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Technical Paper

Geotechnical uncertainty, modeling, and decision making

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Abstract

Modeling only constitutes one aspect of decision making. The prevailing limitation of applying modeling to practice is the absence of explicit consideration of uncertainties. This review paper covers uncertainty quantification (soil properties, stratification, and model performance) and uncertainty calculation with a focus on how it enhances the role of modeling in decision making (reliability analysis, reliability-based design, and inverse analysis). The key output from a reliability analysis is the probability of failure, where "failure" is defined as any condition that does not meet a performance criterion or a set of criteria. In contrast to the global factor of safety, the probability of failure respects both mechanics and statistics, is sensitive to data (thus opening one potential pathway to digital transformation), and it is meaningful for both system and component failures. Resilience engineering requires system level analysis. As such, geotechnical software can provide better decision support by computing the probability of failure/reliability index as one basic output in addition to stresses, strains, forces, and displacements. It is further shown that more critical non-classical failure mechanisms can emerge from spatially variable soils that can escape notice if the engineer were to restrict analysis to conventional homogeneous or layered soil profiles.

Keywords: Uncertainty; Numerical modeling; Decision making; Burland triangle; Risk management

1. Introduction

One distinctive feature of geotechnical engineering is that the engineer has to work with natural materials in an environment that is largely outside his/her control (historical and current conditions). The physical ground reality that is of interest to a geotechnical engineer is very complex and changing at different time scales [geologic (millions of

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years) to seismic (seconds)]. Modeling is frequently understood as a mathematical abstraction of some aspects of reality relevant to the problem at hand. Numerical modeling seeks to draw answers from mathematical equations (typically differential in nature) for a specific problem scenario. In the context of decision making in geotechnical engineering practice, the scenario includes being situated at a specific site. Hence, site investigation is necessary.

Barbour and Krahn (2004) opined that geotechnical engineers rarely focus on "prediction" as it is not unusual for computed and measured responses to differ by one order of magnitude. Tang and Phoon (2021) conducted the largest model validation exercise to date using load test databases and found that geotechnical models are biased (typically on the conservative side) and *imprecise*. Curran and Hammah (2006) shared seven lessons learned from developing software for practice. One lesson listed in the paper is "uncertainty is king; make room for it". The authors elaborated that "because geological materials are formed under a broad variety of complex, physical conditions, the history of which is not known, geomechanics involves large uncertainties. Single-point predictions of quantities have therefore practically zero likelihood of ever being realized in such a world. If room is therefore not made in geomechanics software analysis to accommodate uncertainty, any conclusions reached will be open to question.".

Geotechnical engineering is widely perceived as an art as much as a science, in part because known unknowns and unknown unknowns are not fully addressed in its numerical models (Phoon 2017). Burland Triangle considered "empiricism, precedent, experience, and risk management" to be central in decision making in practice as shown in Fig. 1 (Burland, 1987). As noted above, the reason is that it is not possible to make a *safe decision* based on a single deterministic analysis alone unless the ground profile, soil behaviour, physics, and construction effects are perfectly known, can be computed to perfect accuracy, and there are no unknown unknowns. An experienced engineer is aware that the deterministic answers produced by numerical analyses cannot be applied directly to a real-world project at a specific site without moderation by an ad-hoc combination of informal risk management strategies that include applying a global factor of safety (or partial factors of safety), selecting cautious input values and conservative calculation models, conducting parametric studies, learning from precedents, updating/validating designs and construction procedures based on prototype testing and observations, and keeping engineering judgment as an integral part of the decision making loop. These strategies are effective, but their role in digital transformation is unclear (Phoon et al., 2022b).

The volume of literature in geotechnical risk and reliability since its inception in the sixties is significant (Phoon 2017; Phoon 2020; Chwała et al. 2022). The purpose of this paper is to review how uncertainty quantification and numerical modeling can complement each other *to enhance decision making in practice*. The purpose is not to showcase the power of any methods, be it probabilistic or mechanical. This paper is intended to help practitioners acquire a basic appreciation of current geotechnical risk and reliability methods that can address realistic problems and to guide them to use these methods wisely in conjunction with numerical modeling. The various topics under statistical characterization and probabilistic analysis/design can be organized as enhancements to an uncertainty-informed Burland Triangle as shown in Fig. 1. The elements added to the original triangle are highlighted in grey. This will foreground their relationships more explicitly and highlight where decision making can be further supported in practice. Through the papers reviewed below, it will be shown that uncertainty quantification is not an abstract step divorced from reality. In fact, it brings decision making closer to reality beyond what is offered by the familiar deterministic approach and in closer alignment to current digital transformation (Phoon et al. 2022b).

2. Uncertainties in soil properties

Although numerical models are highly sophisticated, the input ground profile is typically simple and deterministic. This section provides some guidance and key results on the statistical characterization of more realistic *spatially varying* ground profiles.

2.1. Coefficient of variation

The simplest measure of soil/rock property uncertainty is the coefficient of variation (COV), which is defined as the ratio between the standard deviation and the mean. It provides a second-moment characterization of the data scatter. The COV of a design parameter is not an intrinsic statistical property. It depends on the site condition, the measurement method, and the transformation (correlation) model. Hence, the COV takes a range of values rather than a unique value. A comprehensive statistical study on the uncertainty of soil design parameters was conducted by Phoon and Kulhawy (1999a, 1999b). Some guidelines emerging from this classic study are given in Table 1 below. Comparable studies on the uncertainty of rock mass design parameters were conducted by Prakoso (2002) and Aladejare and Wang (2017). Useful summary tables for soil and rock property statistics are given in Phoon et al. (2016), Chapter 2, Tang and Phoon (2021), and Guan et al. (2021). Table 4 in Pan et al. (2018) and Table 9 in Cami et al. (2020) present statistics for cement-mixed soils.

2.2. Transformation uncertainty

One of the most important tasks in geotechnical design is the estimation of pertinent soil parameters, particularly the values governing the behaviour of a geotechnical structure at a limit state. These correlation models are very use-

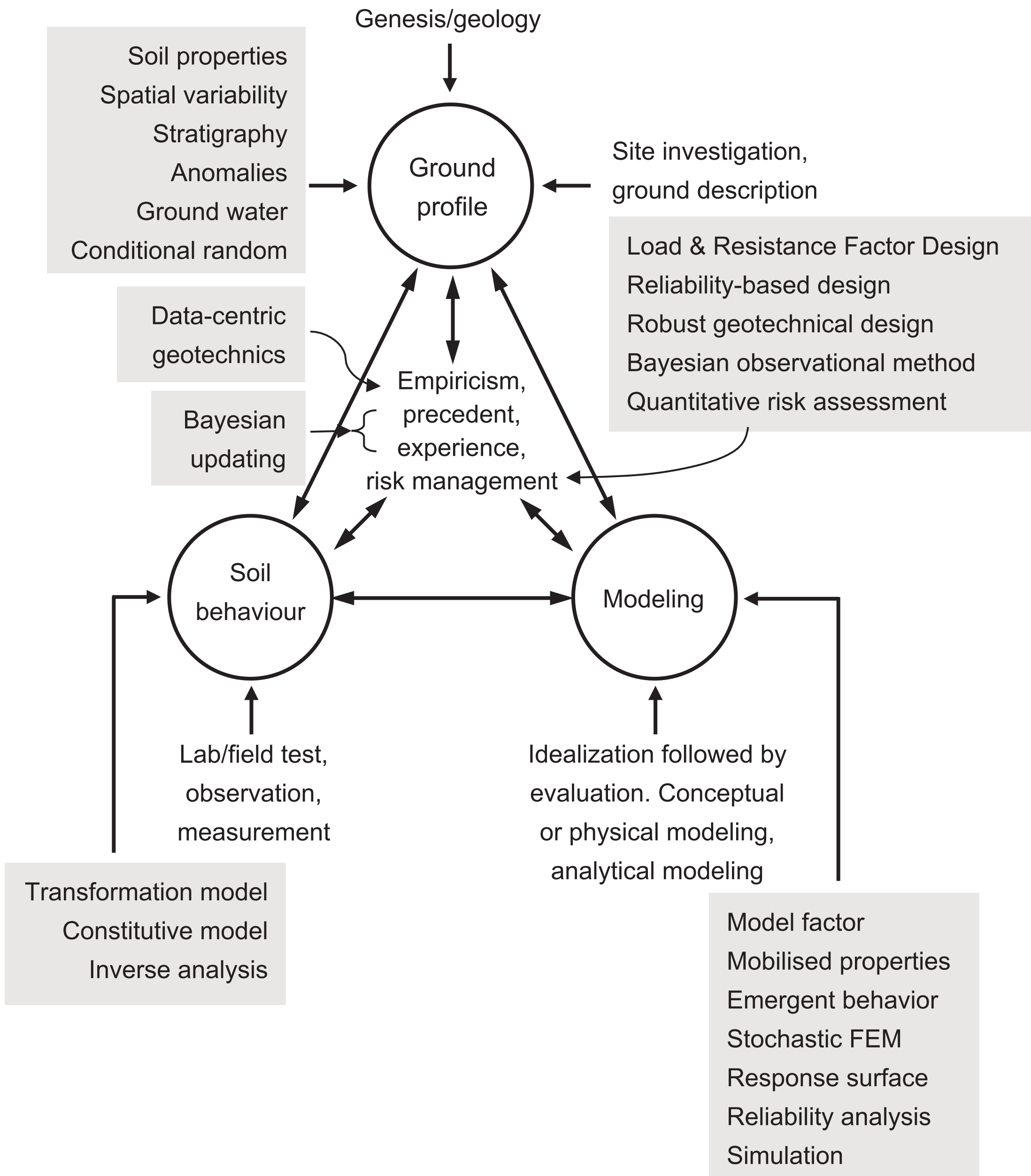

Fig. 1. Uncertainty-informed decision making in Burland Triangle (updated uncertainty elements are highlighted in grey).

ful in practice, because engineers can obtain an estimate of a soil parameter pertinent to design (called “design parameter”) using more commonly available data that are indirectly related to this design parameter but cheaper to acquire, say data from a laboratory index test or a field test. It also should be pointed out that these correlations were generally developed by curve fitting based on laboratory or field data; therefore, they tend to be case-specific and may not generalize to other or new soils/sites. In using these correlations, the caveat is to apply engineering judgment.

Many of these correlations are bivariate in the sense of estimating one desired design parameter (such as undrained shear strength) from one indirect source of data (such as cone tip resistance). The authors are of the opinion that our existing bivariate correlation models can be significantly improved by extending them using a multivariate probability framework. An overview of the multivariate characterization of soil uncertainty was presented in Ching et al. (2016). Machine learning recently has been extensively used to develop soil correlations because it provides the potential to extract useful knowledge from big data (Zhang et al. 2020c). The majority of these machine learning-based correlations are deterministic without physical constraints and the reliability of their predictions cannot be evaluated. Bayesian-based machine learning algorithms, combining the strong mapping capability of machine learning and statistical inference of Bayesian the-

Table 1
Approximate guidelines for coefficients of variation of some design soil parameters (Source: Table 5, Phoon and Kulhawy 1999b).

Design			Point	Spatial Avg.	Correlation
Property[a]	Test[b]	Soil type	COV (%)	COV[c] (%)	Equation[f]
s_u(UC)	Direct (lab)	Clay	20–55	10–40	–
s_u (UU)	Direct (lab)	Clay	10–35	7–25	–
s_u (CIUC)	Direct (lab)	Clay	20–45	10–30	–
s_u (field)	VST	Clay	15–50	15–50	14
s_u (UU)	q_T	Clay	30-40[d]	30-35[d]	18
s_u (CIUC)	q_T	Clay	35-50[d]	35-40[d]	18
s_u (UU)	N	Clay	40–60	40–55	23
s_u^e	K_D	Clay	30–55	30–55	29
s_u(field)	PI	Clay	30-55[d]	–	32
ϕ	Direct (lab)	Clay, sand	7–20	6–20	–
ϕ(TC)	q_T	Sand	10-15[d]	10[d]	38
ϕ_{cv}	PI	Clay	15-20[d]	15-20[d]	43
K_o	Direct (SBPMT)	Clay	20–45	15–45	–
K_o	Direct (SBPMT)	Sand	25–55	20–55	–
K_o	K_D	Clay	35-50[d]	35-50[d]	49
K_o	N	Clay	40-75[d]	–	54
E_{PMT}	Direct (PMT)	Sand	20–70	15–70	–
E_D	Direct (DMT)	Sand	15–70	10–70	–
E_{PMT}	N	Clay	85–95	85–95	61
E_D	N	Silt	40–60	35–55	64

a - s_u = undrained shear strength; UU = unconsolidated-undrained triaxial compression test; UC = unconfined compression test; CIUC = consolidated isotropic undrained triaxial compression test; s_u(field) = corrected s_u from vane shear test; ϕ = effective stress friction angle; TC = triaxial compression; ϕ_{cv} = constant volume ϕ; K_o = in-situ horizontal stress coefficient; E_{PMT} = pressure-meter modulus; E_D = dilatometer modulus.
b - VST = vane shear test; q_T = corrected cone tip resistance; N = standard penetration test blow count; K_D = dilatometer horizontal stress index; PI = plasticity index.
c - averaging over 5 m using Vanmarcke (1983)'s variance reduction function.
d - COV is a function of the mean; refer to COV equations in text for details.
e - mixture of s_u from UU, UC, and VST.
f - Equation numbering in Phoon and Kulhawy (1999b).

ory, have recently emerged as a promising alternative to estimate design parameters (Ching et al. 2021a; Ching et al. 2021b; Zhang et al. 2021b; Zhang et al 2022).

Besides, as soil constitutive models are increasingly adopted in engineering analysis and design, input parameters for these constitutive models become key design parameters that have to be identified from laboratory or field data. Probabilistic methods are competitive in parameter identification if the uncertainty is considered, since parameters of concern should be treated as random variables and expressed in terms of posterior distributions and their statistics. In geotechnical engineering, such probabilistic methods for identifying parameters considering uncertainties have been applied to the linear elastic model (Honjo et al. 1994), one-dimensional elasto-plastic model (Most 2010), an unified soil compression model (Jung et al. 2009), the Hardening Soil model (Miro et al. 2015), and critical state-based constitutive models (Jin et al. 2019a; Jin et al. 2019b).

2.3. Spatial variability

Spatial variability exists in natural deposits or formations because of two reasons: (1) properties and/or geometric features (such as stratification discussed below, voids, discontinuities, etc.) are spatially heterogeneous and (2) data is too limited to produce a single deterministic solution regardless of the site characterization method used. A large part of the literature on spatial variability is founded on random field theory. Vanmarcke (1977a)'s classic paper on "Probabilistic modeling of soil profiles" is arguably the first to introduce random field theory to geotechnical engineering. Vanmarcke (1983)'s key observations in his book "Random fields: analysis and synthesis" are:

1. Spatially averaged soil properties are more relevant, because soil-structure interaction mobilizes a finite volume of the ground.
2. The COV of this spatial average can be much smaller than the COV of the soil property at a point. This uncertainty reduction can be calculated *analytically* using a variance reduction function. Some indicative COV reduction results over a nominal averaging distance of 5 m are shown in Table 1.
3. This variance reduction function is dependent on a key random field parameter called the scale of fluctuation (θ), which can be regarded as a characteristic length parameter that elegantly unifies various common autocorrelation models as shown in Table 2. Some typical values are given in Table 3 and Stuedlein et al. (2021).

The classic research on random field (Vanmarcke 1977a; 1983) has since progressed in theory and practice over the past four decades:

1. The common single exponential autocorrelation model (Table 2) is now regarded as a special case of the Whittle-Matérn (W-M) model with a smoothness parameter $\nu = 0.5$. Hence, the task of identifying an appropriate model from infinite possibilities or from a classical finite subset such as Table 2 is reduced to a simpler curve fitting exercise using θ and ν (Ching et al. 2019).
2. The spatial average is identical to the average value along a *prescribed* curve, say one of the many kinematically admissible slip curves. However, it is not identical to the mobilized value along a *critical* slip curve that is of interest at the ultimate limit state (ULS). This is a special curve resulting in the lowest factor of safety. The mean value needs to be reduced to address this mobilized value, and the variance reduction function needs to be modified (Ching and Phoon 2013; Tabarroki et al. 2022a).
3. For the mobilized value, there exists a "worst case" scale of fluctuation producing the least conservative design scenario as summarized in Table 4. This scenario results in a mean capacity lower than the one produced by homogenizing the spatially variable property at its mean value. In the absence of sufficient site data to estimate the scale of fluctuation, the recommendation is to: (a) pick a typical value from Table 3 or (b) adopt the "worst case" scale in Table 4, rather than to assume an arbitrary value.

3. Geologic uncertainty

As discussed in the previous section, spatial variability exists in natural deposits or formations because properties and/or geometric features are spatially heterogeneous. This section specifically focuses on the uncertainty arising from structural and geometrical features in subsurface modelling (rather than properties), and the uncertainty is defined as "geologic uncertainty" in this paper.

Bárdossy and Fodor (2004) used the terms "structured" and "unstructured" to specify the two types of uncertainties in geologic modelling. The "structured" variability means more or less regular spatial changes that can be described by a trend-surface-analysis, and the examples include gradual compositional transitions of one strata/rock into the other, or cyclic repetitions of sedimentary features in a sequence of layers. On the other hand, "unstructured" variability may occur unexpectedly in a stratum and their spatial position and/or magnitude cannot be exactly predicted. They appear in the trend-surface-analysis as residuals and outliers. An example of geologic uncertainty in the form of soil stratification generated by a coupled Markov Chain model (e.g., Krumbein 1967) is illustrated in Fig. 2. The uncertainty in soil stratification is one form of geologic uncertainty that has attracted attention recently (Phoon et al. 2022a). While the spatial variability of soil properties is considered as "geotechnical uncertainty", Juang et al. (2019) noted that the uncertainty associated with the slope angle, thickness of soil layer above base rock, and the ground water level are forms of the geologic uncertainty for an infinite slope problem. The geotechnical reliability literature has focused primarily on geotechnical uncertainty.

Table 2
Common autocorrelation models in geotechnical engineering (updated Cami et al. 2020; Cami et al. 2021).

Autocorrelation model	**Correlation as a function of lag τ**	**Smoothness ν**	**Frequency of usage**
Single exponential (SExp)	$\rho(\tau) = \exp\{\frac{-2\|\tau\|}{\theta}\}$	0.5	48%
Second-order Markov (SMK)	$\rho(\tau) = (1 + 4\frac{\|\tau\|}{\theta}) exp\{-4\frac{\|\tau\|}{\theta}\}$	1.5	5%
Third-order Markov (TMK)	$\rho(\tau) = (1 + \frac{16}{3}\frac{\|\tau\|}{\theta} + \frac{256}{27}(\frac{\|\tau\|}{\theta})^2)\exp\{-\frac{16}{3}\frac{\|\tau\|}{\theta}\}$	2.5	New to geotechnical practice
Squared exponential (QExp)	$\rho(\tau) = \exp\{-\pi(\frac{\|\tau\|}{\theta})^2\}$	∞ (≈WM with $\nu > 3.5$)	19%
Spherical (Sph)	$\rho(\tau) = \begin{cases} 1 - \frac{9}{8}\|\frac{\tau}{\theta}\| + \frac{27}{128}\|\frac{\tau}{\theta}\|^3, \text{if}\|\tau\| \le \frac{4}{3}\theta; \\ 0, \text{otherwise} \end{cases}$	Outside WM family	7%
Cosine exponential (CosExp)	$\rho(\tau) = \exp\{-\frac{\|\tau\|}{\theta}\}\cos\{\frac{\|\tau\|}{\theta}\}$	0.5	8%
Binary noise (BN)	$\rho(\tau) = \begin{cases} 1 - \|\tau\|/\theta, \text{if}\|\tau\| \le \theta \\ 0, \text{otherwise} \end{cases}$	Outside WM family	12%
Whittle-Mate´rn (WM)	$\rho(\tau) = \frac{2}{\Gamma(\nu)}\left\{\frac{\sqrt{\pi}\Gamma(\nu+0.5)\|\tau\|}{\Gamma(\nu)\theta}\right\}^{\nu} K_{\nu}\left\{\frac{\sqrt{\pi}\Gamma(\nu+0.5)\|\tau\|}{\Gamma(\nu)\theta}\right\}$	All ν	New to geotechnical practice
Cosine Whittle-Mate´rn (CosWM)	$\rho(\tau) = \frac{2^{1-\nu}}{\Gamma(\nu)}\left(\frac{\sqrt{2\nu}\|\tau\|}{s}\right)^{\nu} K_{\nu}\left(\frac{\sqrt{2\nu}\|\tau\|}{s}\right)\cos\left(\frac{\tau}{b}\right)$	All ν	New to geotechnical practice

Note: θ = scale of fluctuation; ν = smoothness parameter that reduces the Whittle-Matérn model to a specific one-parameter autocorrelation model (e.g. $\nu = 0.5$ produces the Markovian exponential model); Γ = gamma function; and K_{ν} = modified Bessel function of second kind with order ν; θ is implicit function of s (scale parameter), ν, and b (hole parameter) for cosine Whittle-Mate´rn model; cosine exponential model can be regarded as a special case of cosine Whittle-Mate´rn model with $\nu = 0.5$ and a certain fixed relationship between s and b.

Table 3
Typical values for the vertical and horizontal scale of fluctuation (Cami et al. 2020).

Soil type	Scale of fluctuation (m)							
	Horizontal				Vertical			
	No. studies	Min	Max	Average	No. studies	Min	Max	Average
Alluvial	9	1.07	49	14.2	13	0.07	1.1	0.36
Ankara Clay	-	-	-	-	4	1	6.2	3.63
Chicago Clay	-	-	-	-	2	0.79	1.25	0.91
Clay	9	0.14	163.8	31.9	16	0.05	3.62	1.29
Clay, Sand, Silt mix	13	1.2	1000	201.5	28	0.06	21	1.58
Hangzhou Clay	2	40.4	45.4	42.9	4	0.49	0.77	0.63
Marine Clay	8	8.37	66	30.9	9	0.11	6.1	1.55
Marine Sand	1	15	15	15	5	0.07	7.2	1.43
Offshore Soil	1	24.6	66.5	45.6	2	0.48	1.62	1.04
Over Consolidated Clay	1	0.14	0.14	0.14	2	0.063	0.255	0.15
Sand	9	1.69	80	24.5	14	0.1	4	1.17
Sensitive Clay	-	-	-	-	2	1.1	2.0	1.55
Silt	3	12.7	45.5	33.2	5	0.14	7.19	2.08
Silty Clay	7	9.65	45.4	29.8	14	0.095	6.47	1.40
Soft Clay	3	22.2	80	47.6	8	0.14	6.2	1.70
Undrained Engineered soil	-	-	-	-	22	0.3	2.7	1.42
Water Content	9	2.8	22.2	12.9	8	0.05	6.2	1.70

In practice, deterministic methods that rely on engineering judgement or experience are widely used for structural modelling because they are somewhat intuitive and simple to calculate. However, relying solely on classical simple profiles can cause potential problems (Griffiths et al. 2012; Li et al. 2014). A deterministic work around without explicit uncertainty quantification is to conduct "what if" parametric studies, but this approach may not be satisfactory (Bárdossy and Fodor 2001). A natural data-driven approach that can exploit valuable data more fully and deal with data constraints such as sparsity consistently is the probabilistic method. Phoon et al. (2019) presented a useful mnemonic, MUSIC-X (Multivariate, Uncertain and Unique, Sparse, Incomplete, and potentially Corrupted with "X" denoting the spatial/temporal dimension) to highlight seven common attributes in real site data. It is clear that deterministic methods cannot deal with real world "ugly" data effectively (Phoon et al. 2022a).

The most well-known method for uncertainty quantification in structural modelling is kriging (e.g., Matheron 1963; Journel and Huijbregts 1978). Several types of kriging estimators such as block kriging, co-kriging, and universal kriging, have been developed and used in many applications (Wackernagel 2003; Li and Heap 2008). Kriging-based geostatistical simulations, however, are not able to evaluate complex and heterogeneous patterns due to the reliance on the variogram (covariance). To overcome this limitation, multiple point geostatistics (MPS) simulations was proposed by Guardiano and Srivastava (1993) and Journel (1993). MPS simulation algorithms borrow high order statistics from a visually and statistically explicit model, which is called a "training image (TI)". The training image can be seen as a prior model, and the prior is updated based on local observation data. Another research direction on geologic uncertainty is to quantify the errors in the engineer's/geologist's interpretations (MacCormack and Eyles 2012; Bond 2015; Lark et al. 2014; Randle et al. 2018).

The above-mentioned studies on geologic uncertainty provide a balance to past emphasis on spatial variabilities of soil properties. However, there are still several impediments to the broader use of probabilistic methods. One important impediment is that engineers desire "solution, not data" and "information in understandable form" (Turner 2003). This observation implies engineers prefer a deterministic approach to a probabilistic approach. Turner (2006) defined "thick" or "thin" clients in terms of their information acceptance capabilities as shown in Fig. 3. A "thick" client can accept and interpret or evaluate a great deal of raw data. In contrast, thin clients desire a relatively simple, concise answer to their questions. Small volumes or carefully selected data or information usually suffice to meet their needs. The important fact is that the "thin" clients are very numerous while the thick clients are much less numerous.

To mitigate this impediment, the value of considering geologic uncertainties in design practice should be explained to the engineers. For example, Yeh et al. (2021) studied the benefit of reducing the geological uncertainty in practice by re-analysing an actual landslide occurred at a -freeway in Northern Taiwan. A wider dissemination of such case studies would encourage the wider use of probabilistic methods in decision making in subsurface structural modelling.

4. Reliability analysis and design

4.1. Reliability analysis

Geotechnical reliability analysis aims to evaluate the probability of failure (P_f) of the geotechnical system given

Table 4
Examples of "worst-case" scale of fluctuations reported in previous studies (updated from Ching et al. 2017; Vessia et al. 2021).

Study	**Problem type**	**"Worst-case" definition**	**Characteristic length**	**"Worst-case" scale of fluctuation**
Fenton and Griffiths (2003)	Bearing capacity of a footing on a c-ϕ soil	Mean bearing capacity is minimal	Footing width (B)	1 × B
Griffiths et al. (2006)	Bearing capacity of footing(s) on a ϕ = 0 soil	Mean bearing capacity is minimal	Footing width (B)	0.5 ~ 2 × B
Vessia et al. (2009)	Bearing capacity of footing on c-ϕ soil	Mean bearing capacity is minimal (anisotropic 2D variability)	Footing width (B)	0.3 ~ 0.5 × B
Fenton and Griffiths (2005)	Differential settlement of footings	Under-design probability is maximal	Footing spacing (S)	1 × S
Breysse et al. (2005)	Settlement of a footing system	Footing rotation is maximal	Footing spacing (S)	0.5 × S
		Mean different settlement between footings is maximal	Footing spacing (S); Footing width (B)	f(S,B) (no simple equation)
Jaksa et al. (2005)	Settlement of a nine-pad footing system	Under-design probability is maximal	Footing spacing (S)	1 × S
Ahmed and Soubra (2014)	Differential settlement of footings	Under-design probability is maximal	Footing spacing (S)	1 × S
Stuedlein and Bong (2017)	Differential settlement of footings	Under-design probability is maximal	Footing spacing (S)	1 × S
Ali et al. (2014)	Risk of infinite slope	Risk of rainfall induced slope failure is maximal	Slope height (H)	1 × H
Hu and Ching (2015)	Active lateral force for a retaining wall	Mean active lateral force is maximal	Wall height (H)	0.2 × H
Fenton et al. (2005)	Active lateral force for a retaining wall	Under-design probability is maximal	Wall height (H)	0.5 ~ 1 × H
Griffiths et al. (2008)	Passive lateral force for a retaining wall	Under-design probability is maximal	Wall height (H)	0.1 to 0.5 × H
Ching and Phoon (2013)	Overall strength of a soil column	Mean strength is Minimal	Column width (W)	1× W (compression) 0× W (simple shear)
Pan et al. (2018)	Stress–strain behaviour of cement-treated clay column	Peak global strength	Column diameter (D)	2 × D

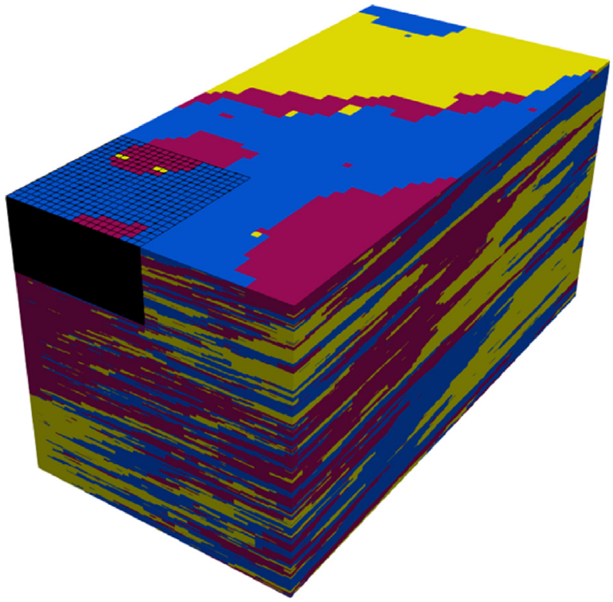

Fig. 2. Soil stratification generated by a coupled Markov Chain model.

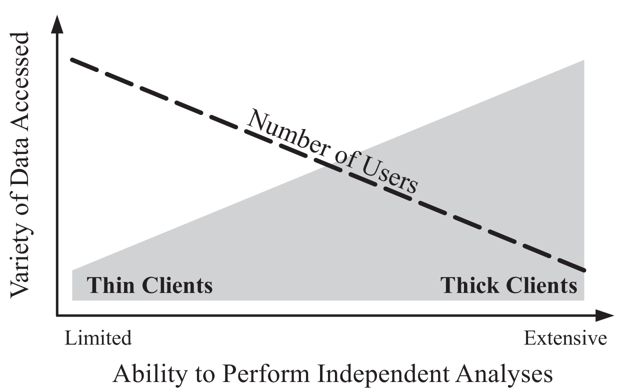

Fig. 3. Concept of thick and thin clients. Adopted from Turner (2006).

that all uncertain parameters of interest (random variables) $\mathbf{x} = [x_1, x_2, \cdots, x_M]$ are properly characterized with their statistical distributions. Mathematically, P_f can be calculated as.

$$P_f = \int \dots \int_{g(\mathbf{x})<0} f(x_1, x_2, \dots, x_M)\mathrm{d}x_1\mathrm{d}x_2\dots\mathrm{d}x_M \tag{1}$$

where $f(x_1, x_2, \dots, x_M)$ is the joint PDF of $\mathbf{x}$, $g(\mathbf{x})$ is the performance function and $g(\mathbf{x}) < 0$ denotes the failure region for which the probability integration is conducted. Note that $g(\mathbf{x}) = 0$ is the limit state that is already widely applied in practice. In most realistic cases, the closed-form solution to Equation (1) is not available. In the literature, analytical approximation methods and Monte Carlo simulation (MCS) methods have been developed to replace this direct integral solution (Shinozuka 1983), depending on the complexity of $g(\mathbf{x})$. Hence, the key output from a reliability analysis is the probability of failure, where "failure" is defined as any condition that does not meet a performance criterion or a set of criteria. In contrast to the global factor of safety, the probability of failure respects both mechanics and statistics, is sensitive to data (thus opening one potential pathway to digital transformation), and it is meaningful for both system and component failures. Resilience engineering requires system level analysis. Geotechnical software can provide better decision support by computing the probability of failure/reliability index as one basic output in addition to stresses, strains, forces, and displacements.

When the performance function $g(\mathbf{x})$ is linear and uncertain parameters are normally distributed, $g(\mathbf{x})$ will also follow the normal distribution. Equation (1) can be evaluated without performing the integration. Let μ_g and σ_g denote the mean and the standard deviation of $g(\mathbf{x})$, respectively. Then, P_f can be calculated as follows.

$$P_f = \Phi(-\beta) \tag{2}$$

where $\beta = \mu_g/\sigma_g$, and Φ denotes the standard normal CDF. The reliability index β was originally proposed by Cornell (1969) for load and resistance reliability analysis, and was generalized to produce the mean value first order second moment method (MVFOSM). The MVFOSM was widely used as the basis for geotechnical reliability-based design because of its analytical simplicity (Christian et al. 1994; Duncan 2000). However, the MVFOSM is subjected to several limitations and, in particular, is sensitive to different but equivalent formulations of the performance function.

Hasofer and Lind (1974) introduced the *first-order reliability method* (FORM) for normal variables. In the space (coordinate system) of uncorrelated standard normal variables $\mathbf{u} = [u_1, u_2, \dots, u_M]^{\mathrm{T}}$, the *Hasofer-Lind reliability index* β_{HL} is defined as the minimum distance from the origin of the axes to the limit state surface/function (LSS, or LSF) (see Fig. 4):

$$\beta_{HL} = \sqrt{(\mathbf{u}_d)^{\mathrm{T}}(\mathbf{u}_d)} = \min_{g(\mathbf{u})=0} D = \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}} \tag{3}$$

where $\mathbf{u}_d$ denotes the minimum distance point on LSS in **u**-*space*, and it is called the *design point* or *checking point*. By statistical transformations (Rackwitz 2001), the Hasofer-Lind method can be extended to correlated nonnormal variables. Mathematically, the reliability index and design point can be solved by a constrained optimization algorithm and/or the Hasofer-Lind-Rackwitz-Fiessler (HLRF) recursive algorithm (Ji et al. 2019; Ji and Kodikara 2015; Liu and Der Kiureghian 1991; Rackwitz and Flessler 1978). Note that the analytical approximation methods can be combined with the Ditlevsen bound solution (Ditlevsen 1979) to calculate the P_f of geotechnical problems involving multiple failure modes.

MCS is the most widely-used simulation-based reliability analysis method, by which the P_f is estimated as:

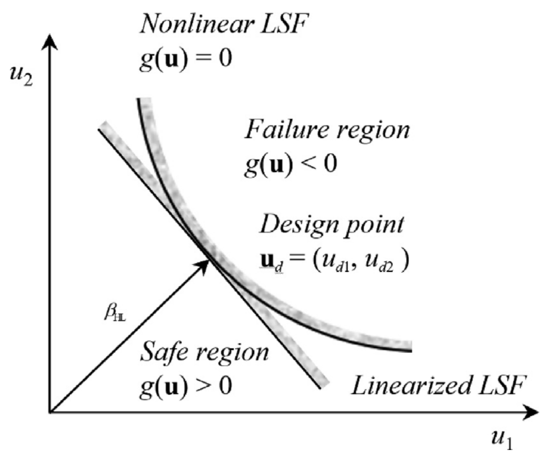

Fig. 4. Hasofer-Lind reliability index for nonlinear performance function.

$$P_f = \int_{g(\mathbf{x})<0} f(\mathbf{x})\mathrm{d}\mathbf{x} = \frac{1}{N}\sum_{k=1}^{N} I(\mathbf{x}^{(k)}) = \frac{N_f}{N} \tag{4}$$

where N is the total number of samples simulated from the statistical distribution of $\mathbf{x}$; N_f is the number of failure samples; $I(\mathbf{x}^{(k)})$ is an indicator function corresponding to the k-th sample $\mathbf{x}^{(k)}$. The $I(\mathbf{x}^{(k)}) = 1$ for $g(\mathbf{x}^{(k)}) < 0$, or $I(\mathbf{x}^{(k)}) = 0$ otherwise. However, direct MCS requires extensive computational efforts for geotechnical reliability problems with small failure probabilities (e.g., $P_f < 0.001$). For example, a total number of $100/P_f$ samples is needed to ensure that the coefficient of variation of P_f is <10 % (Baecher and Christian 2003; Zhang et al. 2021a). The k-th sample $\mathbf{x}^{(k)}$ refers to one specific design scenario, say one value of the undrained shear strength. The $I(\mathbf{x}^{(k)})$ value depends on the outcome of one deterministic finite element run (fails or does not fail). Hence, for $P_f = 0.001$, an engineer will need to execute 100,000 finite element runs. To improve the computational efficiency of MCS, a variety of variance reduction techniques are proposed and applied in geotechnical literature, such as the Latin hypercube sampling (LHS) (Baecher and Christian 2003), the importance sampling (IS) (Ching et al. 2009), the line sampling (LS) (Depina et al. 2016), the subset simulations (SS) (Wang and Cao 2013), the generalized SS (Gao et al. 2019), and the weighted uniform simulations (WUS) (Ji and Wang 2022), as summarized in Table 5.

For many geotechnical problems the performance function is not analytically available and various standalone commercial software are used to evaluate soil-structure interaction. For a standard commercial software that can perform a deterministic run one at a time, practical geotechnical reliability analysis still can be carried out using some techniques for dealing with implicit performance functions, e.g., the response surface method (RSM) (Li et al. 2016), and the FORM with HLRF recursive algorithms. Some geotechnical software have developed with embedded MCS features, such as the *PLAXIS/*

Table 5
Methods, implementations and applications of MCS-based geotechnical reliability analysis.

Methods		**Software implementations**	
Simulation algorithm	**Response surface**	**Probability and statistics**	**Geotechnical numerical simulations**
MCS SS LHS IS LS WUS	CPRS PCE Kriging NN SVM	Matlab Excel Python UQLAB COSSAN	Abaqus FLAC Plaxis GeoStudio Ansys Rocscience
MCS: slopes, retaining structures, foundations, tunnels **SS**: slopes, retaining structures, foundations, tunnels **LHS**: slopes, retaining structures, foundations **IS**: slopes, retaining structures, foundations **LS**: foundations **WUS**: slopes, foundations	**CPRS**: slopes, tunnels **PCE**: slopes,retaining structures, foundations, tunnels **Kriging**: slopes, foundations, **NN**: slopes, retaining structures, foundations, **SVM**: slopes, retaining structures, foundations, tunnels	**Matlab**: slopes, retaining structures, foundations, tunnels **Excel**: slopes, retaining structures, foundations, tunnels **Python**: slopes, **UQLAB**: slopes, foundations, tunnels **COSSAN**: foundations	**ABAQUS**: slopes, foundations, tunnels **FLAC**: slopes, retaining structures, foundations, tunnels **PLAXIS**: slopes, retaining structures, foundations, **GeoStudio**: slopes, **ANSYS**: slopes, tunnels **Rocscience** **RS2**: slopes, retaining structures, foundations, tunnels **Slide2 & Slide3**: 2D & 3D slopes

Note: (a) MCS, Routine Monte Carlo simulation; LHS, Latin hypercube sampling; LS, line sampling; IS, importance sampling; SS, subset simulation; WUS, weighted uniform simulation.
(b) CPRS, classical polynomial response surfaces; PCE, polynomial chaos expansions, SVM, support vector machines; NN, neural networks.
(c) UQLAB (https://www.uqlab.com/); COSSAN (https://cossan.co.uk/); Excel (https://www.microsoft.com/); Matlab (https://ww2.mathworks.cn/); Python (https://www.python.org/).
(d) ABAQUS (https://www.abaqus.com/); ANSYS (https://www.ansys.com/); FLAC (https://www.itascacg.com/); Geostudio (https://www.geoslope.com/); PLAXIS (https://www.bentley.com/); Rocscience (https://www.Rocscience.com).

LE (earlier known to be *SOILVISION/SVSLOPE*), *Geo-slope/W*, *Rocscience* (*Slide2&3*, *RS2*), and $OPTUM^{G2}$, etc. Note that without going through MCS the *PLAXIS/LE* also enables Alternate Point Estimate Methods (APEM) that is based on FOSM, and the *Rocscience (RS2)* enables two-point Point Estimate Method (PEM) for handling geotechnical reliability analysis when the computational cost dominates. In addition, Low and Tang (2007) showed that the Solver, i.e., the optimization tool embedded in Microsoft Excel, is a powerful tool for solving the aforementioned constraint optimization problem, and the application of this user-friendly EXCEL platform to various practical geotechnical applications have been illustrated by Low (2021). The geotechnical reliability analysis in combination with various standalone software implementations has been garnering increasing interests in recent years. Typical geotechnical reliability applications based on commercial software are summarized in Table 5.

4.2. Reliability-based design

Reliability-based design (RBD) is a rational methodology that allows practitioners to design geotechnical elements to achieve a target probability of failure. As such, RBD is considered as the inverse problem of reliability analysis. Reliability analysis computes the probability of failure or reliability index for a given design. RBD searches for a design to produce a target reliability index. Achieving a target reliability index could be done through different simplified methods including: (1) load and resistance factor design (LRFD) format, (2) full RBD format, and (3) expanded or robust RBD formats.

The LRFD format is a simplified RBD approach that allows the designer to achieve the target reliability by adopting properly calibrated load and resistance factors. The application of RBD in geotechnical engineering is traced back to the nineties through the works of Barker et al. (1991) and Phoon et al. (1995). One of the first design codes that adopted the LRFD approach for geotechnical design is the AASHTO LRFD Bridge Design Specification. In AASHTO (2020), the code recommends resistance factors for the ultimate limit state for (1) shallow foundations (Table 10.5.5.2.2-1), (2) driven piles (Table 10.5.5.2.3-1), (3) drilled shafts (Table 10.5.5.2.4-1), (4) micropiles (Table 10.5.5.2.5-1), (5) retaining walls (Table 11.5.7-1), and (6) overall stability of slopes (Article 11.6.3.7). A compilation of some resistance factors in AASHTO (2020) are presented in Table 6. These factors were derived to achieve a target reliability index, β, of 3.5 which corresponds to an approximate probability of failure of 1 in 5,000.

Two important limitations that heavily influenced the LRFD calibration in its geotechnical version are:

1. The inability to maintain a relatively uniform level of reliability over the wide range of COVs that are typically encountered in soil and rock properties (refer to Section 2.1).
2. Lumping all sources of uncertainty in the geotechnical resistance in one bias factor that is generally highly sensitive to geometry, soil properties, and model used to predict the resistance.

The Canadian Highway Bridge Design Code (CSA 2019) realized the importance of maintaining a uniform level of reliability by adopting resistance factors (see Table 7) that depend on the "degree of understanding" of the geotechnical properties and the confidence of the designer in the design model (Fenton et al. 2016; Phoon 2017). The Multiple Load and Resistance Factor Design (MRFD) approach and the Quantile Value Method (QVM) are two creative methods that were proposed by Phoon et al. (2003) and Ching and Phoon (2011) to resolve the inability of simplified RBD to maintain a uniform level of reliability. MRFD allows for utilizing multiple resistance factors to ensure a uniform target reliability level across the whole domain. The Quantile Value Method (QVM) allows for maintaining a uniform level of reliability over a wide range of COVs without changing the resistance factor by adopting "quantile" design values in the simplified RBD problem. Ching et al. (2015) further showed that the reliability uniformity of QVM can be further improved by considering the concept of "effective random dimension" (ERD), which quantifies the degree of redundancy of a geotechnical structure.

Full RBD methods are defined as methods where the designer models all sources of uncertainty in the load and resistance and conducts a reliability analysis to quantify the probability of failure of a given design. If the design does not produce the target reliability level, the designer will repeat the reliability analysis with different design parameters until the target reliability is achieved. In its simplest form, full RBD can be used to back-calculate a "required factor of safety" for a problem-specific target reliability index that is appropriate to the importance of the structure under consideration. This approach was used in Najjar and Gilbert (2009), Bou Diab et al. (2018), Kahiel et al. (2017), Najjar et al. (2017), and Najjar et al. (2020) among others for driven piles, footings on fiber-reinforced clay, footings on aggregate piers, design of pile load test programs, and undrained slopes.

Two separate extensions/developments to the full RBD method were introduced in the last decade by Wang and Cao (2013) and Juang et al. (2013) through the "expanded" RBD method and the "robust" RBD method, respectively. In the expanded full RBD approach, basic design parameters, such as diameter (D) and length (L) of a pile, are formulated artificially as discrete uniform random variables and the design process becomes one in which failure probabilities are developed for various combinations of diameter and length conditional on achieving the target probability of failure using MCS. Feasible designs that satisfy the target reliability levels are defined, and the design with the minimum construction cost is selected as the final design. Cao et al. (2019b) provided a summary of MCS-

Table 6
Sample resistance factors as recommended in AASHTO (2020).

Method/Soil/Condition			Resistance Factor
Shallow Foundations	Bearing Resistance	Theoretical bearing resistance	0.5
		Theoretical method (Munfakh et al., 2001), in sand using CPT	0.5
		Theoretical method (Munfakh et al., 2001), in sand using SPT	0.45
		Semi-empirical methods (Meyerhof, 1957), all soils	0.45
		Footings on rock	0.45
		Plate load test	0.55
	Sliding	Pre-cast concrete placed on sand	0.9
		Cast-in-place concrete on sand	0.8
		Cast-in-place or precast concrete on clay	0.85
		Soil on soil	0.9
		Passive earth pressure component of sliding resistance	0.5
Driven Piles	Nominal Bearing Resistance of Single Pile—Static Analysis Methods	*Side resistance and end bearing: clay and mixed soils*	
		α-method (Tomlinson, 1987; Skempton, 1951)	0.35
		β-method (Esrig & Kirby, 1979; Skempton, 1951)	0.25
		λ-method (Vijayvergiya & Focht, 1972)	0.40
		Side resistance and end bearing: sand	
		Nordlund/Thurman method (Hannigan et al., 2005)	0.45
		SPT-method (Meyerhof)	0.30
		CPT-method (Schmertmann)	0.50
		End bearing in rock (Canadian Geotech. Society, 1985)	0.45
	Uplift Resistance of Single Piles	Nordlund method	0.35
		α-method	0.25
		β-method	0.20
		λ-method	0.30
		SPT-method	0.25
		CPT-method	0.40
		Static load test	0.60
		Dynamic test with signal matching	0.50
Drilled Shafts (single compressive)	Side resistance clay	α-method (Brown et al., 2010)	0.45
	Tip resistance clay	Total stress (Brown et al., 2010)	0.4
	Side resistance sand	σ-method (Brown et al., 2010)	0.55
	Tip resistance sand	Brown et al. (2010)	0.5
	Side resistance IGM	Brown et al. (2010)	0.6
	Tip resistance IGM	Brown et al. (2010)	0.55
	Side resistance rock	Kulhawy et al. (2005), Brown et al. (2010)	0.55
	Side resistance rock	Carter and Kulhawy (1988)	0.5
	Tip resistance rock	Canadian Geotechnical Society and Brown et al. 1985	0.5
Drilled Shafts (uplift)	Clay	α-method (Brown et al., 2010)	0.35
	Sand	β-method (Brown et al., 2010)	0.45
	Rock	Kulhawy et al. (2005), Brown et al. (2010)	0.4
Drilled Shaft Static Test	All soils	Compression	0.7
	All soils	Uplift	0.6

based full RBD methods and their implementation in spreadsheets, and highlighted the value of MCS samples for RBD. On the other hand, the essence of the robust RBD as presented by Juang et al. (2013) is to minimize the variation of the probability of failure caused by the uncertainty in the estimated sample statistics of soil parameters by adjusting the design parameters of the problem under consideration. For the case of a drilled shaft example, the diameter or length is selected so as to increase the robustness of the design. Robustness measures may include the standard deviation of the probability of failure, which together with the cost of the foundation, could be considered as a design objectives. The best design is selected based on a tradeoff relationship between cost and robustness.

The main drawback of full RBD methods, expanded RBD methods, and robust RBD methods is that, unlike simplified RBD methods, they are computationally demanding, theoretically involved, and require adequate knowledge of reliability theory. It is however expected that recent advances in computational power, increasing level of literacy of practicing civil engineers in concepts of probability and statistics, and the increased availability of simple probabilistic tools (spreadsheets, software, or even apps) may pave the way for a wider adoption of full RBD method in geotechnical design practice.

5. Inverse analysis

For geotechnical practitioners, inverse analysis often means "back calculation" that is done for forensic analysis

Table 7
Geotechnical resistance factors in the 2019 Canadian Highway Bridge Code.

Application	Limit State	Test Method/Model	Degree of understanding		
			Low	Typical	High
Shallow foundations	Bearing, φ_{gu}	Analysis	0.45	0.50	0.60
		Scale model test	0.50	0.55	0.65
	Sliding, φ_{gu}	Analysis	0.70	0.80	0.90
	Frictional	Scale model test	0.75	0.85	0.95
	Sliding, φ_{gu}	Analysis	0.55	0.60	0.65
	Cohesive	Scale model test	0.60	0.65	0.70
	Passive resistance, φ_{gu}	Analysis	0.40	0.50	0.55
	Settlement or lateral movement, φ_{gs}	Analysis	0.70	0.80	0.90
		Scale model test	0.80	0.90	1.00
Deep foundations	Compression, φ_{gu}	Static analysis	0.35	0.40	0.45
		Static test	0.50	0.60	0.70
		Dynamic analysis	0.35	0.40	0.45
		Dynamic test	0.45	0.50	0.55
	Tension*, φ_{gu}	Static analysis	0.20	0.30	0.40
		Static test	0.40	0.50	0.60
	Lateral, φ_{gu}	Static analysis	0.45	0.50	0.55
		Static test	0.45	0.50	0.55
	Settlement or lateral deflection, φ_{gs}	Static analysis	0.70	0.80	0.90
		Static test	0.80	0.90	1.00
Ground Anchors	Pull-out, φ_{gu}	Analysis	0.35	0.40	0.50
		Test	0.55	0.60	0.65
Internal MSE reinforcement	Rupture, φ_{gu}	Steel strip	0.65	0.70	0.75
		Steel grid	0.55	0.60	0.65
		Geosynthetic	0.80	0.85	0.90
	Pull-out, φ_{gu}	Analysis	0.80	0.85	0.90
Retaining systems	Bearing, φ_{gu}	Analysis	0.45	0.50	0.60
	Overturning†, φ_{gu}	Analysis	0.45	0.50	0.55
	Base sliding, φ_{gu}	Analysis	0.70	0.80	0.90
	Facing interface sliding, φ_{gu}	Test	0.75	0.85	0.95
	Connections, φ_{gu}	Test	0.65	0.70	0.75
	Settlement, φ_{gs}	Analysis	0.70	0.80	0.90
	Deflection/tilt, φ_{gs}	Analysis	0.70	0.80	0.90
Embankments (fill)	Bearing, φ_{gu}	Analysis	0.45	0.50	0.60
	Sliding, φ_{gu}	Analysis	0.70	0.80	0.90
	Global stability-temporary, φ_{gu}	Analysis	0.70	0.75	0.80
	Global stability-permanent, φ_{gu}	Analysis	0.60	0.65	0.70
	Settlement, φ_{gs}	Analysis	0.70	0.80	0.90
		Test	0.80	0.90	1.00

* Where maximum frost penetration depth is used, a geotechnical resistance factor of 1.0 shall be used to calculate tensile resistance to frost uplift.
† Does not apply to MSE walls.

of slope failure (e.g., Kool et al. 2019). The engineer estimates strength parameters of slope materials that reasonably explain "why/how did that failure happen" through trial-and-error analysis. The task of inferring model parameters and/or boundary conditions based on the observation data is called inverse analysis. In geotechnical engineering, inverse analysis usually means parameter identification. It clearly plays an important role in Bayesian observational approach and one can imagine novel applications in the presence of IoT. This section reviews methods for parameter identification using probability theory.

Bayesian model inference provides a useful framework with probability theory for inverse analysis. Prior knowledge on parameter of interest $\mathbf{x}$ can be updated with observed data $\mathbf{y}$ using Bayes' theorem. Typically, prior information is modeled as a Gaussian distribution, $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{w}$, where $\bar{\mathbf{x}}$ and $\mathbf{w}$ are the mean and the probabilistic component, respectively. The observation equation is generally expressed as a nonlinear function of $\mathbf{x}$ contaminated with Gaussian noise $\mathbf{v}$, i.e., $\mathbf{y} = \mathbf{h}(\mathbf{x}) + \mathbf{v}$, where $\mathbf{w}$ and $\mathbf{v}$ are Gaussian random variable vectors with zero mean whose covariance matrices are $\mathbf{M}$ and $\mathbf{R}$, respectively. The solution that maximizes posterior probability density function (PDF) $p(\mathbf{x}|\mathbf{y})$ is called maximum a posteriori (MAP) estimate which minimize the following objective function J.

$$J = \frac{1}{2}(\mathbf{y} - \mathbf{h}(\mathbf{x}))^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y} - \mathbf{h}(\mathbf{x})) + \frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})^{\mathrm{T}}\mathbf{M}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) \tag{5}$$

Assume that random variable $\mathbf{w}$ in the prior follows Laplace distribution instead of Gaussian. This results in the following objective function.

$$J = (\mathbf{y} - \mathbf{h}(\mathbf{x}))^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y} - \mathbf{h}(\mathbf{x})) + \frac{2}{b}\sum_{i=1}^{n}|x_i - \bar{x}_i| \quad (6)$$

This objective function is known as least absolute shrinkage and selection operator (LASSO) (Bishop 2006; Hastie et al. 2015).

Sample-based method can estimate not only MAP but also its uncertainty which is important for the decision-making. Data assimilation by Particle filter (PF) (e.g., Ristic et al. 2004) has attracted attention for updating model parameters in many research fields. In geotechnical engineering, PF was applied to the long-term consolidation settlement predictions (Shibata et al. 2019; Yoshida and Shuku 2020). Straub and Papaioannou (2015) proposed Bayesian Updating with Structural Reliability Methods (BUS), which converts a Bayesian updating problem into an equivalent reliability analysis problem. For efficiency, subset simulation (SuS) (Au and Wang 2014) is often combined with BUS (DiazDelaO et al. 2017; Betz et al. 2018). BUS has been applied to updating of soil stratigraphy from CPT data (Cao et al. 2019a) and estimation of spatial variability of soil properties (Jiang et al. 2018; 2021). Markov chain Monte Carlo is also a widely used tool to generate samples from posterior PDF. However, sampling directly from the target PDF is difficult and generally not practical if the prior and posterior PDF are significantly different. In order to overcome this, Transitional Markov chain Monte Carlo (TMCMC) was proposed by Ching and Chen (2007). Betz et al. (2016) discussed the properties of TMCMC and proposed modifications to improve efficiency. See, e.g., Angelikopoulos et al. (2015), Wu et al. (2018) for other recent development.

Surrogate model (*meta*-model) is an attractive tool to reduce the number of costly function call (e.g., calculation by 3D finite element method). Teixeira et al. (2021) and Moustapha et al. (2022) reviewed the implementation of adaptive surrogate model for reliability analysis. The surrogate model approach can be also applied to Bayesian updating of model parameters. Giovanis et al. (2017) incorporated artificial neural networks into BUS with SuS as the surrogate model, while Wang and Shafieezadeh (2020), Kitahara et al. (2021), and Liu et al. (2022) incorporated adaptive Kriging. Ni et al. (2021) proposed a variational Bayesian inference approach by using a surrogate model of an adaptive Gaussian process modeling. Song et al. (2022) combined the BUS, adaptive importance sampling and active learning Kriging surrogate model for Bayesian updating.

Table 8 summarizes the methods for inverse problem. When the observation equation is linear, and its noise and the prior are Gaussian, an analytical solution exists. Kalman filter and Gaussian process regression are classified into this category (first row of the table). The second row indicates the inverse analysis with l_1 norm regularization/Laplace prior, which is known as LASSO. The third row indicates the inverse problem with nonlinear observation equation, and/or non-Gaussian noise and/or prior. Many real world inverse problems fall into this category which requires a numerical method to obtain the solutions. The sample-based methods with adaptive surrogate model seems promising, although they need further investigation to clarify its applicability and limitations.

6. Soil-structure interaction in spatially variable soils

To study the propagation of uncertainty in soil and rock properties to the system performance, a logical step beyond the deterministic approach involves probabilistic analyses that model the properties of 'homogenized' material layers as random variables. Uncertainty in material parameters can then be considered either via the Monte Carlo fashion, or through FOSM or FORM discussed earlier (e.g., Vanmarcke 1977b; Li & Lumb 1987; Christian et al. 1994). In early studies where computational power could be a bottleneck for random field simulations, the effects of spatial variability were considered using the homogenized layers and variance reduction factors.

However, spatially variable ground properties can lead to failure mechanisms that cannot be captured by simulations assuming uniform or homogenized geomaterial properties. Kim and Santamarina (2008) showed that at element testing level, spatial variability features could cause strain localization, local drainage and hence affect shear strength development. These are manifested in various ways in soil-structure interaction problems with different stress paths. Considering a foundation with symmetric geometry and load conditions, numerical analyses with homogenized soil properties would yield deformation profiles that are symmetrical. In reality, soil properties vary spatially within the same layer, so foundation tilting or differential settlements among footings may arise. This section describes these aspects of soil-structure interaction problems and summarizes recent findings on the worst-case scales of fluctuation (Vessia et al. 2021) that constitute the most critical scenario for geotechnical problems.

6.1. Shallow footings and piled foundations

Using random field models, Griffiths et al. (2002) and Fenton and Griffiths (2003) investigated the behaviours of strip footings founded on spatially variable soils. For the ultimate limit state, they revealed that, on average, the bearing capacity would be smaller than the deterministic solution adopting mean values of shear strength parameters, while the failure surfaces deviate from classical solutions as soils fail along the "weakest path". For isotropic variability patterns, a pragmatic way to predict the statistics of bearing capacity involves utilizing Prandtl's solution together with the geometric average of soil properties over the domain of plastic failure region. Tabarroki et al. (2022a) calibrated a model called the weakest-path model by random finite element simulation results. The calibrated weakest-path model can capture the behavior of failure along the weakest path (e.g., the bearing capacity

Table 8
Classification of the inverse problem.

Observation equation	Observation noise	Regularization Prior information	Solution	Related method
Linear	l_2-norm Gauss	l_2-norm, Gauss	Analytical	regularized LS Kalman filter, GPR
Linear	l_2-norm Gauss	l_1-norm Laplace	Numerical ADMM, etc.,	LASSO (sparse modeling),
Non-Linear	l_2-norm Gauss (any)	l_1, l_2-norm Gauss, Laplace (any)	Gradient-based method such as GN, BFGS. Global Optimization method such as GA, PSO. Sample-based method such as PF, BUS, TMCMC	regularized nonlinear LS, surrogate model

ADMM: alternating direction method of multipliers, GA: Genetic Algorithm, GN: Gauss-Newton, GPR: Gaussian process regression, LS: least square, LASSO: least absolute shrinkage and selection operator, PF: particle filter, BUS: Bayesian Updating with Structural reliability methods, PSO: Particle Swarm Optimization, TMCMC: Transitional Markov chain Monte Carlo.

is, on average, smaller than the deterministic solution). For the serviceability limit state, Fenton and Griffiths (2002; 2005) further suggested that the statistics of footing settlements can be satisfactorily approximated by such geometric average. Another implication of spatial variability lies on the differential settlements between multiple footings, which had been studied by various researchers (Table 4) with the similar conclusions that the worst scenario entailed the θ value being close to the spacing between footings. Tabarroki et al. (2022b) showed that with the so-called pseudo incremental energy method, the numerical values of footing settlements can be satisfactorily approximated by a weighted geometric average, where the weights can be determined by a single determininsitc finite element analysis. Note that a close approximation in numerical values is stronger than that in statistics because two random variables can have identical statistics and yet be completely uncorrelated.

Deep foundations develop much of the resistance from the surrounding soils, where spatial variability can have substantial impacts on the performance. As soil-pile interactions are often modelled using spring stiffnesses and strengths (‘*t*-z’ curves) along the pile, a natural extension involves varying these parameters based on the random field theory (Fenton and Griffiths 2008). Later, Naghibi et al. (2014; 2016) studied the serviceability limit states of single and two-pile systems in spatially variable ground, by establishing theoretical solutions of the statistics of their settlements. Leung and Lo (2018) then extended the approach by Quek et al. (1991) to incorporate 3D random fields, and suggested that the worst case scenarios for differential settlements of large pile groups involve the horizontal θ_h being close to the foundation width.

6.2. Slopes and retaining structures

Conventional deterministic analyses of slopes evaluate the factor of safety (FOS), while probabilistic analyses produce the reliability index of the slope or its probability of failure. It may be tempting to assume the same critical slip surface in deterministic approach (minimum FOS) and probabilistic approach (maximum probability of failure). This is, however, generally not true unless the slope stability is dominated by particularly weak seams or planes. Griffiths and Fenton (2004) demonstrated the significance of soil spatial variability in slopes, showing that assumptions of homogenized layers could lead to unconservative reliability estimates, in cases of high variability and low mean strength in the soils. Recently, the significance of anisotropic and rotational features of spatial variability is being recognized. A high probability of failure is generally associated with slopes with dip directions aligned with the soil variation patterns. Three-dimensional effects of spatial variability also affect the failure mechanisms and sizes of failure mass (e.g. Griffiths et al. 2009; Huang and Leung 2021). In particular, Hicks et al. (2014) discussed various 3D failure modes depending on the horizontal scale of fluctuation θ_h, showing a tendency for discrete failures to occur as the slope length (e.g. along the longitudinal direction of an embankment) increases relative to θ_h.

Spatial variability in the ground influences retaining structures through the development of lateral earth pressures, as classical solutions do not account for the tendency of soils to fail along the weakest paths in heterogeneous media. Various researchers investigated the critical θ values for active and passive pressures (Table 4). Tabarroki et al. (2022a) showed that, however, the weakest-path seeking for retaining structures is not as significant as that for shallow footings, because the potential failure surfaces for retaining structures are more constrained. For deep excavations, the design considerations extend beyond active and passive pressures, as basal heave, wall deflections and internal forces often govern their performance. Lo and Leung (2019) discussed these aspects by demonstrating the effects of spatially variable ground properties on wall and ground displacements, while Luo et al. (2018; 2020) further analyzed the ensuing failure modes including excessive bending moments or shear forces in the wall and buckling of struts. These risks are often encountered in excavation projects but cannot be predicted by conventional approaches without considering spatial variability of the ground.

7. Model uncertainty

Notwithstanding the advancement of calculation capabilities (e.g., Gibson 1974; Poulos 1989; Potts and Zdravković 1999; 2001), our ability to predict actual behavior is imprecise in many cases because of model imperfection (e.g., D'Appolonia 1990; Focht 1994; Tang and Phoon 2021; Kalenchuk 2022). The predicted behavior will deviate from measurement. The deviation between measured and predicted response is the model uncertainty, which has a strong influence on the calculated probability of failure and thus on the estimation of safety margin (e.g., Tang et al. 1990; Gilbert and Tang 1995; Lacasse and Nadim 1996). The latest edition of ISO 2394 introduced a new Annex D, in which model uncertainty has been identified as one of the critical elements in the development of reliability-based design for geotechnical engineering (ISO 2015). In practice, model uncertainty can be characterized in a relatively straightforward way by using a model factor λ according to Eqn 7(ISO 2015):

$$\lambda = R_m / R_c \tag{7}$$

where R_m and R_c = measured and predicted response.

The response R could be a load, resistance, or displacement, etc. The model factor itself is not constant but takes a range of values that may depend on the scenarios covered in the dataset used for evaluation. It is customary to model λ as a random variable; however, the variation of λ sometimes is explainable by other known variables (e.g., geometry and property) (Tang and Phoon 2021). This could be due in part to the oversimplification of complex real world behavior. In this situation, the model factor can be expressed as $\lambda = \lambda_c \times \varepsilon$, where λ_c = correction factor to capture the variation of λ with each influential parameter and ε = residual that is no longer dependent on these parameters. The correction term λ_c can be applied to the simplified design method and improve its accuracy. Unfortunately, it is not an easy task to establish λ_c, as the test data usually cover a limited range of influential parameters. An alternative way is to use a mechanically consistent numerical method for λ_c that is a ratio of numerical prediction R_p and solution from a simplified design model R_c (Zhang et al. 2015). Because all practical scenarios can be simulated, the numerical results would be a beneficial supplement to the limited test data. As shown in Fig. 5, λ_c is characterized as the product of f (systematic variation) and η (residual), and ε is evaluated as the ratio of R_m and R_p that is the model factor of the adopted numerical method, which is likely to be random. The model uncertainty of the simplified design method is characterized by $\varepsilon' = \varepsilon \times \eta$. This framework is currently the best in terms of providing physical insights, correcting the bias in the original model, handling problems with highly sensitive input parameters and making efficient use of limited test data (e.g., Zhang et al. 2015; Tang and Phoon 2021).

A comprehensive review of model factor statistics was presented in Tang and Phoon (2021), covering a variety of geotechnical structures in wide range of material types. On this basis, a practical and informative three-tier scheme to classify model uncertainty by the model factor mean and COV was proposed (Fig. 6). For capacity, the mean is interpreted as "unconservative" when mean < 1, "moderately conservative" when mean = 1–3, and "highly conservative" when mean > 3. Many calculated models of capacity are moderately conservative. For displacement, the mean is interpreted as "unconservative" when mean > 1. The calculation models of displacement are typically conservative where model factor mean < 1. The dispersion of the model factor is classified as low (COV < 0.3), medium (COV = 0.3–0.6), and high (COV > 0.6). Otake and Honjo (2022) presented a comparison between generic and Japanese data on model factors of shallow and deep foundations with this classification scheme. The three-tier classification scheme may provide designers with an empirically grounded framework for developing resistance factors as a function of the degree of site/model understanding – a concept already adopted in Canadian Highway Bridge Design Code as shown in Table 7 (CSA 2019).

Advanced numerical analyses would be expected to perform better than simplified empirical and semi-empirical methods, because of the adoption of more realistic constitutive models and more consistent physics. Poulos et al. (2001) argued that the gap between theory and practice is larger than the last review conducted by Terzaghi in 1951. Whyte (2018) and Ramsey (2019) argued that theoretical studies were centred on the development of various "philosopher" constitutive models, but few efforts were undertaken to calibrate a robust model for practical engineering applications ("engineer" model). These studies pay less attention to the fact that the degree of complexity in the adopted numerical model must be justified by the quality and quantity of geomaterial and performance data available. In her 2019 Canadian Geotechnical Colloquium paper, Kalenchuk (2022) provided an in-depth discussion of practical limitations in the day-to-day application of numerical methods in geotechnical engineering. It is largely associated with practitioners' overconfidence of the ability of numerical tools to carry out sophisticated computations and disregarding (or lack of understanding) of the uncertainty in the results obtained. Abchir et al. (2016) and Briaud and Wang (2018) evaluated the model uncertainty of *t-z* and *p-y* methods – numerical analyses of the load and settlement of axially and laterally loaded piles, respectively. For settlement, the results appear to deviate from measurements by significant margins. Model calibration is the process of correlating the observations of actual

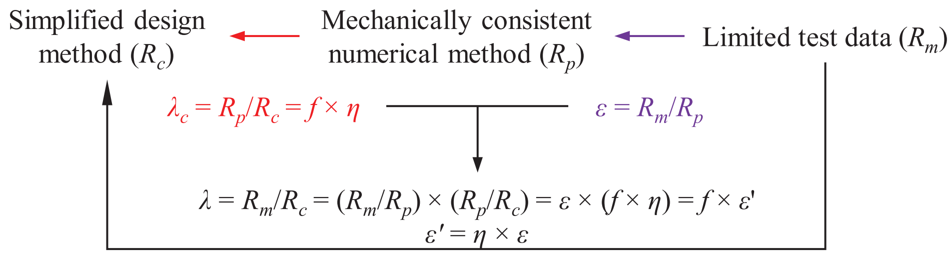

Fig. 5. Two-step procedure for model factor assessment with limited test data in the presence of statistical dependency (after Tang and Phoon 2021).

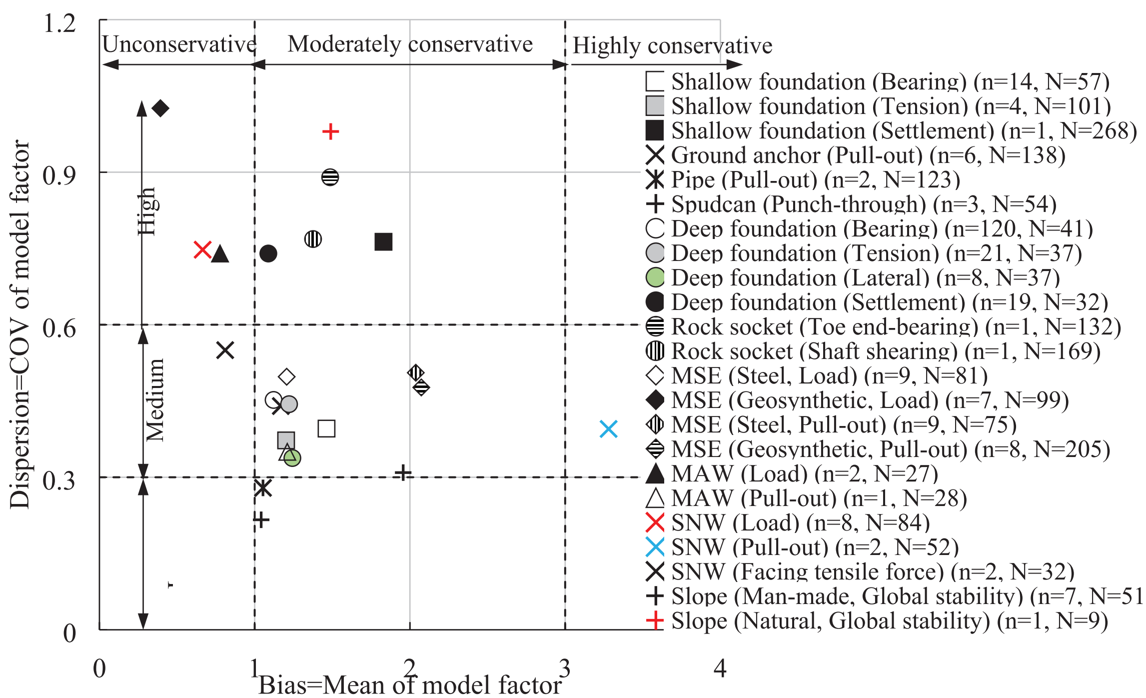

Fig. 6. Classification of model uncertainty based on model factor mean and COV for a variety of geotechnical structures, where n = number of data groups, N = number of load tests averaged over the n-groups, MSE = mechanically stabilized earth wall, MAW = multi-anchor wall, and SNW = soil nail wall (after Tang and Phoon 2021).

ground behavior to the numerical output. By doing so, the analysis/design models can be tailored to the problem and site of interest. This is an important step to reduce the bias and uncertainty in numerical analysis. For example, Jeanjean et al. (2017) and Zhang and Andersen (2017) proposed a framework for monotonic backbone *p-y* curves in cohesive materials. The bearing capacity factor is calculated from limit analysis and finite element analysis. The shape of the *p-y* curves is deduced from an extensive database of direct simple shear tests with calibration using finite element analysis. The proposed monotonic *p-y* curves were simultaneously modified to account for cyclic loading by Zhang et al. (2017) and were recently verified with centrifuge and field laterally loaded pile tests by Zhang et al. (2020a; b). Compared with current industry practice, such data-based and physics-informed *p-y* methods provided noticeable improvement in calculating pile response in very soft to stiff clay.

8. Conclusions

The purpose of numerical modeling is to enhance decision making in practice rather than to enhance our prediction ability. The Burland Triangle places “soil behaviour”, “ground profile”, and “modeling” at its three vertices, but situates “empiricism, precedent, experience, and risk management” at the core of decision making because geotechnical engineers must cope with uncertainties arising from the variable nature of soil and rock, changeable environmental conditions, and imprecision in predicting field performance from models. Prevailing practice does not consider these uncertainties explicitly and as such, risk as recognized by the Burland Triangle is managed informally through an ad-hoc combination of strategies that include applying a global factor of safety (or partial factors of safety), selecting cautious input values and conservative calculation models, conducting parametric studies, learning from precedents, updating/validating designs and construction procedures based on prototype testing and observations, and keeping engineering judgment as an integral part of the decision making loop. It is accurate to say that the value of applying probabilistic methods to manage risk more formally is not widely recognized.

The purpose of this review paper is to present probabilistic research conducted over the past several decades that can complement numerical analysis to support increasingly complex decision making in practice. Complexity is not related to the project details alone, but to system level issues (e.g., resilience) and to new design goals (e.g., sustainability). Currently, numerical analysis is conducted using deterministic inputs and simple soil profiles that are inconsistent with sparse site data. This deterministic mapping (one set of inputs to one set of outputs) does provide valuable physical insights, but cannot support risk-informed decision making on its own without appealing to engineering judgment. Ironically, the usefulness of numerical analysis to decision making and its role in digital transformation are diminished in the absence of explicit uncertainty quantification.

Statistical guidelines for modeling soil properties as random variables and random fields are available. The characterization of geologic uncertainties is more challenging than the aforementioned geotechnical uncertainties, but the field of data-driven site characterization (DDSC) is advancing rapidly using modern machine learning methods (Phoon et al. 2022a). Geotechnical models are biased and imprecise. It is possible to quantify the ratio of the measured response and the calculated response as a random variable called a model factor using extensive load test or other performance databases. The propagation of uncertainties from inputs to responses can be carried out more correctly than parametric studies (cannot address correlations between inputs, spatial variability, and statistical uncertainty) and more efficiently than Monte Carlo simulation. For design, it is more important to calculate the probability of a response not satisfying a given performance criterion or a set of criteria rather than to calculate the probability of achieving a general response. This is called the probability of failure. Decision making based on the probability of failure is more consistent than one based on the global factor of safety or partial factors of safety with respect to both mechanics and statistics. In addition, the probability of failure is sensitive to data (thus opening one potential pathway to digital transformation) and meaningful for both system and component failures. Resilience engineering requires system level analysis. Hence, to provide better decision support, geotechncical software should compute the probability of failure/reliability index as one basic output in addition to stresses, strains, forces, and displacements. However, for the engineer to understand why the probability of failure is high or low, the uncertainty of the explanatory variables/mechanisms should be provided in the form of simulated outcomes. These simulated outcomes display a more complete and more consistent picture of "what if" design scenarios than prevailing parametric studies. Simplified reliability-based design approaches that can produce solutions meeting a target probability of failure for different degrees of understanding of the site and structures of different importance have been developed and are already adopted in new design codes. This review paper further shows that probabilistic methods can lead to novel applications with potential to lead to digital transformation (data-driven site characterization, inverse analysis) and novel understanding of soil behaviours (non-classical failure mechanisms in spatially variable soils) outside the reach of deterministic methods.

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