}}{\partial Y}\,,$$

$$\beta_{27}=2\left(\Sigma_{11}+\Sigma_{12}\right)\left\{\left[\left(\frac{\partial\bar{Y}}{\partial X}\right)^2+\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2\right]\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{X}}{\partial Y}+\left[\left(\frac{\partial\bar{X}}{\partial X}\right)^2+\left(\frac{\partial\bar{X}}{\partial Y}\right)^2\right]\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}\right\},$$

$$\beta_{28}=\Sigma_8\left(\frac{\partial\bar{Y}}{\partial X}\right)^3+\left(\Sigma_5+\Sigma_6\right)\frac{\partial\bar{Y}}{\partial X}\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2,\quad \beta_{30}=\Sigma_8\left(\frac{\partial\bar{Y}}{\partial Y}\right)^3+\left(\Sigma_5+\Sigma_6\right)\left(\frac{\partial\bar{Y}}{\partial X}\right)^2\frac{\partial\bar{Y}}{\partial Y},$$

$$\beta_{29}=\Sigma_6\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}\frac{\partial\bar{X}}{\partial Y}+\frac{\partial\bar{X}}{\partial X}\left[\Sigma_8\left(\frac{\partial\bar{Y}}{\partial X}\right)^2+\Sigma_5\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2\right],\quad \beta_{31}=\Sigma_6\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}\frac{\partial\bar{X}}{\partial X}+\frac{\partial\bar{X}}{\partial Y}\left[\Sigma_8\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2+\Sigma_5\left(\frac{\partial\bar{Y}}{\partial X}\right)^2\right],$$

$$\beta_{32}=\Sigma_6\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial Y}+\frac{\partial\bar{Y}}{\partial X}\left[\Sigma_8\left(\frac{\partial\bar{X}}{\partial X}\right)^2+\Sigma_5\left(\frac{\partial\bar{X}}{\partial Y}\right)^2\right],\quad \beta_{33}=\Sigma_8\left(\frac{\partial\bar{X}}{\partial X}\right)^3+\left(\Sigma_5+\Sigma_6\right)\left(\frac{\partial\bar{X}}{\partial Y}\right)^2\frac{\partial\bar{X}}{\partial X},$$

$$\beta_{34}=\Sigma_6\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial X}+\frac{\partial\bar{Y}}{\partial Y}\left[\Sigma_8\left(\frac{\partial\bar{X}}{\partial Y}\right)^2+\Sigma_5\left(\frac{\partial\bar{X}}{\partial X}\right)^2\right],\quad \beta_{35}=\Sigma_8\left(\frac{\partial\bar{X}}{\partial Y}\right)^3+\left(\Sigma_5+\Sigma_6\right)\frac{\partial\bar{X}}{\partial Y}\left(\frac{\partial\bar{X}}{\partial X}\right)^2,$$

$$\beta_{36}=\left(2\Sigma_5+\Sigma_6\right)\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}+\frac{\partial\bar{X}}{\partial X}\left[2\Sigma_8\left(\frac{\partial\bar{Y}}{\partial X}\right)^2+\Sigma_6\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2\right],\quad \beta_{40}=\left(\Sigma_{17}+2\Sigma_{18}+\Sigma_3+\Sigma_6\right)\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}\,,$$

$$\beta_{37}=\left(2\Sigma_5+\Sigma_6\right)\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}+\frac{\partial\bar{Y}}{\partial X}\left[2\Sigma_8\left(\frac{\partial\bar{X}}{\partial X}\right)^2+\Sigma_6\left(\frac{\partial\bar{X}}{\partial Y}\right)^2\right],\quad \beta_{41}=\left(\Sigma_{17}+\Sigma_3\right)\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{X}}{\partial Y}+\left(2\Sigma_{18}+\Sigma_6\right)\frac{\partial\bar{Y}}{\partial Y}\frac{\partial\bar{X}}{\partial X}\,,$$

$$\beta_{38}=\left(2\Sigma_5+\Sigma_6\right)\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{Y}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}+\frac{\partial\bar{X}}{\partial Y}\left[2\Sigma_8\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2+\Sigma_6\left(\frac{\partial\bar{Y}}{\partial X}\right)^2\right],\quad \beta_{42}=2\left(\Sigma_{16}+\Sigma_4\right)\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{Y}}{\partial X}+2\left(\Sigma_{18}+\Sigma_7\right)\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial Y}\,,$$

$$\beta_{39}=\left(2\Sigma_5+\Sigma_6\right)\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial X}+\frac{\partial\bar{Y}}{\partial Y}\left[2\Sigma_8\left(\frac{\partial\bar{X}}{\partial Y}\right)^2+\Sigma_6\left(\frac{\partial\bar{X}}{\partial X}\right)^2\right],\quad \beta_{43}=\left(\Sigma_{17}+\Sigma_3\right)\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{Y}}{\partial Y}+\left(2\Sigma_{18}+\Sigma_6\right)\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial X}\,,$$

$$\beta_{44}=\left(\Sigma_{17}+2\Sigma_{18}+\Sigma_3+\Sigma_6\right)\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{X}}{\partial Y},\quad \beta_{45}=2\left(\Sigma_{16}+\Sigma_4\right)\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial Y}+2\left(\Sigma_{18}+\Sigma_7\right)\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{Y}}{\partial X},$$

$$\beta_{46}=\Sigma_9\left[\left(\frac{\partial\bar{X}}{\partial X}\right)^2+\left(\frac{\partial\bar{X}}{\partial Y}\right)^2\right],\ \beta_{47}=\Sigma_9\left[\left(\frac{\partial\bar{Y}}{\partial X}\right)^2+\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2\right],\quad \beta_{48}=\left(\Sigma_4+\Sigma_{16}\right)\left(\frac{\partial\bar{X}}{\partial X}\right)^2+\left(\Sigma_7+\Sigma_{18}\right)\left(\frac{\partial\bar{X}}{\partial Y}\right)^2,$$

$$\beta_{49}=\left(\Sigma_4+\Sigma_{16}\right)\left(\frac{\partial\bar{Y}}{\partial X}\right)^2+\left(\Sigma_7+\Sigma_{18}\right)\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2,\ \beta_{50}=\left(\Sigma_7+\Sigma_{18}\right)\left(\frac{\partial\bar{X}}{\partial X}\right)^2+\left(\Sigma_4+\Sigma_{16}\right)\left(\frac{\partial\bar{X}}{\partial Y}\right)^2,$$

$$\beta_{51}=\left(\Sigma_4+\Sigma_{16}\right)\left(\frac{\partial\bar{Y}}{\partial Y}\right)^2+\left(\Sigma_7+\Sigma_{18}\right)\left(\frac{\partial\bar{Y}}{\partial X}\right)^2,\quad \beta_{53}=2\Sigma_9\frac{\partial\bar{X}}{\partial X},\quad \beta_{52}=2\Sigma_9\left(\frac{\partial\bar{X}}{\partial X}\frac{\partial\bar{Y}}{\partial X}+\frac{\partial\bar{X}}{\partial Y}\frac{\partial\bar{Y}}{\partial Y}\right),$$

$$\beta_{54}=2\Sigma_9\frac{\partial\bar{Y}}{\partial X},\quad \beta_{55}=2\Sigma_9\frac{\partial\bar{X}}{\partial Y},\quad \beta_{56}=2\Sigma_9\frac{\partial\bar{Y}}{\partial Y},\quad \beta_{57}=\Sigma_9\bar{X}\bar{Y}\,. \qquad \text{(A. 1)}$$

Appendix B

The sub-block matrices $\boldsymbol{B}_{(ij)(kl)}$ are presented as follows:

$$\boldsymbol{B}_{(11)(11)}=\begin{bmatrix}1 & 0 & 0 & 0\\ -3 & \frac{5\left(1+\sqrt{5}\right)}{4} & \frac{5\left(1-\sqrt{5}\right)}{4} & \frac{1}{2}\\ -3 & 0 & 0 & 0\\ 9 & -\frac{15\left(1+\sqrt{5}\right)}{4} & -\frac{15\left(1-\sqrt{5}\right)}{4} & -\frac{3}{2}\end{bmatrix},\quad \boldsymbol{B}_{(14)(14)}=\begin{bmatrix}1 & 0 & 0 & 0\\ -3 & \frac{5\left(1+\sqrt{5}\right)}{4} & \frac{5\left(1-\sqrt{5}\right)}{4} & \frac{1}{2}\\ 3 & 0 & 0 & 0\\ -9 & \frac{15\left(1+\sqrt{5}\right)}{4} & \frac{15\left(1-\sqrt{5}\right)}{4} & \frac{3}{2}\end{bmatrix},$$

$$\boldsymbol{B}_{(44)(14)}=\begin{bmatrix}0 & 0 & 0 & 1\\ -\frac{1}{2} & -\frac{5\left(1-\sqrt{5}\right)}{4} & -\frac{5\left(1+\sqrt{5}\right)}{4} & 3\\ 0 & 0 & 0 & 3\\ -\frac{3}{2} & -\frac{15\left(1-\sqrt{5}\right)}{4} & -\frac{15\left(1+\sqrt{5}\right)}{4} & 9\end{bmatrix},\quad \boldsymbol{B}_{(41)(11)}=\begin{bmatrix}0 & 0 & 0 & 1\\ -\frac{1}{2} & -\frac{5\left(1-\sqrt{5}\right)}{4} & -\frac{5\left(1+\sqrt{5}\right)}{4} & 3\\ 0 & 0 & 0 & -3\\ \frac{3}{2} & \frac{15\left(1-\sqrt{5}\right)}{4} & \frac{15\left(1+\sqrt{5}\right)}{4} & -9\end{bmatrix},$$

$$\boldsymbol{B}_{(11)(44)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\\frac{5\left(1-\sqrt{5}\right)}{4}&0&0&0\\-\frac{15\left(1-\sqrt{5}\right)}{4}&-\frac{25}{4}&\frac{25\left(3-\sqrt{5}\right)}{8}&\frac{5\left(1-\sqrt{5}\right)}{8}\end{bmatrix},\quad \boldsymbol{B}_{(11)(14)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\\frac{1}{2}&0&0&0\\-\frac{3}{2}&\frac{5\left(1+\sqrt{5}\right)}{8}&\frac{5\left(1-\sqrt{5}\right)}{8}&\frac{1}{4}\end{bmatrix},$$

$$\boldsymbol{B}_{(41)(41)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\frac{5\left(1+\sqrt{5}\right)}{4}\\-\frac{5\left(1+\sqrt{5}\right)}{8}&\frac{25}{4}&-\frac{25\left(3+\sqrt{5}\right)}{8}&\frac{15\left(1+\sqrt{5}\right)}{4}\end{bmatrix},\quad \boldsymbol{B}_{(41)(14)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\frac{1}{2}\\-\frac{1}{4}&-\frac{5\left(1-\sqrt{5}\right)}{8}&-\frac{5\left(1+\sqrt{5}\right)}{8}&\frac{3}{2}\end{bmatrix},$$

$$\boldsymbol{B}_{(41)(44)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\frac{5\left(1-\sqrt{5}\right)}{4}\\-\frac{5\left(1-\sqrt{5}\right)}{8}&-\frac{25\left(3-\sqrt{5}\right)}{8}&\frac{25}{4}&\frac{15\left(1-\sqrt{5}\right)}{4}\end{bmatrix},\ \boldsymbol{B}_{(14)(41)}=-\boldsymbol{B}_{(11)(44)},\ \boldsymbol{B}_{(14)(11)}=-\boldsymbol{B}_{(11)(14)},$$

$$\boldsymbol{B}_{(44)(44)}=-\boldsymbol{B}_{(41)(41)},\quad \boldsymbol{B}_{(44)(11)}=\boldsymbol{B}_{(41)(14)},\quad \boldsymbol{B}_{(44)(41)}=-\boldsymbol{B}_{(41)(44)}. \tag{B. 1}$$

Appendix C

With the help of MAPLE symbolic computation system, the explicit algebraic expressions of element stiffness matrix $\boldsymbol{K}_e$ are as follows:

$$\boldsymbol{K}_e=\begin{bmatrix}\boldsymbol{k}_{18}^{(e)}&\boldsymbol{k}_{19}^{(e)}&\boldsymbol{k}_{20}^{(e)}&\boldsymbol{k}_{24}^{(e)}&\boldsymbol{k}_{25}^{(e)}&\boldsymbol{k}_{26}^{(e)}&\boldsymbol{k}_{30}^{(e)}&\boldsymbol{k}_{31}^{(e)}&\boldsymbol{k}_{32}^{(e)}&\boldsymbol{k}_{36}^{(e)}&\boldsymbol{k}_{37}^{(e)}&\boldsymbol{k}_{38}^{(e)}\\ &\boldsymbol{k}_{21}^{(e)}&\boldsymbol{k}_{22}^{(e)}&\mathbf{0}&\boldsymbol{k}_{27}^{(e)}&\boldsymbol{k}_{28}^{(e)}&\mathbf{0}&\boldsymbol{k}_{33}^{(e)}&\boldsymbol{k}_{34}^{(e)}&\mathbf{0}&\boldsymbol{k}_{39}^{(e)}&\boldsymbol{k}_{40}^{(e)}\\ &&\boldsymbol{k}_{23}^{(e)}&\mathbf{0}&\mathbf{0}&\boldsymbol{k}_{29}^{(e)}&\mathbf{0}&\boldsymbol{k}_{34}^{(e)}&\boldsymbol{k}_{35}^{(e)}&\mathbf{0}&\boldsymbol{k}_{52}^{(e)}&\boldsymbol{k}_{41}^{(e)}\\ &&&\boldsymbol{k}_{42}^{(e)}&\boldsymbol{k}_{43}^{(e)}&\boldsymbol{k}_{44}^{(e)}&\boldsymbol{k}_{48}^{(e)}&\boldsymbol{k}_{49}^{(e)}&\boldsymbol{k}_{50}^{(e)}&\boldsymbol{k}_{54}^{(e)}&\boldsymbol{k}_{55}^{(e)}&\boldsymbol{k}_{56}^{(e)}\\ &&&&\boldsymbol{k}_{45}^{(e)}&\boldsymbol{k}_{46}^{(e)}&\mathbf{0}&\boldsymbol{k}_{51}^{(e)}&\boldsymbol{k}_{52}^{(e)}&\mathbf{0}&\boldsymbol{k}_{57}^{(e)}&\boldsymbol{k}_{58}^{(e)}\\ &&&&&\boldsymbol{k}_{47}^{(e)}&\mathbf{0}&\boldsymbol{k}_{40}^{(e)}&\boldsymbol{k}_{53}^{(e)}&\mathbf{0}&\boldsymbol{k}_{58}^{(e)}&\boldsymbol{k}_{59}^{(e)}\\ &&&&&&\boldsymbol{k}_{60}^{(e)}&\boldsymbol{k}_{1}^{(e)}&\boldsymbol{k}_{2}^{(e)}&\boldsymbol{k}_{6}^{(e)}&\boldsymbol{k}_{7}^{(e)}&\boldsymbol{k}_{8}^{(e)}\\ &&&&&&&\boldsymbol{k}_{3}^{(e)}&\boldsymbol{k}_{4}^{(e)}&\mathbf{0}&\boldsymbol{k}_{9}^{(e)}&\boldsymbol{k}_{10}^{(e)}\\ &&&&&&&&\boldsymbol{k}_{5}^{(e)}&\mathbf{0}&\mathbf{0}&\boldsymbol{k}_{11}^{(e)}\\ &&\textbf{\textit{Sym}}&&&&&&&\boldsymbol{k}_{12}^{(e)}&\boldsymbol{k}_{13}^{(e)}&\boldsymbol{k}_{14}^{(e)}\\ &&&&&&&&&&\boldsymbol{k}_{15}^{(e)}&\boldsymbol{k}_{16}^{(e)}\\ &&&&&&&&&&&\boldsymbol{k}_{17}^{(e)}\end{bmatrix}, \tag{C. 1}$$

where

$$\boldsymbol{k}_1^{(e)}=\begin{bmatrix}\kappa_{17}&-\kappa_{43}&-\kappa_{169}&\kappa_{24}\\\kappa_{43}&\kappa_{9}&-\kappa_{24}&-\kappa_{11}\\-\kappa_{165}&\kappa_{26}&\kappa_{180}&-\kappa_{41}\\-\kappa_{26}&-\kappa_{11}&\kappa_{41}&\kappa_{12}\end{bmatrix},\quad \boldsymbol{k}_2^{(e)}=\begin{bmatrix}\kappa_{16}&-\kappa_{168}&-\kappa_{45}&\kappa_{29}\\-\kappa_{164}&\kappa_{176}&\kappa_{31}&-\kappa_{39}\\\kappa_{45}&-\kappa_{29}&\kappa_{3}&-\kappa_{5}\\-\kappa_{31}&\kappa_{39}&-\kappa_{5}&\kappa_{6}\end{bmatrix},\quad \boldsymbol{k}_3^{(e)}=\begin{bmatrix}\kappa_{120}&-\kappa_{117}&-\kappa_{116}&\kappa_{114}\\-\kappa_{117}&\kappa_{109}&\kappa_{115}&-\kappa_{95}\\-\kappa_{116}&\kappa_{115}&\kappa_{110}&-\kappa_{96}\\\kappa_{114}&-\kappa_{95}&-\kappa_{96}&\kappa_{84}\end{bmatrix},$$

$$\boldsymbol{k}_4^{(e)}=\begin{bmatrix}\kappa_{185}&-\kappa_{32}&\kappa_{34}&-\kappa_{83}\\\kappa_{32}&\kappa_{159}&\kappa_{83}&\kappa_{162}\\-\kappa_{34}&\kappa_{83}&\kappa_{159}&-\kappa_{160}\\-\kappa_{83}&-\kappa_{162}&\kappa_{160}&\kappa_{1}\end{bmatrix},\quad \boldsymbol{k}_5^{(e)}=\begin{bmatrix}\kappa_{157}&-\kappa_{154}&-\kappa_{153}&\kappa_{151}\\-\kappa_{154}&\kappa_{146}&\kappa_{152}&-\kappa_{132}\\-\kappa_{153}&\kappa_{152}&\kappa_{147}&-\kappa_{133}\\\kappa_{151}&-\kappa_{132}&-\kappa_{133}&\kappa_{121}\end{bmatrix},\quad \boldsymbol{k}_6^{(e)}=\begin{bmatrix}\kappa_{57}&\kappa_{55}&-\kappa_{51}&-\kappa_{50}\\-\kappa_{55}&-\kappa_{54}&\kappa_{50}&\kappa_{49}\\-\kappa_{51}&-\kappa_{50}&\kappa_{58}&\kappa_{66}\\\kappa_{50}&\kappa_{49}&-\kappa_{66}&-\kappa_{65}\end{bmatrix},$$

$$\boldsymbol{k}_7^{(e)}=\begin{bmatrix}\kappa_{21}&\kappa_{43}&-\kappa_{173}&-\kappa_{24}\\-\kappa_{43}&-\kappa_{25}&\kappa_{24}&\kappa_{187}\\-\kappa_{175}&-\kappa_{26}&\kappa_{183}&\kappa_{41}\\\kappa_{26}&\kappa_{188}&-\kappa_{41}&-\kappa_{195}\end{bmatrix},\quad \boldsymbol{k}_8^{(e)}=\begin{bmatrix}\kappa_{18}&\kappa_{170}&-\kappa_{44}&-\kappa_{28}\\-\kappa_{170}&-\kappa_{177}&\kappa_{28}&\kappa_{38}\\\kappa_{44}&\kappa_{28}&\kappa_{2}&\kappa_{4}\\-\kappa_{28}&-\kappa_{38}&-\kappa_{4}&-\kappa_{7}\end{bmatrix},\quad \boldsymbol{k}_9^{(e)}=\begin{bmatrix}\kappa_{119}&\kappa_{108}&-\kappa_{112}&-\kappa_{104}\\-\kappa_{108}&-\kappa_{105}&\kappa_{104}&\kappa_{101}\\-\kappa_{112}&-\kappa_{104}&\kappa_{100}&\kappa_{89}\\\kappa_{104}&\kappa_{101}&-\kappa_{89}&-\kappa_{86}\end{bmatrix},$$

$$\boldsymbol{k}_{10}^{(e)}=\begin{bmatrix}\kappa_{184} & \kappa_{32} & \kappa_{35} & \kappa_{83}\\ -\kappa_{32} & -\kappa_{196} & -\kappa_{83} & -\kappa_{37}\\ -\kappa_{35} & -\kappa_{83} & \kappa_{158} & \kappa_{160}\\ \kappa_{83} & \kappa_{37} & -\kappa_{160} & -\kappa_{161}\end{bmatrix},\quad \boldsymbol{k}_{11}^{(e)}=\begin{bmatrix}\kappa_{156} & \kappa_{145} & -\kappa_{149} & -\kappa_{141}\\ -\kappa_{145} & -\kappa_{142} & \kappa_{141} & \kappa_{138}\\ -\kappa_{149} & -\kappa_{141} & \kappa_{137} & \kappa_{126}\\ \kappa_{141} & \kappa_{138} & -\kappa_{126} & -\kappa_{123}\end{bmatrix},\quad \boldsymbol{k}_{12}^{(e)}=\begin{bmatrix}\kappa_{78} & \kappa_{81} & -\kappa_{79} & -\kappa_{73}\\ \kappa_{81} & \kappa_{77} & -\kappa_{74} & -\kappa_{48}\\ -\kappa_{79} & -\kappa_{74} & \kappa_{82} & \kappa_{80}\\ -\kappa_{73} & -\kappa_{48} & \kappa_{80} & \kappa_{67}\end{bmatrix},$$

$$\boldsymbol{k}_{13}^{(e)}=\begin{bmatrix}-\kappa_{17} & -\kappa_{43} & \kappa_{169} & \kappa_{24}\\ \kappa_{43} & -\kappa_{9} & -\kappa_{24} & \kappa_{11}\\ \kappa_{165} & \kappa_{26} & -\kappa_{180} & -\kappa_{41}\\ -\kappa_{26} & \kappa_{11} & \kappa_{41} & -\kappa_{12}\end{bmatrix},\quad \boldsymbol{k}_{14}^{(e)}=\begin{bmatrix}\kappa_{16} & \kappa_{168} & -\kappa_{45} & -\kappa_{29}\\ \kappa_{164} & \kappa_{176} & -\kappa_{31} & -\kappa_{39}\\ \kappa_{45} & \kappa_{29} & \kappa_{3} & \kappa_{5}\\ \kappa_{31} & \kappa_{39} & \kappa_{5} & \kappa_{6}\end{bmatrix},\quad \boldsymbol{k}_{15}^{(e)}=\begin{bmatrix}\kappa_{120} & \kappa_{117} & -\kappa_{116} & -\kappa_{114}\\ \kappa_{117} & \kappa_{109} & -\kappa_{115} & -\kappa_{95}\\ -\kappa_{116} & -\kappa_{115} & \kappa_{110} & \kappa_{96}\\ -\kappa_{114} & -\kappa_{95} & \kappa_{96} & \kappa_{84}\end{bmatrix},$$

$$\boldsymbol{k}_{16}^{(e)}=\begin{bmatrix}-\kappa_{185} & -\kappa_{32} & -\kappa_{34} & -\kappa_{83}\\ \kappa_{32} & -\kappa_{159} & \kappa_{83} & -\kappa_{162}\\ \kappa_{34} & \kappa_{83} & -\kappa_{159} & -\kappa_{160}\\ -\kappa_{83} & \kappa_{162} & \kappa_{160} & \kappa_{1}\end{bmatrix},\quad \boldsymbol{k}_{17}^{(e)}=\begin{bmatrix}\kappa_{157} & \kappa_{154} & -\kappa_{153} & -\kappa_{151}\\ \kappa_{154} & \kappa_{146} & -\kappa_{152} & -\kappa_{132}\\ -\kappa_{153} & -\kappa_{152} & \kappa_{147} & \kappa_{133}\\ -\kappa_{151} & -\kappa_{132} & \kappa_{133} & \kappa_{121}\end{bmatrix},\quad \boldsymbol{k}_{18}^{(e)}=\begin{bmatrix}\kappa_{78} & \kappa_{81} & \kappa_{79} & \kappa_{73}\\ \kappa_{81} & \kappa_{77} & \kappa_{74} & \kappa_{48}\\ \kappa_{79} & \kappa_{74} & \kappa_{82} & \kappa_{80}\\ \kappa_{73} & \kappa_{48} & \kappa_{80} & \kappa_{67}\end{bmatrix},$$

$$\boldsymbol{k}_{19}^{(e)}=\begin{bmatrix}-\kappa_{17} & -\kappa_{43} & -\kappa_{169} & -\kappa_{24}\\ \kappa_{43} & -\kappa_{9} & \kappa_{24} & -\kappa_{11}\\ -\kappa_{165} & -\kappa_{26} & -\kappa_{180} & -\kappa_{41}\\ \kappa_{26} & -\kappa_{11} & \kappa_{41} & -\kappa_{12}\end{bmatrix},\quad \boldsymbol{k}_{20}^{(e)}=\begin{bmatrix}-\kappa_{16} & -\kappa_{168} & -\kappa_{45} & -\kappa_{29}\\ -\kappa_{164} & -\kappa_{176} & -\kappa_{31} & -\kappa_{39}\\ \kappa_{45} & \kappa_{29} & -\kappa_{3} & -\kappa_{5}\\ \kappa_{31} & \kappa_{39} & -\kappa_{5} & -\kappa_{6}\end{bmatrix},\quad \boldsymbol{k}_{21}^{(e)}=\begin{bmatrix}\kappa_{120} & \kappa_{117} & \kappa_{116} & \kappa_{114}\\ \kappa_{117} & \kappa_{109} & \kappa_{115} & \kappa_{95}\\ \kappa_{116} & \kappa_{115} & \kappa_{110} & \kappa_{96}\\ \kappa_{114} & \kappa_{95} & \kappa_{96} & \kappa_{84}\end{bmatrix},$$

$$\boldsymbol{k}_{22}^{(e)}=\begin{bmatrix}\kappa_{185} & \kappa_{32} & -\kappa_{34} & -\kappa_{83}\\ -\kappa_{32} & \kappa_{159} & \kappa_{83} & -\kappa_{162}\\ \kappa_{34} & \kappa_{83} & \kappa_{159} & \kappa_{160}\\ -\kappa_{83} & \kappa_{162} & -\kappa_{160} & \kappa_{1}\end{bmatrix},\quad \boldsymbol{k}_{23}^{(e)}=\begin{bmatrix}\kappa_{157} & \kappa_{154} & \kappa_{153} & \kappa_{151}\\ \kappa_{154} & \kappa_{146} & \kappa_{152} & \kappa_{132}\\ \kappa_{153} & \kappa_{152} & \kappa_{147} & \kappa_{133}\\ \kappa_{151} & \kappa_{132} & \kappa_{133} & \kappa_{121}\end{bmatrix},\quad \boldsymbol{k}_{24}^{(e)}=\begin{bmatrix}\kappa_{57} & -\kappa_{55} & \kappa_{51} & -\kappa_{50}\\ \kappa_{55} & -\kappa_{54} & \kappa_{50} & -\kappa_{49}\\ \kappa_{51} & -\kappa_{50} & \kappa_{58} & -\kappa_{66}\\ \kappa_{50} & -\kappa_{49} & \kappa_{66} & -\kappa_{65}\end{bmatrix},$$

$$\boldsymbol{k}_{25}^{(e)}=\begin{bmatrix}-\kappa_{21} & \kappa_{43} & -\kappa_{173} & \kappa_{24}\\ -\kappa_{43} & \kappa_{25} & -\kappa_{24} & \kappa_{187}\\ -\kappa_{175} & \kappa_{26} & -\kappa_{183} & \kappa_{41}\\ -\kappa_{26} & \kappa_{188} & -\kappa_{41} & \kappa_{195}\end{bmatrix},\quad \boldsymbol{k}_{26}^{(e)}=\begin{bmatrix}-\kappa_{18} & \kappa_{170} & -\kappa_{44} & \kappa_{28}\\ -\kappa_{170} & \kappa_{177} & -\kappa_{28} & \kappa_{38}\\ \kappa_{44} & -\kappa_{28} & -\kappa_{2} & \kappa_{4}\\ \kappa_{28} & -\kappa_{38} & -\kappa_{4} & \kappa_{7}\end{bmatrix},\quad \boldsymbol{k}_{27}^{(e)}=\begin{bmatrix}\kappa_{119} & -\kappa_{108} & \kappa_{112} & -\kappa_{104}\\ \kappa_{108} & -\kappa_{105} & \kappa_{104} & -\kappa_{101}\\ \kappa_{112} & -\kappa_{104} & \kappa_{100} & -\kappa_{89}\\ \kappa_{104} & -\kappa_{101} & \kappa_{89} & -\kappa_{86}\end{bmatrix},$$

$$\boldsymbol{k}_{28}^{(e)}=\begin{bmatrix}\kappa_{184} & -\kappa_{32} & -\kappa_{35} & \kappa_{83}\\ \kappa_{32} & -\kappa_{196} & -\kappa_{83} & \kappa_{37}\\ \kappa_{35} & -\kappa_{83} & \kappa_{158} & -\kappa_{160}\\ \kappa_{83} & -\kappa_{37} & \kappa_{160} & -\kappa_{161}\end{bmatrix},\quad \boldsymbol{k}_{29}^{(e)}=\begin{bmatrix}\kappa_{156} & -\kappa_{145} & \kappa_{149} & -\kappa_{141}\\ \kappa_{145} & -\kappa_{142} & \kappa_{141} & -\kappa_{138}\\ \kappa_{149} & -\kappa_{141} & \kappa_{137} & -\kappa_{126}\\ \kappa_{141} & -\kappa_{138} & \kappa_{126} & -\kappa_{123}\end{bmatrix},\quad \boldsymbol{k}_{30}^{(e)}=\begin{bmatrix}-\kappa_{75} & \kappa_{76} & \kappa_{52} & -\kappa_{71}\\ -\kappa_{76} & \kappa_{46} & \kappa_{71} & -\kappa_{70}\\ -\kappa_{52} & \kappa_{71} & -\kappa_{61} & \kappa_{63}\\ -\kappa_{71} & \kappa_{70} & -\kappa_{63} & \kappa_{64}\end{bmatrix},$$

$$\boldsymbol{k}_{31}^{(e)}=\begin{bmatrix}-\kappa_{15} & \kappa_{42} & \kappa_{167} & -\kappa_{23}\\ -\kappa_{42} & \kappa_{22} & \kappa_{23} & -\kappa_{186}\\ -\kappa_{167} & \kappa_{23} & \kappa_{182} & -\kappa_{40}\\ -\kappa_{23} & \kappa_{186} & \kappa_{40} & -\kappa_{194}\end{bmatrix},\quad \boldsymbol{k}_{32}^{(e)}=\begin{bmatrix}-\kappa_{14} & \kappa_{166} & \kappa_{44} & -\kappa_{28}\\ -\kappa_{166} & \kappa_{178} & \kappa_{28} & -\kappa_{38}\\ -\kappa_{44} & \kappa_{28} & \kappa_{27} & -\kappa_{189}\\ -\kappa_{28} & \kappa_{38} & \kappa_{189} & -\kappa_{192}\end{bmatrix},\quad \boldsymbol{k}_{33}^{(e)}=\begin{bmatrix}\kappa_{113} & -\kappa_{97} & -\kappa_{99} & \kappa_{92}\\ \kappa_{97} & -\kappa_{93} & -\kappa_{92} & \kappa_{90}\\ \kappa_{99} & -\kappa_{92} & -\kappa_{94} & \kappa_{91}\\ \kappa_{92} & -\kappa_{90} & -\kappa_{91} & \kappa_{87}\end{bmatrix},$$

$$\boldsymbol{k}_{34}^{(e)}=\begin{bmatrix}-\kappa_{185} & \kappa_{33} & \kappa_{35} & -\kappa_{83}\\ -\kappa_{33} & \kappa_{197} & \kappa_{83} & -\kappa_{37}\\ -\kappa_{35} & \kappa_{83} & \kappa_{199} & -\kappa_{36}\\ -\kappa_{83} & \kappa_{37} & \kappa_{36} & -\kappa_{200}\end{bmatrix},\quad \boldsymbol{k}_{35}^{(e)}=\begin{bmatrix}\kappa_{150} & -\kappa_{134} & -\kappa_{136} & \kappa_{129}\\ \kappa_{134} & -\kappa_{130} & -\kappa_{129} & \kappa_{127}\\ \kappa_{136} & -\kappa_{129} & -\kappa_{131} & \kappa_{128}\\ \kappa_{129} & -\kappa_{127} & -\kappa_{128} & \kappa_{124}\end{bmatrix},\quad \boldsymbol{k}_{36}^{(e)}=\begin{bmatrix}-\kappa_{60} & -\kappa_{56} & \kappa_{59} & \kappa_{53}\\ -\kappa_{56} & -\kappa_{47} & \kappa_{53} & \kappa_{62}\\ -\kappa_{59} & -\kappa_{53} & -\kappa_{69} & -\kappa_{72}\\ -\kappa_{53} & -\kappa_{62} & -\kappa_{72} & -\kappa_{68}\end{bmatrix},$$

$$\boldsymbol{k}_{37}^{(e)}=\begin{bmatrix}-\kappa_{19} & -\kappa_{42} & \kappa_{171} & \kappa_{23}\\ \kappa_{42} & -\kappa_{8} & -\kappa_{23} & \kappa_{10}\\ -\kappa_{171} & -\kappa_{23} & \kappa_{181} & \kappa_{40}\\ \kappa_{23} & -\kappa_{10} & -\kappa_{40} & \kappa_{13}\end{bmatrix},\quad \boldsymbol{k}_{38}^{(e)}=\begin{bmatrix}-\kappa_{20} & -\kappa_{172} & \kappa_{45} & \kappa_{29}\\ -\kappa_{174} & -\kappa_{179} & \kappa_{31} & \kappa_{39}\\ -\kappa_{45} & -\kappa_{29} & \kappa_{30} & \kappa_{190}\\ -\kappa_{31} & -\kappa_{39} & \kappa_{191} & \kappa_{193}\end{bmatrix},\quad \boldsymbol{k}_{39}^{(e)}=\begin{bmatrix}\kappa_{118} & \kappa_{111} & -\kappa_{107} & -\kappa_{103}\\ \kappa_{111} & \kappa_{98} & -\kappa_{103} & -\kappa_{88}\\ \kappa_{107} & \kappa_{103} & -\kappa_{106} & -\kappa_{102}\\ \kappa_{103} & \kappa_{88} & -\kappa_{102} & -\kappa_{85}\end{bmatrix},$$

$$\boldsymbol{k}_{40}^{(e)}=\begin{bmatrix}-\kappa_{184} & -\kappa_{33} & \kappa_{34} & \kappa_{83}\\ \kappa_{33} & -\kappa_{158} & -\kappa_{83} & \kappa_{162}\\ -\kappa_{34} & -\kappa_{83} & \kappa_{198} & \kappa_{36}\\ \kappa_{83} & -\kappa_{162} & -\kappa_{36} & \kappa_{163}\end{bmatrix},\quad \boldsymbol{k}_{41}^{(e)}=\begin{bmatrix}\kappa_{155} & \kappa_{148} & -\kappa_{144} & -\kappa_{140}\\ \kappa_{148} & \kappa_{135} & -\kappa_{140} & -\kappa_{125}\\ \kappa_{144} & \kappa_{140} & -\kappa_{143} & -\kappa_{139}\\ \kappa_{140} & \kappa_{125} & -\kappa_{139} & -\kappa_{122}\end{bmatrix},\quad \boldsymbol{k}_{42}^{(e)}=\begin{bmatrix}\kappa_{78} & -\kappa_{81} & \kappa_{79} & -\kappa_{73}\\ -\kappa_{81} & \kappa_{77} & -\kappa_{74} & \kappa_{48}\\ \kappa_{79} & -\kappa_{74} & \kappa_{82} & -\kappa_{80}\\ -\kappa_{73} & \kappa_{48} & -\kappa_{80} & \kappa_{67}\end{bmatrix},$$

$$\boldsymbol{k}_{43}^{(e)}=\begin{bmatrix}\kappa_{17} & -\kappa_{43} & \kappa_{169} & -\kappa_{24}\\ \kappa_{43} & \kappa_{9} & \kappa_{24} & \kappa_{11}\\ \kappa_{165} & -\kappa_{26} & \kappa_{180} & -\kappa_{41}\\ \kappa_{26} & \kappa_{11} & \kappa_{41} & \kappa_{12}\end{bmatrix},\quad \boldsymbol{k}_{44}^{(e)}=\begin{bmatrix}-\kappa_{16} & \kappa_{168} & -\kappa_{45} & \kappa_{29}\\ \kappa_{164} & -\kappa_{176} & \kappa_{31} & -\kappa_{39}\\ \kappa_{45} & -\kappa_{29} & -\kappa_{3} & \kappa_{5}\\ -\kappa_{31} & \kappa_{39} & \kappa_{5} & -\kappa_{6}\end{bmatrix},\quad \boldsymbol{k}_{45}^{(e)}=\begin{bmatrix}\kappa_{120} & -\kappa_{117} & \kappa_{116} & -\kappa_{114}\\ -\kappa_{117} & \kappa_{109} & -\kappa_{115} & \kappa_{95}\\ \kappa_{116} & -\kappa_{115} & \kappa_{110} & -\kappa_{96}\\ -\kappa_{114} & \kappa_{95} & -\kappa_{96} & \kappa_{84}\end{bmatrix},$$

$$\boldsymbol{k}_{46}^{(e)}=\begin{bmatrix}-\kappa_{185} & \kappa_{32} & \kappa_{34} & -\kappa_{83}\\ -\kappa_{32} & -\kappa_{159} & \kappa_{83} & \kappa_{162}\\ -\kappa_{34} & \kappa_{83} & -\kappa_{159} & \kappa_{160}\\ -\kappa_{83} & -\kappa_{162} & -\kappa_{160} & \kappa_{1}\end{bmatrix},\quad \boldsymbol{k}_{47}^{(e)}=\begin{bmatrix}\kappa_{157} & -\kappa_{154} & \kappa_{153} & -\kappa_{151}\\ -\kappa_{154} & \kappa_{146} & -\kappa_{152} & \kappa_{132}\\ \kappa_{153} & -\kappa_{152} & \kappa_{147} & -\kappa_{133}\\ -\kappa_{151} & \kappa_{132} & -\kappa_{133} & \kappa_{121}\end{bmatrix},\quad \boldsymbol{k}_{48}^{(e)}=\begin{bmatrix}-\kappa_{60} & \kappa_{56} & \kappa_{59} & -\kappa_{53}\\ \kappa_{56} & -\kappa_{47} & -\kappa_{53} & \kappa_{62}\\ -\kappa_{59} & \kappa_{53} & -\kappa_{69} & \kappa_{72}\\ \kappa_{53} & -\kappa_{62} & \kappa_{72} & -\kappa_{68}\end{bmatrix},$$

$$\boldsymbol{k}_{49}^{(e)}=\begin{bmatrix}\kappa_{19} & -\kappa_{42} & -\kappa_{171} & \kappa_{23}\\ \kappa_{42} & \kappa_{8} & -\kappa_{23} & -\kappa_{10}\\ \kappa_{171} & -\kappa_{23} & -\kappa_{181} & \kappa_{40}\\ \kappa_{23} & \kappa_{10} & -\kappa_{40} & -\kappa_{13}\end{bmatrix},\quad \boldsymbol{k}_{50}^{(e)}=\begin{bmatrix}-\kappa_{20} & \kappa_{172} & \kappa_{45} & -\kappa_{29}\\ \kappa_{174} & -\kappa_{179} & -\kappa_{31} & \kappa_{39}\\ -\kappa_{45} & \kappa_{29} & \kappa_{30} & -\kappa_{190}\\ \kappa_{31} & -\kappa_{39} & -\kappa_{191} & \kappa_{193}\end{bmatrix},\quad \boldsymbol{k}_{51}^{(e)}=\begin{bmatrix}\kappa_{118} & -\kappa_{111} & -\kappa_{107} & \kappa_{103}\\ -\kappa_{111} & \kappa_{98} & \kappa_{103} & -\kappa_{88}\\ \kappa_{107} & -\kappa_{103} & -\kappa_{106} & \kappa_{102}\\ -\kappa_{103} & \kappa_{88} & \kappa_{102} & -\kappa_{85}\end{bmatrix},$$

$$\boldsymbol{k}_{52}^{(e)}=\begin{bmatrix}\kappa_{184} & -\kappa_{33} & -\kappa_{34} & \kappa_{83}\\ \kappa_{33} & \kappa_{158} & -\kappa_{83} & -\kappa_{162}\\ \kappa_{34} & -\kappa_{83} & -\kappa_{198} & \kappa_{36}\\ \kappa_{83} & \kappa_{162} & -\kappa_{36} & -\kappa_{163}\end{bmatrix},\quad \boldsymbol{k}_{53}^{(e)}=\begin{bmatrix}\kappa_{155} & -\kappa_{148} & -\kappa_{144} & \kappa_{140}\\ -\kappa_{148} & \kappa_{135} & \kappa_{140} & -\kappa_{125}\\ \kappa_{144} & -\kappa_{140} & -\kappa_{143} & \kappa_{139}\\ -\kappa_{140} & \kappa_{125} & \kappa_{139} & -\kappa_{122}\end{bmatrix},\quad \boldsymbol{k}_{54}^{(e)}=\begin{bmatrix}-\kappa_{75} & -\kappa_{76} & \kappa_{52} & \kappa_{71}\\ \kappa_{76} & \kappa_{46} & -\kappa_{71} & -\kappa_{70}\\ -\kappa_{52} & -\kappa_{71} & -\kappa_{61} & -\kappa_{63}\\ \kappa_{71} & \kappa_{70} & \kappa_{63} & \kappa_{64}\end{bmatrix},$$

$$\boldsymbol{k}_{55}^{(e)}=\begin{bmatrix}\kappa_{15} & \kappa_{42} & -\kappa_{167} & -\kappa_{23}\\ -\kappa_{42} & -\kappa_{22} & \kappa_{23} & \kappa_{186}\\ \kappa_{167} & \kappa_{23} & -\kappa_{182} & -\kappa_{40}\\ -\kappa_{23} & -\kappa_{186} & \kappa_{40} & \kappa_{194}\end{bmatrix},\quad \boldsymbol{k}_{56}^{(e)}=\begin{bmatrix}-\kappa_{14} & -\kappa_{166} & \kappa_{44} & \kappa_{28}\\ \kappa_{166} & \kappa_{178} & -\kappa_{28} & -\kappa_{38}\\ -\kappa_{44} & -\kappa_{28} & \kappa_{27} & \kappa_{189}\\ \kappa_{28} & \kappa_{38} & -\kappa_{189} & -\kappa_{192}\end{bmatrix},\quad \boldsymbol{k}_{57}^{(e)}=\begin{bmatrix}\kappa_{113} & \kappa_{97} & -\kappa_{99} & -\kappa_{92}\\ -\kappa_{97} & -\kappa_{93} & \kappa_{92} & \kappa_{90}\\ \kappa_{99} & \kappa_{92} & -\kappa_{94} & -\kappa_{91}\\ -\kappa_{92} & -\kappa_{90} & \kappa_{91} & \kappa_{87}\end{bmatrix},$$

$$\boldsymbol{k}_{58}^{(e)}=\begin{bmatrix}\kappa_{185} & \kappa_{33} & -\kappa_{35} & -\kappa_{83}\\ -\kappa_{33} & -\kappa_{197} & \kappa_{83} & \kappa_{37}\\ \kappa_{35} & \kappa_{83} & -\kappa_{199} & -\kappa_{36}\\ -\kappa_{83} & -\kappa_{37} & \kappa_{36} & \kappa_{200}\end{bmatrix},\ \boldsymbol{k}_{59}^{(e)}=\begin{bmatrix}\kappa_{150} & \kappa_{134} & -\kappa_{136} & -\kappa_{129}\\ -\kappa_{134} & -\kappa_{130} & \kappa_{129} & \kappa_{127}\\ \kappa_{136} & \kappa_{129} & -\kappa_{131} & -\kappa_{128}\\ -\kappa_{129} & -\kappa_{127} & \kappa_{128} & \kappa_{124}\end{bmatrix},\quad \boldsymbol{k}_{60}^{(e)}=\begin{bmatrix}\kappa_{78} & -\kappa_{81} & -\kappa_{79} & \kappa_{73}\\ -\kappa_{81} & \kappa_{77} & \kappa_{74} & -\kappa_{48}\\ -\kappa_{79} & \kappa_{74} & \kappa_{82} & -\kappa_{80}\\ \kappa_{73} & -\kappa_{48} & -\kappa_{80} & \kappa_{67}\end{bmatrix},\qquad \text{(C. 2)}$$

where

$$\kappa_1=0,\quad \hat{\kappa}_8=\kappa_2=\frac{19\Sigma_8 a}{150},\quad \hat{\kappa}_9=\kappa_3=\frac{28\Sigma_8 a}{75},\quad \hat{\kappa}_{10}=\kappa_4=\frac{3\Sigma_8 a^2}{50},\quad \hat{\kappa}_{11}=\kappa_5=\frac{8\Sigma_8 a^2}{75},\quad \hat{\kappa}_{12}=\kappa_6=\frac{\Sigma_8 a^3}{25},$$

$$\hat{\kappa}_{13}=\kappa_7=\frac{2\Sigma_8 a^3}{75},\quad \hat{\kappa}_{15}=\kappa_{14}=\frac{19\Sigma_9 a}{75}-\frac{3(\Sigma_5+\Sigma_6)}{10a},\quad \hat{\kappa}_{17}=\kappa_{16}=\frac{56\Sigma_9 a}{75}-\frac{3(\Sigma_5-\Sigma_6)}{10a},\quad \hat{\kappa}_{19}=\kappa_{18}=\frac{3(\Sigma_5-\Sigma_6)}{10a}+\frac{19\Sigma_9 a}{75},$$

$$\hat{\kappa}_{21}=\kappa_{20}=\frac{3(\Sigma_5+\Sigma_6)}{10a}+\frac{56\Sigma_9 a}{75},\quad \hat{\kappa}_{27}=\kappa_{22}=\frac{19b\left(15\Sigma_8+4\Sigma_9 a^2\right)}{2250}-\frac{a^2(\Sigma_5+\Sigma_6)}{25b},\quad \hat{\kappa}_{28}=\kappa_{23}=\frac{3b^2\left(5\Sigma_8+4\Sigma_9 a^2\right)}{250a}-\frac{a(\Sigma_5+\Sigma_6)}{50},$$

$$\hat{\kappa}_{29}=\kappa_{24}=\frac{a(\Sigma_5+\Sigma_6)}{50}+\frac{8b^2\left(5\Sigma_8+4\Sigma_9 a^2\right)}{375a},\quad \hat{\kappa}_{30}=\kappa_{25}=\frac{a^2(\Sigma_5+\Sigma_6)}{25b}+\frac{28b\left(15\Sigma_8+4\Sigma_9 a^2\right)}{1125},$$

$$\hat{\kappa}_{31}=\kappa_{26}=\frac{a(11\Sigma_5+\Sigma_6)}{50}+\frac{8b^2\left(5\Sigma_8+4\Sigma_9 a^2\right)}{375a},\quad \hat{\kappa}_{34}=\kappa_{32}=\frac{a}{10}(\Sigma_3-\Sigma_6+\Sigma_{17}-2\Sigma_{18})+\frac{(\Sigma_{11}-\Sigma_{12})}{4a},$$

$$\hat{\kappa}_{35}=\kappa_{33}=\frac{a}{10}(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18})+\frac{\Sigma_{11}+\Sigma_{12}}{4a},\quad \hat{\kappa}_{37}=\kappa_{36}=\frac{ab^2(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18})}{75}+\frac{(\Sigma_{11}+\Sigma_{12})}{10}\left(a+\frac{b^2}{3a}\right),$$

$$\hat{\kappa}_{40}=\kappa_{38}=\frac{ab(\Sigma_5+\Sigma_6)}{75}+\frac{2a^3\left(5\Sigma_8+4\Sigma_9 b^2\right)}{375b},\quad \hat{\kappa}_{41}=\kappa_{39}=\frac{4ab(\Sigma_5+\Sigma_6)}{75}+\frac{a^3\left(5\Sigma_8+4\Sigma_9 b^2\right)}{125b},$$

$$\hat{\kappa}_{44}=\kappa_{42}=-\frac{3a(\Sigma_5+\Sigma_6)}{25b}+\frac{19b\left(5\Sigma_8+4\Sigma_9 a^2\right)}{750a},\quad \hat{\kappa}_{45}=\kappa_{43}=\frac{3a(\Sigma_5+\Sigma_6)}{25b}+\frac{28b\left(5\Sigma_8+4\Sigma_9 a^2\right)}{375a},$$

$$\kappa_{46}=\frac{4a^3\left(5\Sigma_1+2\Sigma_9 b^2\right)}{125b^3}+\frac{2a(\Sigma_2+\Sigma_7)}{25b}+\frac{38b\left(15\Sigma_1-\Sigma_9 a^2\right)}{1125a},\quad \kappa_{47}=\frac{6a^3\left(5\Sigma_1+2\Sigma_9 b^2\right)}{125b^3}+\frac{8a(\Sigma_2+\Sigma_7)}{25b}-\frac{76b\left(15\Sigma_1+2\Sigma_9 a^2\right)}{1125a},$$

$$\hat{\kappa}_{80}=\kappa_{48}=\frac{2a^3\left(15\Sigma_1+\Sigma_9 b^2\right)}{125b^2}+\frac{64b^2\left(15\Sigma_1+2\Sigma_9 a^2\right)}{1125a}+\frac{4a(6\Sigma_2+\Sigma_7)}{75},\quad \hat{\kappa}_{72}=\kappa_{49}=\frac{4a^3\left(15\Sigma_1+\Sigma_9 b^2\right)}{375b^2}-\frac{32b^2\left(15\Sigma_1-\Sigma_9 a^2\right)}{1125a}+\frac{a(6\Sigma_2+\Sigma_7)}{75},$$

$$\kappa_{50}=\frac{3a^2\left(15\Sigma_1+\Sigma_9 b^2\right)}{125b^2}-\frac{16b^2\left(15\Sigma_1+\Sigma_9 a^2\right)}{375a^2}-\frac{6\Sigma_2+\Sigma_7}{50},\quad \kappa_{51}=\frac{19a\left(15\Sigma_1+\Sigma_9 b^2\right)}{375b^2}-\frac{16b^2\left(5\Sigma_1+2\Sigma_9 a^2\right)}{125a^3}-\frac{3(6\Sigma_2+\Sigma_7)}{25a},$$

$$\hat{\kappa}_{76}=\kappa_{52}=\frac{19a\left(15\Sigma_1+\Sigma_9 b^2\right)}{375b^2}+\frac{9b^2\left(5\Sigma_1+2\Sigma_9 a^2\right)}{125a^3}-\frac{3(\Sigma_2+\Sigma_7)}{25a},\quad \kappa_{53}=\frac{16a^2\left(15\Sigma_1+\Sigma_9 b^2\right)}{375b^2}-\frac{3b^2\left(15\Sigma_1+\Sigma_9 a^2\right)}{125a^2}+\frac{6\Sigma_2+\Sigma_7}{50},$$

$$\hat{\kappa}_{69}=\kappa_{54}=\frac{2a(\Sigma_2+\Sigma_7)}{25b}-\frac{112b\left(15\Sigma_1-\Sigma_9 a^2\right)}{1125a}+\frac{4a^3\left(5\Sigma_1+2\Sigma_9 b^2\right)}{125b^3},\quad \kappa_{55}=\frac{9a^2\left(5\Sigma_1+2\Sigma_9 b^2\right)}{125b^3}-\frac{56b\left(15\Sigma_1+\Sigma_9 a^2\right)}{375a^2}-\frac{3(\Sigma_2+\Sigma_7)}{25b},$$

$$\kappa_{56}=-\frac{19b\left(15\Sigma_1+\Sigma_9 a^2\right)}{375a^2}+\frac{16a^2\left(5\Sigma_1+2\Sigma_9 b^2\right)}{125b^3}+\frac{3(6\Sigma_2+\Sigma_7)}{25b},\quad \kappa_{57}=-\frac{56b\left(5\Sigma_1+2\Sigma_9 a^2\right)}{125a^3}+\frac{19a\left(5\Sigma_1+2\Sigma_9 b^2\right)}{125b^3}-\frac{18(\Sigma_2+\Sigma_7)}{25ab},$$

$$\kappa_{58}=\frac{76a\left(15\Sigma_1+2\Sigma_9 b^2\right)}{1125b}-\frac{6b^3\left(5\Sigma_1+2\Sigma_9 a^2\right)}{125a^3}-\frac{8b(\Sigma_2+\Sigma_7)}{25a},\quad \kappa_{59}=\frac{56a\left(15\Sigma_1+\Sigma_9 b^2\right)}{375b^2}-\frac{9b^2\left(5\Sigma_1+2\Sigma_9 a^2\right)}{125a^3}+\frac{3(\Sigma_2+\Sigma_7)}{25a},$$

$$\kappa_{60}=\frac{56a\left(5\Sigma_1+2\Sigma_9 b^2\right)}{125b^3}-\frac{19b\left(5\Sigma_1+2\Sigma_9 a^2\right)}{125a^3}+\frac{18(\Sigma_2+\Sigma_7)}{25ab},\ \kappa_{61}=-\frac{4b^3\left(5\Sigma_1+2\Sigma_9 a^2\right)}{125a^3}-\frac{38a\left(15\Sigma_1-\Sigma_9 b^2\right)}{1125b}-\frac{2b(\Sigma_2+\Sigma_7)}{25a},$$

$$\kappa_{62}=\frac{2a^3\left(15\Sigma_1+\Sigma_9 b^2\right)}{125b^2}+\frac{4a(\Sigma_2+\Sigma_7)}{75}-\frac{4b^2\left(15\Sigma_1+2\Sigma_9 a^2\right)}{125a},\quad \kappa_{63}=-\frac{4b^3\left(15\Sigma_1+\Sigma_9 a^2\right)}{375a^2}-\frac{2a^2\left(15\Sigma_1-\Sigma_9 b^2\right)}{125b}-\frac{b(\Sigma_2+\Sigma_7)}{75},$$

$$\kappa_{64}=\frac{2ab(\Sigma_2+\Sigma_7)}{225}-\frac{8}{125}\left[\frac{b^3}{a}(15\Sigma_1-\Sigma_9a^2)+\frac{a^3}{b}(15\Sigma_1-\Sigma_9b^2)\right],\quad \hat{\kappa}_{68}=\kappa_{65}=\frac{16a^3(15\Sigma_1+2\Sigma_9b^2)}{1125b}+\frac{8ab(\Sigma_2+\Sigma_7)}{225}-\frac{4b^3(15\Sigma_1-\Sigma_9a^2)}{375a},$$

$$\kappa_{66}=\frac{4a^2(15\Sigma_1+2\Sigma_9b^2)}{125b}-\frac{2b^3(15\Sigma_1+\Sigma_9a^2)}{125a^2}-\frac{4b(\Sigma_2+\Sigma_7)}{75},\quad \kappa_{67}=\frac{8}{375}\left[\frac{b^3}{a}(15\Sigma_1+2\Sigma_9a^2)+\frac{a^3}{b}(15\Sigma_1+2\Sigma_9b^2)\right]+\frac{32ab(\Sigma_2+\Sigma_7)}{225},$$

$$\kappa_{70}=\frac{2b^2(15\Sigma_1-\Sigma_9a^2)}{125a}+\frac{4a^3(15\Sigma_1+\Sigma_9b^2)}{375b^2}+\frac{a(\Sigma_2+\Sigma_7)}{75},\quad \kappa_{71}=\frac{3}{125}\left[\frac{a^2}{b^2}(15\Sigma_1+\Sigma_9b^2)+\frac{b^2}{a^2}(15\Sigma_1+\Sigma_9a^2)\right]-\frac{\Sigma_2+\Sigma_7}{50},$$

$$\kappa_{73}=\frac{16}{375}\left[\frac{a^2}{b^2}(15\Sigma_1+\Sigma_9b^2)+\frac{b^2}{a^2}(15\Sigma_1+\Sigma_9a^2)\right]+\frac{11\Sigma_2+\Sigma_7}{50},\quad \kappa_{74}=\frac{16}{375}\left[\frac{a^2}{b^2}(15\Sigma_1+\Sigma_9b^2)+\frac{b^2}{a^2}(15\Sigma_1+\Sigma_9a^2)\right]+\frac{61\Sigma_2+\Sigma_7}{50},$$

$$\kappa_{75}=\frac{19}{125}\left[\frac{a}{b^3}(5\Sigma_1+2\Sigma_9b^2)+\frac{b}{a^3}(5\Sigma_1+2\Sigma_9a^2)\right]-\frac{18(\Sigma_2+\Sigma_7)}{25ab},\quad \hat{\kappa}_{82}=\kappa_{77}=\frac{6a^3(5\Sigma_1+2\Sigma_9b^2)}{125b^3}+\frac{224b(15\Sigma_1+2\Sigma_9a^2)}{1125a}+\frac{8a(\Sigma_2+\Sigma_7)}{25b},$$

$$\kappa_{78}=\frac{56}{125}\left[\frac{a}{b^3}(5\Sigma_1+2\Sigma_9b^2)+\frac{b}{a^3}(5\Sigma_1+2\Sigma_9a^2)\right]+\frac{18(\Sigma_2+\Sigma_7)}{25ab},\quad \hat{\kappa}_{81}=\kappa_{79}=\frac{56a(15\Sigma_1+\Sigma_9b^2)}{375b^2}+\frac{16b^2(5\Sigma_1+2\Sigma_9a^2)}{125a^3}+\frac{3(6\Sigma_2+\Sigma_7)}{25a},$$

$$\kappa_{83}=\frac{ab(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18})}{25}+\frac{(\Sigma_{11}+\Sigma_{12})}{10}\left(\frac{a}{b}+\frac{b}{a}\right),\quad \begin{aligned}\hat{\kappa}_{121}=\kappa_{84}&=\frac{8\Sigma_9a^3b^3}{625}+\frac{16ab}{375}\left[b^2(\Sigma_4+\Sigma_{16})+a^2(\Sigma_7+\Sigma_{18})\right]\\&+\frac{32ab}{225}(\Sigma_{10}+\Sigma_{15})+\frac{8}{25}\left(\frac{\Sigma_{14}a^3}{b}+\frac{\Sigma_{13}b^3}{a}\right)\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{123}=\kappa_{85}&=\frac{16\Sigma_9a^3b^3}{1875}+\frac{8ab}{225}(\Sigma_{10}+\Sigma_{15})-\frac{4}{75}\left(\frac{3\Sigma_{14}a^3}{b}-\frac{4\Sigma_{13}b^3}{a}\right)\\&+\frac{4ab}{1125}\left[3a^2(\Sigma_7+\Sigma_{18})+8b^2(\Sigma_4+\Sigma_{16})\right]\end{aligned},\quad \begin{aligned}\hat{\kappa}_{122}=\kappa_{86}&=\frac{16\Sigma_9a^3b^3}{1875}+\frac{4ab}{1125}\left[8a^2(\Sigma_7+\Sigma_{18})+3b^2(\Sigma_4+\Sigma_{16})\right]\\&+\frac{8ab}{225}(\Sigma_{10}+\Sigma_{15})+\frac{4}{75}\left(\frac{4\Sigma_{14}a^3}{b}-\frac{3\Sigma_{13}b^3}{a}\right)\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{124}=\kappa_{87}&=\frac{32\Sigma_9a^3b^3}{5625}+\frac{2ab(\Sigma_{10}+\Sigma_{15})}{225}-\frac{8}{75}\left(\frac{\Sigma_{14}a^3}{b}+\frac{\Sigma_{13}b^3}{a}\right)\\&\frac{8ab}{1125}\left[b^2(\Sigma_4+\Sigma_{16})+a^2(\Sigma_7+\Sigma_{18})\right]\end{aligned},\quad \begin{aligned}\hat{\kappa}_{126}=\kappa_{88}&=\frac{2a}{125}\left[4b^2(\Sigma_4+\Sigma_{16})-a^2(\Sigma_7+\Sigma_{18})\right]+\frac{12\Sigma_9a^3b^2}{625}\\&+\frac{6}{25}\left(\frac{2\Sigma_{13}b^2}{a}-\frac{\Sigma_{14}a^3}{b^2}\right)-\frac{4a(\Sigma_{10}+\Sigma_{15})}{75}\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{125}=\kappa_{89}&=\frac{2b}{125}\left[4a^2(\Sigma_7+\Sigma_{18})-b^2(\Sigma_4+\Sigma_{16})\right]+\frac{12\Sigma_9a^2b^3}{625}\\&+\frac{6}{25}\left(\frac{2\Sigma_{14}a^2}{b}-\frac{\Sigma_{13}b^3}{a^2}\right)-\frac{4b(\Sigma_{10}+\Sigma_{15})}{75}\end{aligned},\quad \begin{aligned}\hat{\kappa}_{128}=\kappa_{90}&=\frac{2a}{375}\left[3b^2(\Sigma_4+\Sigma_{16})-2a^2(\Sigma_7+\Sigma_{18})\right]+\frac{8\Sigma_9a^3b^2}{625}\\&-\frac{2}{25}\left(\frac{2\Sigma_{14}a^3}{b^2}+\frac{3\Sigma_{13}b^2}{a}\right)-\frac{a(\Sigma_{10}+\Sigma_{15})}{75}\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{127}=\kappa_{91}&=\frac{2b}{375}\left[3a^2(\Sigma_7+\Sigma_{18})-2b^2(\Sigma_4+\Sigma_{16})\right]+\frac{8\Sigma_9a^2b^3}{625}\\&-\frac{2}{25}\left(\frac{2\Sigma_{13}b^3}{a^2}+\frac{3\Sigma_{14}a^2}{b}\right)-\frac{b(\Sigma_{10}+\Sigma_{15})}{75}\end{aligned},\quad \begin{aligned}\hat{\kappa}_{129}=\kappa_{92}&=\frac{(\Sigma_{10}+\Sigma_{15})}{50}-\frac{3\left[b^2(\Sigma_4+\Sigma_{16})+a^2(\Sigma_7+\Sigma_{18})\right]}{125}\\&+\frac{18\Sigma_9a^2b^2}{625}-\frac{9}{25}\left(\frac{\Sigma_{14}a^2}{b^2}+\frac{\Sigma_{13}b^2}{a^2}\right)\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{131}=\kappa_{93}&=\frac{152a^3b\Sigma_9}{5625}-\frac{8a^3(\Sigma_7+\Sigma_{18})}{125b}+\frac{38ab(\Sigma_4+\Sigma_{16})}{1125}\\&-\frac{2}{75}\left(\frac{19b\Sigma_{13}}{a}+\frac{6\Sigma_{14}a^3}{b^3}\right)-\frac{2a(\Sigma_{10}+\Sigma_{15})}{25b}\end{aligned},\quad \begin{aligned}\hat{\kappa}_{130}=\kappa_{94}&=\frac{152ab^3\Sigma_9}{5625}+\frac{38ab(\Sigma_7+\Sigma_{18})}{1125}-\frac{8b^3(\Sigma_4+\Sigma_{16})}{125a}\\&-\frac{2}{75}\left(\frac{6\Sigma_{13}b^3}{a^3}+\frac{19a\Sigma_{14}}{b}\right)-\frac{2b(\Sigma_{10}+\Sigma_{15})}{25a}\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{133}=\kappa_{95}&=\frac{64\Sigma_9a^3b^2}{1875}\;\frac{128ab^2(\Sigma_4+\Sigma_{16})}{1125}+\frac{2a^3(\Sigma_7+\Sigma_{18})}{125}\\&+\frac{2}{75}\left(\frac{32\Sigma_{13}b^2}{a}+\frac{9\Sigma_{14}a^3}{b^2}\right)++\frac{4a(6\Sigma_{15}+\Sigma_{10})}{75}\end{aligned},\quad \begin{aligned}\hat{\kappa}_{132}=\kappa_{96}&=\frac{64\Sigma_9a^2b^3}{1875}+\frac{2b^3(\Sigma_4+\Sigma_{16})}{125}+\frac{4b(6\Sigma_{15}+\Sigma_{10})}{75}\\&+\frac{2}{75}\left(\frac{32\Sigma_{14}a^2}{b}+\frac{9\Sigma_{13}b^3}{a^2}\right)+\frac{128a^2b(\Sigma_7+\Sigma_{18})}{1125}\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{136}=\kappa_{97}&=\frac{3(\Sigma_{10}+\Sigma_{15})}{25b}-\frac{1}{25}\left(\frac{19b\Sigma_{13}}{a^2}+\frac{9\Sigma_{14}a^2}{b^3}\right)+\frac{38\Sigma_9a^2b}{625}\\&-\frac{19b(\Sigma_4+\Sigma_{16})}{375}-\frac{18a^2(\Sigma_7+\Sigma_{18})}{125b}\end{aligned},\quad \begin{aligned}\hat{\kappa}_{137}=\kappa_{98}&=\frac{152ab(\Sigma_4+\Sigma_{16})}{1125}+\frac{76b(a^4\Sigma_9+25\Sigma_{13})}{1875a}\\&-\frac{12a^3(\Sigma_7+\Sigma_{18})}{125b}-\frac{6\Sigma_{14}a^3}{25b^3}-\frac{8a(\Sigma_{10}+\Sigma_{15})}{25b}\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{134}=\kappa_{99}&=\frac{19a(2b^4\Sigma_9-25\Sigma_{14})}{625b^2}-\frac{18b^2(\Sigma_4+\Sigma_{16})}{125a}\\&-\frac{9\Sigma_{13}b^2}{25a^3}+\frac{3(\Sigma_{10}+\Sigma_{15})}{25a}-\frac{19a(\Sigma_7+\Sigma_{18})}{375}\end{aligned},\quad \begin{aligned}\hat{\kappa}_{135}=\kappa_{100}&=-\frac{12b^3(\Sigma_4+\Sigma_{16})}{125a}+\frac{76a(b^4\Sigma_9+25\Sigma_{14})}{1875b}\\&-\frac{6\Sigma_{13}b^3}{25a^3}-\frac{8b(\Sigma_{10}+\Sigma_{15})}{25a}+\frac{152ab(\Sigma_7+\Sigma_{18})}{1125}\end{aligned},$$

$$\begin{aligned}\hat{\kappa}_{139}=\kappa_{101}&=\frac{4a}{1125}\left[8b^2(\Sigma_4+\Sigma_{16})+3a^2(\Sigma_7+\Sigma_{18})\right]+\frac{4\Sigma_{14}a^3}{25b^2}\\&+\frac{32b^2(4a^4\Sigma_9-75\Sigma_{13})}{5625a}+\frac{a(6\Sigma_{15}+\Sigma_{10})}{75}\end{aligned},\quad \begin{aligned}\hat{\kappa}_{138}=\kappa_{102}&=\frac{4b}{1125}\left[3b^2(\Sigma_4+\Sigma_{16})+8a^2(\Sigma_7+\Sigma_{18})\right]+\frac{4\Sigma_{13}b^3}{25a^2}\\&+\frac{32a^2(4b^4\Sigma_9-75\Sigma_{14})}{5625b}+\frac{b(6\Sigma_{15}+\Sigma_{10})}{75}\end{aligned},$$

$$\hat{\kappa}_{141}=\kappa_{103}=\frac{1}{375}\left[9b^2\left(\Sigma_4+\Sigma_{16}\right)-16a^2\left(\Sigma_7+\Sigma_{18}\right)\right]+\frac{16a^2\left(2b^4\Sigma_9-25\Sigma_{14}\right)}{625b^2}+\frac{9\Sigma_{13}b^2}{25a^2}-\frac{\Sigma_{10}+6\Sigma_{15}}{50},\quad \hat{\kappa}_{140}=\kappa_{104}=\frac{1}{375}\left[9a^2\left(\Sigma_7+\Sigma_{18}\right)-16b^2\left(\Sigma_4+\Sigma_{16}\right)\right]+\frac{16b^2\left(2a^4\Sigma_9-25\Sigma_{13}\right)}{625a^2}+\frac{9\Sigma_{14}a^2}{25b^2}-\frac{\Sigma_{10}+6\Sigma_{15}}{50},$$

$$\hat{\kappa}_{143}=\kappa_{105}=\frac{8a^2}{1125}\left[\frac{9a}{b}\left(\Sigma_7+\Sigma_{18}\right)+\frac{14b}{a}\left(\Sigma_4+\Sigma_{16}\right)\right]+\frac{4\Sigma_{14}a^3}{25b^3}+\frac{112b\left(4a^4\Sigma_9-75\Sigma_{13}\right)}{5625a}+\frac{2a\left(\Sigma_{10}+\Sigma_{15}\right)}{25b},\quad \hat{\kappa}_{142}=k_{106}=\frac{8b^2}{1125}\left[\frac{14a}{b}\left(\Sigma_7+\Sigma_{18}\right)+\frac{9b}{a}\left(\Sigma_4+\Sigma_{16}\right)\right]+\frac{4\Sigma_{13}b^3}{25a^3}+\frac{112a\left(4b^4\Sigma_9-75\Sigma_{14}\right)}{5625b}+\frac{2b\left(\Sigma_{10}+\Sigma_{15}\right)}{25a},$$

$$\hat{\kappa}_{145}=\kappa_{107}=\frac{18b^2\left(\Sigma_4+\Sigma_{16}\right)}{125a}+\frac{56a}{625b^2}\left(2b^4\Sigma_9-25\Sigma_{14}\right)+\frac{9\Sigma_{13}b^2}{25a^3}-\frac{3\left(\Sigma_{10}+\Sigma_{15}\right)}{25a}-\frac{56a}{375}\left(\Sigma_7+\Sigma_{18}\right),\quad \hat{\kappa}_{144}=\kappa_{108}=\frac{18a^2\left(\Sigma_7+\Sigma_{18}\right)}{125b}+\frac{56b\left(2a^4\Sigma_9-25\Sigma_{13}\right)}{625a^2}+\frac{9\Sigma_{14}a^2}{25b^3}-\frac{3\left(\Sigma_{10}+\Sigma_{15}\right)}{25b}-\frac{56b\left(\Sigma_4+\Sigma_{16}\right)}{375},$$

$$\hat{\kappa}_{147}=\kappa_{109}=\frac{448ab}{1125}\left(\Sigma_4+\Sigma_{16}\right)+\frac{224b\left(a^4\Sigma_9+25\Sigma_{13}\right)}{1875a}+\frac{12a^3\left(\Sigma_7+\Sigma_{18}\right)}{125b}+\frac{8a\left(\Sigma_{10}+\Sigma_{15}\right)}{25b}+\frac{6\Sigma_{14}a^3}{24b^3},\quad \hat{\kappa}_{146}=\kappa_{110}=\frac{448ab}{1125}\left(\Sigma_7+\Sigma_{18}\right)+\frac{12b^3\left(\Sigma_4+\Sigma_{16}\right)}{125a}+\frac{6\Sigma_{13}b^3}{25a^3}+\frac{224a\left(b^4\Sigma_9+25\Sigma_{14}\right)}{1875b}+\frac{8b\left(\Sigma_{10}+\Sigma_{15}\right)}{25a},$$

$$\hat{\kappa}_{149}=\kappa_{111}=\frac{19b}{375}\left(\Sigma_4+\Sigma_{16}\right)-\frac{32a^2}{125b}\left(\Sigma_7+\Sigma_{18}\right)-\frac{16\Sigma_{14}a^2}{25b^3}+\frac{19b\left(32a^4\Sigma_9+225\Sigma_{13}\right)}{5625a^2}-\frac{3\left(6\Sigma_{15}+\Sigma_{10}\right)}{25b},\quad \hat{\kappa}_{148}=\kappa_{112}=\frac{19a\left(32b^4\Sigma_9+225\Sigma_{14}\right)}{5625b^2}-\frac{32b^2}{125a}\left(\Sigma_4+\Sigma_{16}\right)-\frac{16\Sigma_{13}b^2}{25a^3}+\frac{19a}{375}\left(\Sigma_7+\Sigma_{18}\right)-\frac{3\left(6\Sigma_{15}+\Sigma_{10}\right)}{25a},$$

$$\hat{\kappa}_{150}=\kappa_{113}=-\frac{38b}{125a}\left(\Sigma_4+\Sigma_{16}\right)-\frac{38a}{125b}\left(\Sigma_7+\Sigma_{18}\right)+\frac{722\Sigma_9ab}{5625}-\frac{19}{25}\left(\frac{\Sigma_{14}a}{b^3}+\frac{\Sigma_{13}b}{a^3}\right)+\frac{18\left(\Sigma_{10}+\Sigma_{15}\right)}{25ab},\quad \hat{\kappa}_{151}=\kappa_{114}=\frac{512\Sigma_9a^2b^2}{5625}+\frac{16}{375}\left[b^2\left(\Sigma_4+\Sigma_{16}\right)+a^2\left(\Sigma_7+\Sigma_{18}\right)\right]+\frac{16}{25}\left(\frac{\Sigma_{14}a^2}{b^2}+\frac{b^2\Sigma_{13}}{a^2}\right)+\frac{\Sigma_{10}+11\Sigma_{15}}{50},$$

$$\hat{\kappa}_{152}=\kappa_{115}=\frac{16}{375}\left[b^2\left(\Sigma_4+\Sigma_{16}\right)+a^2\left(\Sigma_7+\Sigma_{18}\right)\right]+\frac{512\Sigma_9a^2b^2}{5625}+\frac{16}{25}\left(\frac{\Sigma_{14}a^2}{b^2}+\frac{b^2\Sigma_{13}}{a^2}\right)+\frac{\Sigma_{10}+61\Sigma_{15}}{50},\quad \hat{\kappa}_{154}=\kappa_{116}=\frac{32b^2}{125a}\left(\Sigma_4+\Sigma_{16}\right)+\frac{56a}{375}\left(\Sigma_7+\Sigma_{18}\right)+\frac{16b^2\Sigma_{13}}{25a^3}+\frac{56a\left(32\Sigma_9b^4+225\Sigma_{14}\right)}{5625b^2}+\frac{3\left(6\Sigma_{15}+\Sigma_{10}\right)}{25a},$$

$$\hat{\kappa}_{153}=\kappa_{117}=\frac{32a^2}{125b}\left(\Sigma_7+\Sigma_{18}\right)+\frac{56b}{375}\left(\Sigma_4+\Sigma_{16}\right)+\frac{16\Sigma_{14}a^2}{25b^3}+\frac{56b\left(32a^4\Sigma_9+225\Sigma_{13}\right)}{5625a^2}+\frac{3\left(6\Sigma_{15}+\Sigma_{10}\right)}{25b},\quad \hat{\kappa}_{156}=\kappa_{118}=\frac{38b}{125a}\left(\Sigma_4+\Sigma_{16}\right)-\frac{112a}{125b}\left(\Sigma_7+\Sigma_{18}\right)+\frac{19b\Sigma_{13}}{25a^3}+\frac{56a\left(38\Sigma_9b^4-225\Sigma_{14}\right)}{5625b^3}-\frac{18\left(\Sigma_{10}+\Sigma_{15}\right)}{25ab},$$

$$\hat{\kappa}_{155}=\kappa_{119}=\frac{38a}{125b}\left(\Sigma_7+\Sigma_{18}\right)-\frac{112b}{125a}\left(\Sigma_4+\Sigma_{16}\right)+\frac{19\Sigma_{14}a}{25b^3}+\frac{56b\left(38a^4\Sigma_9-225\Sigma_{13}\right)}{5625a^3}-\frac{18\left(\Sigma_{10}+\Sigma_{15}\right)}{25ab},\quad \hat{\kappa}_{157}=\kappa_{120}=\frac{112}{125}\left[\frac{b}{a}\left(\Sigma_4+\Sigma_{16}\right)+\frac{a}{b}\left(\Sigma_7+\Sigma_{18}\right)\right]+\frac{6272\Sigma_9ab}{5625}+\frac{56}{25}\left(\frac{\Sigma_{14}a}{b^3}+\frac{b\Sigma_{13}}{a^3}\right)+\frac{18\left(\Sigma_{10}+\Sigma_{15}\right)}{25ab},$$

$$\kappa_{158}=\frac{1}{4}\left(\Sigma_{11}-\Sigma_{12}\right),\quad \kappa_{159}=\frac{1}{4}\left(\Sigma_{11}+\Sigma_{12}\right),\quad \hat{\kappa}_{162}=\kappa_{160}=\frac{a\left(\Sigma_{11}-\Sigma_{12}\right)}{10},\quad \hat{\kappa}_{163}=\kappa_{161}=\frac{a^2\left(\Sigma_{11}-\Sigma_{12}\right)}{30},$$

$$\hat{\kappa}_{165}=\kappa_{164}=\frac{16\Sigma_9a^2}{75}-\frac{11\Sigma_5-\Sigma_6}{20},\quad \hat{\kappa}_{167}=\kappa_{166}=\frac{3\Sigma_9a^2}{25}-\frac{\Sigma_5+\Sigma_6}{20},\quad \hat{\kappa}_{169}=\kappa_{168}=\frac{16\Sigma_9a^2}{75}-\frac{\Sigma_5-\Sigma_6}{20},$$

$$\hat{\kappa}_{171}=\kappa_{170}=\frac{\Sigma_5-\Sigma_6}{20}+\frac{3\Sigma_9a^2}{25},\quad \hat{\kappa}_{173}=\kappa_{172}=\frac{\Sigma_5+\Sigma_6}{20}+\frac{16\Sigma_9a^2}{75},\quad \hat{\kappa}_{175}=\kappa_{174}=\frac{11\Sigma_5+\Sigma_6}{20}+\frac{16\Sigma_9a^2}{75},$$

$$,\ \hat{\kappa}_{180}=\kappa_{176}=\frac{2\Sigma_9a^3}{25}-\frac{2a\left(\Sigma_5-\Sigma_6\right)}{15},\quad \hat{\kappa}_{181}=\kappa_{177}=\frac{4\Sigma_9a^3}{75}-\frac{a\left(\Sigma_5-\Sigma_6\right)}{30},\quad \hat{\kappa}_{182}=\kappa_{178}=\frac{a\left(\Sigma_5+\Sigma_6\right)}{30}+\frac{4\Sigma_9a^3}{75},$$

$$\hat{\kappa}_{183}=\kappa_{179}=\frac{2a}{15}\left(\Sigma_5+\Sigma_6\right)+\frac{2\Sigma_9a^3}{25},\quad \kappa_{184}=\frac{1}{4}\left(\Sigma_3-\Sigma_6+\Sigma_{17}-2\Sigma_{18}\right),\quad \kappa_{185}=\frac{1}{4}\left(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18}\right),$$

$$\hat{\kappa}_{189}=\kappa_{186}=\frac{3\Sigma_8b^2}{50}+\frac{2\Sigma_9a^2b^2}{125}-\frac{a^2\left(\Sigma_5+\Sigma_6\right)}{150},\quad \hat{\kappa}_{190}=\kappa_{187}=\frac{a^2\left(\Sigma_5+\Sigma_6\right)}{150}+\frac{8\Sigma_8b^2}{75}+\frac{32\Sigma_9a^2b^2}{1125}.$$

$$\hat{\kappa}_{191}=\kappa_{188}=\frac{a^2\left(11\Sigma_5+\Sigma_6\right)}{150}+\frac{8\Sigma_8b^2}{75}+\frac{32\Sigma_9a^2b^2}{1125},\quad \hat{\kappa}_{194}=\kappa_{192}=\frac{ab^2}{225}\left(\Sigma_5+\Sigma_6\right)+\frac{2\Sigma_8a^3}{75}+\frac{8\Sigma_9a^3b^2}{1125},$$

$$\hat{\kappa}_{195}=\kappa_{193}=\frac{4ab^2\left(\Sigma_5+\Sigma_6\right)}{225}+\frac{\Sigma_8a^3}{25}+\frac{4\Sigma_9a^3b^2}{375},\ \hat{\kappa}_{198}=\kappa_{196}=\frac{a^2\left(\Sigma_3-\Sigma_6+\Sigma_{17}-2\Sigma_{18}\right)}{30}+\frac{\Sigma_{11}-\Sigma_{12}}{4},$$

$$\hat{\kappa}_{199}=\kappa_{197}=\frac{a^2\left(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18}\right)}{30}+\frac{\Sigma_{11}+\Sigma_{12}}{4},\quad \kappa_{200}=\frac{a^2b^2\left(\Sigma_3+\Sigma_6+\Sigma_{17}+2\Sigma_{18}\right)}{225}+\frac{\left(a^2+b^2\right)\left(\Sigma_{11}+\Sigma_{12}\right)}{30}. \qquad \text{(C. 3)}$$

where $\hat{\kappa}_n$ denote the result of exchanging a and b in κ_n.

Appendix D

With the help of MAPLE symbolic computation system, the explicit algebraic expressions of element mass matrix $\boldsymbol{M}_e$ are as follows:

$$\boldsymbol{M}_e=\begin{bmatrix} \boldsymbol{m}_1^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_3^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_5^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_7^{(e)} & \mathbf{0} & \mathbf{0} \\ & \boldsymbol{m}_2^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_4^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_6^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_8^{(e)} & \mathbf{0} \\ & & \boldsymbol{m}_2^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_4^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_6^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_8^{(e)} \\ & & & \boldsymbol{m}_9^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{11}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{13}^{(e)} & \mathbf{0} & \mathbf{0} \\ & & & & \boldsymbol{m}_{10}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{12}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{14}^{(e)} & \mathbf{0} \\ & & & & & \boldsymbol{m}_{10}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{12}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{14}^{(e)} \\ & & & & & & \boldsymbol{m}_{15}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{17}^{(e)} & \mathbf{0} & \mathbf{0} \\ & & & & & & & \boldsymbol{m}_{16}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{18}^{(e)} & \mathbf{0} \\ & & & & & & & & \boldsymbol{m}_{16}^{(e)} & \mathbf{0} & \mathbf{0} & \boldsymbol{m}_{18}^{(e)} \\ & & \textbf{\textit{Sym}} & & & & & & & \boldsymbol{m}_{19}^{(e)} & \mathbf{0} & \mathbf{0} \\ & & & & & & & & & & \boldsymbol{m}_{20}^{(e)} & \mathbf{0} \\ & & & & & & & & & & & \boldsymbol{m}_{20}^{(e)} \end{bmatrix}, \tag{D. 1}$$

where

$$\boldsymbol{m}_1^{(e)}=\begin{bmatrix}\gamma_4 & \gamma_{16} & \gamma_8 & \gamma_{19}\\ \gamma_{16} & \gamma_{26} & \gamma_{19} & \gamma_{30}\\ \gamma_8 & \gamma_{19} & \gamma_{11} & \gamma_{22}\\ \gamma_{19} & \gamma_{30} & \gamma_{22} & \gamma_{32}\end{bmatrix},\quad \boldsymbol{m}_2^{(e)}=\begin{bmatrix}\gamma_{37} & \gamma_{49} & \gamma_{41} & \gamma_{52}\\ \gamma_{49} & \gamma_{59} & \gamma_{52} & \gamma_{63}\\ \gamma_{41} & \gamma_{52} & \gamma_{44} & \gamma_{55}\\ \gamma_{52} & \gamma_{63} & \gamma_{55} & \gamma_{65}\end{bmatrix},\quad \boldsymbol{m}_3^{(e)}=\begin{bmatrix}\gamma_3 & -\gamma_{14} & \gamma_7 & -\gamma_{18}\\ \gamma_{14} & -\gamma_{27} & \gamma_{18} & -\gamma_{31}\\ \gamma_7 & -\gamma_{18} & \gamma_9 & -\gamma_{21}\\ \gamma_{18} & -\gamma_{31} & \gamma_{21} & -\gamma_{33}\end{bmatrix},$$

$$\boldsymbol{m}_4^{(e)}=\begin{bmatrix}\gamma_{35} & -\gamma_{46} & \gamma_{40} & -\gamma_{51}\\ \gamma_{46} & -\gamma_{60} & \gamma_{51} & -\gamma_{64}\\ \gamma_{40} & -\gamma_{51} & \gamma_{42} & -\gamma_{53}\\ \gamma_{51} & -\gamma_{64} & \gamma_{53} & -\gamma_{66}\end{bmatrix},\quad \boldsymbol{m}_5^{(e)}=\begin{bmatrix}\gamma_2 & -\gamma_{13} & -\gamma_5 & \gamma_{17}\\ \gamma_{13} & -\gamma_{25} & -\gamma_{17} & \gamma_{28}\\ \gamma_5 & -\gamma_{17} & -\gamma_{10} & \gamma_{20}\\ \gamma_{17} & -\gamma_{28} & -\gamma_{20} & \gamma_{34}\end{bmatrix},\quad \boldsymbol{m}_6^{(e)}=\begin{bmatrix}\gamma_{36} & -\gamma_{47} & -\gamma_{39} & \gamma_{50}\\ \gamma_{47} & -\gamma_{58} & -\gamma_{50} & \gamma_{62}\\ \gamma_{39} & -\gamma_{50} & -\gamma_{43} & \gamma_{54}\\ \gamma_{50} & -\gamma_{62} & -\gamma_{54} & \gamma_{67}\end{bmatrix},$$

$$\boldsymbol{m}_7^{(e)}=\begin{bmatrix}\gamma_3 & \gamma_{15} & -\gamma_6 & -\gamma_{18}\\ \gamma_{15} & \gamma_{24} & -\gamma_{18} & -\gamma_{29}\\ \gamma_6 & \gamma_{18} & -\gamma_{12} & -\gamma_{23}\\ \gamma_{18} & \gamma_{29} & -\gamma_{23} & -\gamma_{33}\end{bmatrix},\quad \boldsymbol{m}_8^{(e)}=\begin{bmatrix}\gamma_{35} & \gamma_{48} & -\gamma_{38} & -\gamma_{51}\\ \gamma_{48} & \gamma_{57} & -\gamma_{51} & -\gamma_{61}\\ \gamma_{38} & \gamma_{51} & -\gamma_{45} & -\gamma_{56}\\ \gamma_{51} & \gamma_{61} & -\gamma_{56} & -\gamma_{66}\end{bmatrix},\quad \boldsymbol{m}_9^{(e)}=\begin{bmatrix}\gamma_4 & -\gamma_{16} & \gamma_8 & -\gamma_{19}\\ -\gamma_{16} & \gamma_{26} & -\gamma_{19} & \gamma_{30}\\ \gamma_8 & -\gamma_{19} & \gamma_{11} & -\gamma_{22}\\ -\gamma_{19} & \gamma_{30} & -\gamma_{22} & \gamma_{32}\end{bmatrix},$$

$$\boldsymbol{m}_{10}^{(e)}=\begin{bmatrix}\gamma_{37} & -\gamma_{49} & \gamma_{41} & -\gamma_{52}\\ -\gamma_{49} & \gamma_{59} & -\gamma_{52} & \gamma_{63}\\ \gamma_{41} & -\gamma_{52} & \gamma_{44} & -\gamma_{55}\\ -\gamma_{52} & \gamma_{63} & -\gamma_{55} & \gamma_{65}\end{bmatrix},\quad \boldsymbol{m}_{11}^{(e)}=\begin{bmatrix}\gamma_3 & -\gamma_{15} & -\gamma_6 & \gamma_{18}\\ -\gamma_{15} & \gamma_{24} & \gamma_{18} & -\gamma_{29}\\ \gamma_6 & -\gamma_{18} & -\gamma_{12} & \gamma_{23}\\ -\gamma_{18} & \gamma_{29} & \gamma_{23} & -\gamma_{33}\end{bmatrix},\quad \boldsymbol{m}_{12}^{(e)}=\begin{bmatrix}\gamma_{35} & -\gamma_{48} & -\gamma_{38} & \gamma_{51}\\ -\gamma_{48} & \gamma_{57} & \gamma_{51} & -\gamma_{61}\\ \gamma_{38} & -\gamma_{51} & -\gamma_{45} & \gamma_{56}\\ -\gamma_{51} & \gamma_{61} & \gamma_{56} & -\gamma_{66}\end{bmatrix},$$

$$\boldsymbol{m}_{13}^{(e)}=\begin{bmatrix}\gamma_2 & \gamma_{13} & -\gamma_5 & -\gamma_{17}\\ -\gamma_{13} & -\gamma_{25} & \gamma_{17} & \gamma_{28}\\ \gamma_5 & \gamma_{17} & -\gamma_{10} & -\gamma_{20}\\ -\gamma_{17} & -\gamma_{28} & \gamma_{20} & \gamma_{34}\end{bmatrix},\quad \boldsymbol{m}_{14}^{(e)}=\begin{bmatrix}\gamma_{36} & \gamma_{47} & -\gamma_{39} & -\gamma_{50}\\ -\gamma_{47} & -\gamma_{58} & \gamma_{50} & \gamma_{62}\\ \gamma_{39} & \gamma_{50} & -\gamma_{43} & -\gamma_{54}\\ -\gamma_{50} & -\gamma_{62} & \gamma_{54} & \gamma_{67}\end{bmatrix},\quad \boldsymbol{m}_{15}^{(e)}=\begin{bmatrix}\gamma_4 & -\gamma_{16} & -\gamma_8 & \gamma_{19}\\ -\gamma_{16} & \gamma_{26} & \gamma_{19} & -\gamma_{30}\\ -\gamma_8 & \gamma_{19} & \gamma_{11} & -\gamma_{22}\\ \gamma_{19} & -\gamma_{30} & -\gamma_{22} & \gamma_{32}\end{bmatrix},$$

$$\boldsymbol{m}_{16}^{(e)}=\begin{bmatrix}\gamma_{37} & -\gamma_{49} & -\gamma_{41} & \gamma_{52}\\ -\gamma_{49} & \gamma_{59} & \gamma_{52} & -\gamma_{63}\\ -\gamma_{41} & \gamma_{52} & \gamma_{44} & -\gamma_{55}\\ \gamma_{52} & -\gamma_{63} & -\gamma_{55} & \gamma_{65}\end{bmatrix},\quad \boldsymbol{m}_{17}^{(e)}=\begin{bmatrix}\gamma_3 & \gamma_{14} & -\gamma_7 & -\gamma_{18}\\ -\gamma_{14} & -\gamma_{27} & \gamma_{18} & \gamma_{31}\\ -\gamma_7 & -\gamma_{18} & \gamma_9 & \gamma_{21}\\ \gamma_{18} & \gamma_{31} & -\gamma_{21} & -\gamma_{33}\end{bmatrix},\quad \boldsymbol{m}_{18}^{(e)}=\begin{bmatrix}\gamma_{35} & \gamma_{46} & -\gamma_{40} & -\gamma_{51}\\ -\gamma_{46} & -\gamma_{60} & \gamma_{51} & \gamma_{64}\\ -\gamma_{40} & -\gamma_{51} & \gamma_{42} & \gamma_{53}\\ \gamma_{51} & \gamma_{64} & -\gamma_{53} & -\gamma_{66}\end{bmatrix},$$

$$\boldsymbol{m}_{19}^{(e)}=\begin{bmatrix}\gamma_4 & \gamma_{16} & -\gamma_8 & -\gamma_{19}\\ \gamma_{16} & \gamma_{26} & -\gamma_{19} & -\gamma_{30}\\ -\gamma_8 & -\gamma_{19} & \gamma_{11} & \gamma_{22}\\ -\gamma_{19} & -\gamma_{30} & \gamma_{22} & \gamma_{32}\end{bmatrix},\quad \boldsymbol{m}_{20}^{(e)}=\begin{bmatrix}\gamma_{37} & \gamma_{49} & -\gamma_{41} & -\gamma_{52}\\ \gamma_{49} & \gamma_{59} & -\gamma_{52} & -\gamma_{63}\\ -\gamma_{41} & -\gamma_{52} & \gamma_{44} & \gamma_{55}\\ -\gamma_{52} & -\gamma_{63} & \gamma_{55} & \gamma_{65}\end{bmatrix}, \tag{D. 2}$$

where

$\gamma_0=\dfrac{\rho abh}{625}$, $\gamma_1=0$, $\gamma_2=\dfrac{361\gamma_0}{9}$, $\gamma_3=\dfrac{1064\gamma_0}{9}$, $\gamma_4=\dfrac{3136\gamma_0}{9}$, $\gamma_5=19\gamma_0 b$, $\gamma_6=56\gamma_0 b$, $\gamma_7=\dfrac{304\gamma_0 b}{9}$, $\gamma_8=\dfrac{896\gamma_0 b}{9}$,

$\gamma_9=\dfrac{38\gamma_0 b^2}{3}$, $\gamma_{10}=\dfrac{76\gamma_0 b^2}{9}$, $\gamma_{11}=\dfrac{112\gamma_0 b^2}{3}$, $\gamma_{12}=\dfrac{224\gamma_0 b^2}{9}$, $\gamma_{13}=19\gamma_0 a$, $\gamma_{14}=56\gamma_0 a$, $\gamma_{15}=\dfrac{304\gamma_0 a}{9}$, $\gamma_{16}=\dfrac{896\gamma_0 a}{9}$,

$\gamma_{17}=9\gamma_0 ab$, $\gamma_{18}=16\gamma_0 ab$, $\gamma_{19}=\frac{256\gamma_0 ab}{9}$, $\gamma_{20}=4\gamma_0 ab^2$, $\gamma_{21}=6\gamma_0 ab^2$, $\gamma_{22}=\frac{32\gamma_0 ab^2}{3}$, $\gamma_{23}=\frac{64\gamma_0 ab^2}{9}$, $\gamma_{24}=\frac{38\gamma_0 a^2}{3}$,

$\gamma_{25}=\frac{76\gamma_0 a^2}{9}$, $\gamma_{26}=\frac{112\gamma_0 a^2}{3}$, $\gamma_{27}=\frac{224\gamma_0 a^2}{9}$, $\gamma_{28}=4\gamma_0 a^2 b$, $\gamma_{29}=6\gamma_0 a^2 b$, $\gamma_{30}=\frac{32\gamma_0 a^2 b}{3}$, $\gamma_{31}=\frac{64\gamma_0 a^2 b}{9}$,

$\gamma_{32}=4\gamma_0 a^2 b^2$, $\gamma_{33}=\frac{8}{3}\gamma_0 a^2 b^2$, $\gamma_{34}=\frac{16\gamma_0 a^2 b^2}{9}$, $\gamma_{35}=\frac{266\gamma_0 h^2}{27}$, $\gamma_{36}=\frac{361\gamma_0 h^2}{108}$, $\gamma_{37}=\frac{784\gamma_0 h^2}{27}$, $\gamma_{38}=\frac{14}{3}\gamma_0 bh^2$,

$\gamma_{39}=\frac{19\gamma_0 bh^2}{12}$, $\gamma_{40}=\frac{76\gamma_0 bh^2}{27}$, $\gamma_{41}=\frac{224\gamma_0 bh^2}{27}$, $\gamma_{42}=\frac{19\gamma_0 b^2 h^2}{18}$, $\gamma_{43}=\frac{19\gamma_0 b^2 h^2}{27}$, $\gamma_{44}=\frac{28\gamma_0 b^2 h^2}{9}$, $\gamma_{45}=\frac{56\gamma_0 b^2 h^2}{27}$,

$\gamma_{46}=\frac{14}{3}\gamma_0 ah^2$, $\gamma_{47}=\frac{19\gamma_0 ah^2}{12}$, $\gamma_{48}=\frac{76\gamma_0 ah^2}{27}$, $\gamma_{49}=\frac{224\gamma_0 ah^2}{27}$, $\gamma_{50}=\frac{3}{4}\gamma_0 abh^2$, $\gamma_{51}=\frac{4}{3}\gamma_0 abh^2$, $\gamma_{52}=\frac{64\gamma_0 abh^2}{27}$,

$\gamma_{53}=\frac{1}{2}\gamma_0 ab^2 h^2$, $\gamma_{54}=\frac{1}{3}\gamma_0 ab^2 h^2$, $\gamma_{55}=\frac{8\gamma_0 ab^2 h^2}{9}$, $\gamma_{59}=\frac{28\gamma_0 a^2 h^2}{9}$, $\gamma_{60}=\frac{56\gamma_0 a^2 h^2}{27}$, $\gamma_{56}=\frac{16\gamma_0 ab^2 h^2}{27}$, $\gamma_{57}=\frac{19\gamma_0 a^2 h^2}{18}$,

$\gamma_{58}=\frac{19\gamma_0 a^2 h^2}{27}$, $\gamma_{61}=\frac{1}{2}\gamma_0 a^2 bh^2$, $\gamma_{62}=\frac{1}{3}\gamma_0 a^2 bh^2$, $\gamma_{63}=\frac{8\gamma_0 a^2 bh^2}{9}$, $\gamma_{64}=\frac{16\gamma_0 a^2 bh^2}{27}$, $\gamma_{65}=\frac{1}{3}\gamma_0 a^2 b^2 h^2$, $\gamma_{66}=\frac{2}{9}\gamma_0 a^2 b^2 h^2$,

$$\gamma_{67}=\frac{4\gamma_0 a^2 b^2 h^2}{27}. \tag{D. 3}$$

Appendix E

With the help of MAPLE symbolic computation system, the explicit algebraic expressions of element stiffness and mass matrices $\boldsymbol{K}_e$ and $\boldsymbol{M}_e$ are as follows:

$$\boldsymbol{K}_e=\begin{bmatrix}
\kappa_{24} & \kappa_{17} & \kappa_9 & \kappa_4 & \kappa_{22} & -\kappa_{14} & \kappa_6 & -\kappa_1 & -\kappa_{23} & \kappa_{15} & \kappa_7 & -\kappa_3 & -\kappa_{25} & -\kappa_{16} & \kappa_8 & \kappa_2 \\
 & \kappa_{33} & \kappa_5 & \kappa_{11} & \kappa_{14} & -\kappa_{30} & \kappa_1 & -\kappa_{10} & -\kappa_{15} & \kappa_{31} & \kappa_3 & -\kappa_{12} & -\kappa_{16} & -\kappa_{32} & \kappa_2 & \kappa_{13} \\
 & & \kappa_{29} & \kappa_{19} & \kappa_6 & -\kappa_1 & \kappa_{26} & -\kappa_{20} & -\kappa_7 & \kappa_3 & \kappa_{28} & -\kappa_{21} & -\kappa_8 & -\kappa_2 & \kappa_{27} & \kappa_{18} \\
 & & & \kappa_{37} & \kappa_1 & -\kappa_{10} & \kappa_{20} & -\kappa_{36} & -\kappa_3 & \kappa_{12} & \kappa_{21} & -\kappa_{35} & -\kappa_2 & -\kappa_{13} & \kappa_{18} & \kappa_{34} \\
 & & & & \kappa_{24} & -\kappa_{17} & \kappa_9 & -\kappa_4 & -\kappa_{25} & \kappa_{16} & \kappa_8 & -\kappa_2 & -\kappa_{23} & -\kappa_{15} & \kappa_7 & \kappa_3 \\
 & & & & & \kappa_{33} & -\kappa_5 & \kappa_{11} & \kappa_{16} & -\kappa_{32} & -\kappa_2 & \kappa_{13} & \kappa_{15} & \kappa_{31} & -\kappa_3 & -\kappa_{12} \\
 & & & & & & \kappa_{29} & -\kappa_{19} & -\kappa_8 & \kappa_2 & \kappa_{27} & -\kappa_{18} & -\kappa_7 & -\kappa_3 & \kappa_{28} & \kappa_{21} \\
 & & & & & & & \kappa_{37} & \kappa_2 & -\kappa_{13} & -\kappa_{18} & \kappa_{34} & \kappa_3 & \kappa_{12} & -\kappa_{21} & -\kappa_{35} \\
 & & & & & & & & \kappa_{24} & -\kappa_{17} & -\kappa_9 & \kappa_4 & \kappa_{22} & \kappa_{14} & -\kappa_6 & -\kappa_1 \\
 & & & & & & & & & \kappa_{33} & \kappa_5 & -\kappa_{11} & -\kappa_{14} & -\kappa_{30} & \kappa_1 & \kappa_{10} \\
 & & & & & & & & & & \kappa_{29} & -\kappa_{19} & -\kappa_6 & -\kappa_1 & \kappa_{26} & \kappa_{20} \\
 & & & Sym & & & & & & & & \kappa_{37} & \kappa_1 & \kappa_{10} & -\kappa_{20} & -\kappa_{36} \\
 & & & & & & & & & & & & \kappa_{24} & \kappa_{17} & -\kappa_9 & -\kappa_4 \\
 & & & & & & & & & & & & & \kappa_{33} & -\kappa_5 & -\kappa_{11} \\
 & & & & & & & & & & & & & & \kappa_{29} & \kappa_{19} \\
 & & & & & & & & & & & & & & & \kappa_{37}
\end{bmatrix}, \tag{E. 1}$$

where

$\kappa_1=\frac{D}{25}\left(\frac{9a^2}{2b^2}-\frac{8b^2}{a^2}\right)-\frac{D}{50}(5\nu+1)$, $\kappa_2=\frac{D}{50}(5\nu+1)+\frac{D}{25}\left(\frac{8a^2}{b^2}-\frac{9b^2}{2a^2}\right)$, $\kappa_3=\frac{9D}{50}\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)-\frac{D}{50}$,

$\kappa_4=\frac{D}{50}(10\nu+1)+\frac{8D}{25}\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)$, $\kappa_5=\frac{D}{50}(60\nu+1)+\frac{8D}{25}\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)$, $\kappa_6=\frac{D}{25}\left(\frac{19a}{2b^2}-\frac{8b^2}{a^3}\right)-\frac{3D}{25a}(5\nu+1)$,

$\kappa_7=\frac{D}{25}\left(\frac{19a}{2b^2}+\frac{9b^2}{2a^3}-\frac{3}{a}\right)$, $\kappa_8=\frac{D}{25}\left(\frac{3}{a}-\frac{9b^2}{2a^3}+\frac{28a}{b^2}\right)$, $\kappa_9=\frac{3D}{25a}(5\nu+1)+\frac{4D}{25}\left(\frac{2b^2}{a^3}+\frac{7a}{b^2}\right)$,

$\kappa_{10}=\frac{Da}{75}(5\nu+1)+\frac{2D}{25}\left(\frac{a^3}{b^2}-\frac{8b^2}{3a}\right)$, $\kappa_{11}=\frac{4Da}{75}(5\nu+1)+\frac{D}{25}\left(\frac{32b^2}{3a}+\frac{3a^3}{b^2}\right)$, $\kappa_{12}=\frac{D}{25}\left(\frac{a}{3}+\frac{2a^3}{b^2}+\frac{3b^2}{a}\right)$,

$\kappa_{13}=\frac{D}{25}\left(\frac{4a}{3}-\frac{6b^2}{a}+\frac{3a^3}{b^2}\right)$, $\kappa_{14}=\frac{D}{25}\left(\frac{9a^2}{2b^3}-\frac{3}{b}-\frac{28b}{a^2}\right)$, $\kappa_{15}=\frac{D}{25}\left(\frac{9a^2}{2b^3}+\frac{19b}{2a^2}-\frac{3}{b}\right)$,

$$\kappa_{16}=\frac{3D}{25b}(5\nu+1)+\frac{D}{25}\left(\frac{8a^2}{b^3}-\frac{19b}{a^2}\right),\quad \kappa_{17}=\frac{3D}{25b}(5\nu+1)+\frac{4D}{25}\left(\frac{2a^2}{b^3}+\frac{7b}{a^2}\right),\quad \kappa_{18}=\frac{2D}{25}\left(\frac{8a^2}{3b}-\frac{b^3}{a^2}\right)-\frac{Db}{75}(5\nu+1),$$

$$\kappa_{19}=\frac{4Db}{75}(5\nu+1)+\frac{D}{25}\left(\frac{32a^2}{3b}+\frac{3b^3}{a^2}\right),\quad \kappa_{20}=\frac{D}{25}\left(\frac{6a^2}{b}-3-\frac{3b^3}{a^2}\right),\quad \kappa_{21}=\frac{D}{25}\left(\frac{b}{3}+\frac{3a^2}{b}+\frac{3b^3}{a^2}\right),$$

$$\kappa_{22}=\frac{D}{25}\left(\frac{19a}{2b^3}-\frac{18}{ab}-\frac{28b}{a^3}\right),\quad \kappa_{23}=\frac{19D}{50}\left(\frac{a}{b^3}+\frac{b}{a^3}\right)-\frac{18D}{25ab},\quad \kappa_{24}=\frac{18D}{25ab}+\frac{28D}{25}\left(\frac{a}{b^3}+\frac{b}{a^3}\right),$$

$$\kappa_{25}=\frac{D}{25}\left(\frac{18}{ab}-\frac{19b}{2a^3}+\frac{28a}{b^3}\right),\quad \kappa_{26}=\frac{D}{25}\left(\frac{38a}{3b}-\frac{3b^3}{a^3}-\frac{8b}{a}\right),\quad \kappa_{27}=\frac{56Da}{75b}-\frac{2D}{25}\left(\frac{b}{a}+\frac{b^3}{a^3}\right),$$

$$\kappa_{28}=\frac{D}{25}\left(\frac{2b}{a}+\frac{19a}{3b}+\frac{2b^3}{a^3}\right),\quad \kappa_{29}=\frac{D}{25}\left(\frac{8b}{a}+\frac{112a}{3b}+\frac{3b^3}{a^3}\right),\quad \kappa_{30}=\frac{D}{25}\left(\frac{a}{b}-\frac{28b}{3a}+\frac{a^3}{b^3}\right),$$

$$\kappa_{31}=\frac{D}{25}\left(\frac{2a}{b}+\frac{19b}{3a}+\frac{2a^3}{b^3}\right),\kappa_{32}=\frac{D}{25}\left(\frac{8a}{b}-\frac{38b}{3a}+\frac{3a^3}{b^3}\right),\quad \kappa_{33}=\frac{D}{25}\left(\frac{8a}{b}+\frac{3a^3}{b^3}+\frac{112b}{3a}\right),$$

$$\kappa_{34}=\frac{2D}{25}\left(\frac{a^3}{b}-\frac{4b^3}{3a}-\frac{4ab}{9}\right),\quad \kappa_{36}=\frac{2D}{25}\left(\frac{4ab}{9}+\frac{4a^3}{3b}-\frac{b^3}{a}\right),\quad \kappa_{35}=\frac{4D}{75}\left(\frac{a^3}{b}+\frac{b^3}{a}\right)-\frac{2Dab}{225},$$

$$\kappa_{37}=\frac{32Dab}{225}+\frac{4D}{25}\left(\frac{a^3}{b}+\frac{b^3}{a}\right). \tag{E. 2}$$

$$\boldsymbol{M}_e=\begin{bmatrix}
\gamma_3 & \gamma_{15} & \gamma_7 & \gamma_{18} & \gamma_2 & -\gamma_{13} & \gamma_6 & -\gamma_{17} & \gamma_1 & -\gamma_{12} & -\gamma_4 & \gamma_{16} & \gamma_2 & \gamma_{14} & -\gamma_5 & -\gamma_{17}\\
 & \gamma_{25} & \gamma_{18} & \gamma_{29} & \gamma_{13} & -\gamma_{26} & \gamma_{17} & -\gamma_{30} & \gamma_{12} & -\gamma_{24} & -\gamma_{16} & \gamma_{27} & \gamma_{14} & \gamma_{23} & -\gamma_{17} & -\gamma_{28}\\
 & & \gamma_{10} & \gamma_{21} & \gamma_6 & -\gamma_{17} & \gamma_8 & -\gamma_{20} & \gamma_4 & -\gamma_{16} & -\gamma_9 & \gamma_{19} & \gamma_5 & \gamma_{17} & -\gamma_{11} & -\gamma_{22}\\
 & & & \gamma_{31} & \gamma_{17} & -\gamma_{30} & \gamma_{20} & -\gamma_{32} & \gamma_{16} & -\gamma_{27} & -\gamma_{19} & \gamma_{33} & \gamma_{17} & \gamma_{28} & -\gamma_{22} & -\gamma_{32}\\
 & & & & \gamma_3 & -\gamma_{15} & \gamma_7 & -\gamma_{18} & \gamma_2 & -\gamma_{14} & -\gamma_5 & \gamma_{17} & \gamma_1 & \gamma_{12} & -\gamma_4 & -\gamma_{16}\\
 & & & & & \gamma_{25} & -\gamma_{18} & \gamma_{29} & -\gamma_{14} & \gamma_{23} & \gamma_{17} & -\gamma_{28} & -\gamma_{12} & -\gamma_{24} & \gamma_{16} & \gamma_{27}\\
 & & & & & & \gamma_{10} & -\gamma_{21} & \gamma_5 & -\gamma_{17} & -\gamma_{11} & \gamma_{22} & \gamma_4 & \gamma_{16} & -\gamma_9 & -\gamma_{19}\\
 & & & & & & & \gamma_{31} & -\gamma_{17} & \gamma_{28} & \gamma_{22} & -\gamma_{32} & -\gamma_{16} & -\gamma_{27} & \gamma_{19} & \gamma_{33}\\
 & & & & & & & & \gamma_3 & -\gamma_{15} & -\gamma_7 & \gamma_{18} & \gamma_2 & \gamma_{13} & -\gamma_6 & -\gamma_{17}\\
 & & & & & & & & & \gamma_{25} & \gamma_{18} & -\gamma_{29} & -\gamma_{13} & -\gamma_{26} & \gamma_{17} & \gamma_{30}\\
 & & & & & & & & & & \gamma_{10} & -\gamma_{21} & -\gamma_6 & -\gamma_{17} & \gamma_8 & \gamma_{20}\\
 & & & & & & & & & & & \gamma_{31} & \gamma_{17} & \gamma_{30} & -\gamma_{20} & -\gamma_{32}\\
 & & & & \mathit{Sym} & & & & & & & & \gamma_3 & \gamma_{15} & -\gamma_7 & -\gamma_{18}\\
 & & & & & & & & & & & & & \gamma_{25} & -\gamma_{18} & -\gamma_{29}\\
 & & & & & & & & & & & & & & \gamma_{10} & \gamma_{21}\\
 & & & & & & & & & & & & & & & \gamma_{31}
\end{bmatrix}, \tag{E. 3}$$

where

$$\gamma_1=\frac{361\rho hab}{5625},\quad \gamma_2=\frac{1064\rho hab}{5625},\quad \gamma_3=\frac{3136\rho hab}{5625},\quad \gamma_4=\frac{19\rho hab^2}{625},\quad \gamma_5=\frac{56\rho hab^2}{625},\quad \gamma_6=\frac{304\rho hab^2}{5625},$$

$$\gamma_7=\frac{896\rho hab^2}{5625},\quad \gamma_8=\frac{38\rho hab^3}{1875},\quad \gamma_9=\frac{76\rho hab^3}{5625},\quad \gamma_{10}=\frac{112\rho hab^3}{1875},\quad \gamma_{11}=\frac{224\rho hab^3}{5625},\quad \gamma_{12}=\frac{19\rho ha^2b}{625},$$

$$\gamma_{13}=\frac{56\rho ha^2b}{625},\quad \gamma_{14}=\frac{304\rho ha^2b}{5625},\quad \gamma_{15}=\frac{896\rho ha^2b}{5625},\quad \gamma_{16}=\frac{9\rho ha^2b^2}{625},\quad \gamma_{17}=\frac{16\rho ha^2b^2}{625},\quad \gamma_{18}=\frac{256\rho ha^2b^2}{5625},$$

$$\gamma_{19}=\frac{4\rho ha^2b^3}{625},\quad \gamma_{20}=\frac{6\rho ha^2b^3}{625},\quad \gamma_{21}=\frac{32\rho ha^2b^3}{1875},\quad \gamma_{22}=\frac{64\rho ha^2b^3}{5625},\quad \gamma_{23}=\frac{38\rho ha^3b}{1875},\quad \gamma_{24}=\frac{76\rho ha^3b}{5625},$$

$$\gamma_{25}=\frac{112\rho ha^3b}{1875},\quad \gamma_{26}=\frac{224\rho ha^3b}{5625},\quad \gamma_{27}=\frac{4\rho ha^3b^2}{625},\quad \gamma_{28}=\frac{6\rho ha^3b^2}{625},\quad \gamma_{29}=\frac{32\rho ha^3b^2}{1875},\quad \gamma_{30}=\frac{64\rho ha^3b^2}{5625},$$

$$\gamma_{31}=\frac{4\rho ha^3b^3}{625},\quad \gamma_{33}=\frac{16\rho ha^3b^3}{5625},\quad \gamma_{32}=\frac{8\rho ha^3b^3}{1875}. \tag{E. 4}$$

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