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CLASSICAL PATHWISE SOLUTIONS TO NONLINEAR STOCHASTIC AGGREGATION-DIFFUSION EQUATIONS

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Abstract. It is well-known that solutions of deterministic nonlocal aggregation-diffusion models may blow up in two or higher dimensions. Various mechanisms hence have been proposed to "regularize" the deterministic aggregation-diffusion equations in a manner that allows pattern formation without blow-up. However, stochastic effect has not been ever considered among other things. In this work, we consider a nonlocal aggregation-diffusion model with multiplicative noise and establish the local existence and uniqueness of classical pathwise solutions on $\mathbb{R}^d (d \geq 2)$. If the noise is non-autonomous and linear, we establish the global existence and large-time behavior of pathwise solutions with decay properties by combining the Moser-Alikakos iteration technique and some decay estimates of Girsanov type processes. If the noise is nonlinear and strong enough, we show that blow-up can be prevented. As such, our results assert that certain multiplicative noise can also regularize the aggregation-diffusion model.

Contents

1. Introduction

Aggregation-diffusion equations via nonlocal interactions are ubiquitous in the modeling of various biological processes/phenomenon from microscopic to macroscopic levels. Among a large class of equations, the following nonlocal aggregation-diffusion equation has recently received extensive attention

$$\frac{\partial u}{\partial t} - \Delta u^m + \chi \mathrm{div}(u \nabla G * u) = 0,\ x \in \mathbb{R}^d,\ t > 0, \tag{1.1}$$

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where $m \geq 1$ is the diffusion parameter, $\chi \in \mathbb{R}$ is the aggregation coefficient, $u(x,t)$ represents the density of species (cells) at position $x \in \mathbb{R}^d$ $(d \geq 2)$ at time t, and $G : \mathbb{R}^d \to \mathbb{R}$ is an interaction kernel. The equation (1.1) can be derived as the continuum limit of many particle system [6, 41] and has a range of applications arising in physics and biology depending on the choice of interaction kernel and diffusion parameter $m \geq 1$, such as self-organization of chemotactic movement [4, 33, 43], biological swarm [9, 54], cancer invasion [18, 23], and so on (see a survey article [10]). While the linear random motion is indicated by $m = 1$, the nonlinear degenerate diffusion with $m > 1$ describes the repulsion between species to account for the over-crowding effect. When the interaction kernel G is a Newtonian or Bessel potential, equation (1.1) is well-known as the Keller–Segel chemotaxis model, for which many interesting results have been available. Among other things, the most prominent feature of the Keller–Segel model is that there is a critical mass in the critical regime $m = 2 - 2/d$ such that the solution to (1.1) may blow up in finite time for super-critical mass and exist globally for sub-critical mass. This was established first for the case $m = 1$ in [5, 17, 39, 40], and later extended to any $m > 0$ (see [2, 3, 34, 49] for subcritical case $m > 2 - 2/d$, [3, 49] for critical case $m = 2-2/d$, and [2, 3, 49] for super-critical case $m < 2-2/d$). Moreover, various modifications/mechanisms have been proposed to "regularize" the equation (1.1) with $m = 1$ in a manner that allows pattern formation but without blow-up (see a survey article [28]).

As is well-known, an additional logistic term, as one of the mechanisms shown in [28], has been shown to being able to regularize the (1.1) in the literature (cf. [42, 55, 56]). However, in a fluctuating or noisy environment, an additional stochastic process may be more appropriate to capture the reality (cf. [38]). The purpose of this paper is to consider the aggregation-diffusion model (1.1) with a multiplicative noise and investigate whether this randomness can affect the global dynamics of the system such as global well-posededness/blow-up and asymptotic behavior of solutions. For simplicity we consider $m = 1$ in this paper and for definiteness we assume G is the Bessel kernel, i.e., G is the Green function of the Helmholtz operator $I - \Delta$ (namely $(I - \Delta)^{-1}u = G * u$). Then, we consider the following stochastic aggregation-diffusion model

$$\mathrm{d}u - \Delta u \,\mathrm{d}t + \chi \mathrm{div}(u \nabla G * u)\,\mathrm{d}t = \sigma(t,u)\,\mathrm{d}\mathcal{W}, \quad x \in \mathbb{R}^d \ (d \geq 2),\ t > 0, \tag{1.2}$$

where $\mathcal{W}$ is a cylindrical Wiener process which will be specified in next section, $\sigma(t,u)\,\mathrm{d}\mathcal{W}$ accounts for the noise arising from the fluctuating or noisy environment. To simplify notations, we define a linear differential operator $Q(\cdot)$ with order -1 and the nonlocal nonlinear term $F(u)$ as follows

$$\begin{cases} Q(u) = \nabla G * u = \ \nabla(I - \Delta)^{-1}u, \\ F(u) = \mathrm{div}\,(uQ(u)) = \ (Q(u) \cdot \nabla)u + u\mathrm{div}Q(u). \end{cases} \tag{1.3}$$

Then (1.2) can be reformulated as

$$\mathrm{d}u - \Delta u \,\mathrm{d}t + \chi F(u)\,\mathrm{d}t = \sigma(t,u)\,\mathrm{d}\mathcal{W}, \quad x \in \mathbb{R}^d \ (d \geq 2),\ t > 0.$$

In contrast to abundant results available to its deterministic counterpart, the stochastic aggregation-diffusion model (1.2) has not been studied and basic questions like well-posedness (even local well-posedness) and large-time behavior of solutions are still unknown. Hence it would be of interest to establish some analytical results for the stochastic aggregation-diffusion models. Therefore the first goal of this paper is to

- Establish local existence and uniqueness of pathwise solutions to the following stochastic aggregation-diffusion model

$$\begin{cases} \mathrm{d}u - \Delta u \,\mathrm{d}t + \chi F(u)\,\mathrm{d}t = \sigma(t,u)\,\mathrm{d}\mathcal{W}, & x \in \mathbb{R}^d,\ t > 0, \\ u(\omega, 0, x) = u_0(\omega, x) \in H^s, \end{cases} \tag{1.4}$$

  where ω belongs to some sample space Ω. The relevant results are stated in Theorem 2.1.

On the other hand, what kind of effects that the noise may bring is a question worthwhile to study. For example, it is known that the well-posedness of linear stochastic transport equation with noise can be established under weaker hypotheses than its deterministic counterpart (cf. [19, 20]). For stochastic Euler equations, certain noise may prevent coalescence of vortices (singularity) in two-dimensional space [21]. With a focus on (1.2), it is natural to study how the noise affect its global dynamics. As shown in [25, 35, 50], the linear noise $\sigma(t,u)\,\mathrm{d}\mathcal{W} = \beta u \,\mathrm{d}W$, where $\beta \in \mathbb{R} \setminus \{0\}$ and W is a standard 1-D Brownian motion, is a dissipative factor for many SPDEs. Motivated by these works, we consider the global dynamics of (1.2) with non-autonomous linear multiplicative noise, namely $\sigma(t,u)\,\mathrm{d}\mathcal{W}=\beta(t)u\,\mathrm{d}W$. Therefore our second goal is set to

- Establish the global boundedness and large-time behavior of pathwise solutions to the following stochastic aggregation-diffusion model with non-autonomous linear noise

$$\begin{cases}\mathrm{d}u-\Delta u\,\mathrm{d}t+\chi F(u)\,\mathrm{d}t=\beta(t)u\,\mathrm{d}W,\ x\in\mathbb{R}^d,\ t>0\\ u(\omega,0,x)=u_0(\omega,x)\in H^s,\end{cases}\tag{1.5}$$

  where W is a standard 1-D Brownian motion. The detailed results for (1.5) are stated in Theorem 2.2 ($d=2, s>5$) and Theorem 2.3 ($d\geq 2, s>\frac{d}{2}+4$).

We outline here that linear noise can bring some "regularization" effects on the aggregation-diffusion model as stated in Theorems 2.2 and Theorem 2.3. Without noise, it is well-known that the solution to (1.5) may blow up in two dimensions with a critical mass and three (or higher) dimensions for small mass (cf. [16, 37, 39]). In Theorem 2.2, if the initial data is small in L^1 sense, then L^∞ norm of the solution decays exponentially almost surely. In Theorem 2.3, a linear large noise can guarantee exponential decay of H^s norm with high probability.

However, the above results hold true either with some smallness conditions on initial data (Theorem 2.2) or with probability (Theorem 2.3). It is therefore very natural to ask *when does global solvability hold without smallness condition on initial data or with probability one?* Theorems 2.2 and 2.3 indicates that linear noise is not enough. Our final goal in this paper is to find out such noise structure. The mathematical interest of finding such noise is important because it is helpful to understand the mechanisms which stabilize the equation, and this is the first step as searching for the real correct and physical noise which provides regularization effect.

As we will see in (2) in Theorem 2.1 below, for the solution to (1.4), its H^s-norm blows up if and only if its H^γ-norm blows up, where $\gamma\in(\frac{d}{2}+1,s]$. This suggests choosing a noise coefficient involving the H^r-norm of u. Therefore, to be consistent with Theorem 2.1, in this work we consider the case that $\sigma(t,u)\,\mathrm{d}\mathcal{W}=a\,(1+\|u\|_{H^r})^q\,u\,\mathrm{d}W$, where $a\in\mathbb{R}\setminus\{0\}$, W is a standard 1-D Brownian motion, $r\in(\frac{d}{2}+1,s-3]$ and $q>0$ is a parameter to be determined. That is, we will consider

- Determine the range of a and q such that the solution to the following problem exists globally in time:

$$\begin{cases}\mathrm{d}u-\Delta u\,\mathrm{d}t+\chi F(u)\,\mathrm{d}t=a(1+\|u\|_{H^r})^q u\,\mathrm{d}W,\ x\in\mathbb{R}^d,\ t>0\\ u(\omega,0,x)=u_0(\omega,x),\end{cases}\tag{1.6}$$

  where $r\in(\frac{d}{2}+1,s-3]$ and W is a standard 1-D Brownian motion. This problem is solved in Theorem 2.4.

The rest of this paper is organized as follows. In Section 2, we introduce some notations, state our main results and then briefly sketch the proof strategies. In Section 3, we present some basic results that will be frequently used. In Section 4, we prove Theorem 2.1. In Section 5, we consider the problems (1.5) and (1.6). We prove Theorems 2.2 and 2.3 in subsection 5.1 and prove Theorem 2.4 in subsection 5.2.

2. Main results

In this section, we shall introduce some notions regarding martingale and pathwise solutions to (1.2), recall some results from abstract probability theory and functional analysis, and then state our main results.

2.1. **Notations and background.** The set of test functions defined on $\mathbb{R}^d$ ($d\geq 1$) is denoted by $\mathcal{D}(\mathbb{R}^d)$. The space of distributions on $\mathbb{R}^d$ is denoted by $\mathcal{D}'(\mathbb{R}^d)$ that is the continuous dual space of $\mathcal{D}(\mathbb{R}^d)$. Let $L^p(\mathbb{R}^d)$ with $d\geq 1$ and $1\leq p<\infty$ be the standard Lebesgue space of real valued measurable p-integrable defined on $\mathbb{R}^n$ and let $L^\infty(\mathbb{R}^d)$ be the space of essentially bounded functions. Particularly, $L^2(\mathbb{R}^d)$ has an inner product $(f,g)_{L^2}=\int_{\mathbb{R}^d} f\cdot\overline{g}\,\mathrm{d}x$, where $\overline{g}$ denotes the complex conjugation of g. The Fourier transform and inverse Fourier transform of $f(x)\in L^2(\mathbb{R}^d)$ are defined by $\widehat{f}(\xi)=\int_{\mathbb{R}^d} f(x)\mathrm{e}^{-ix\cdot\xi}\,\mathrm{d}x$, and

$f(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi$, respectively. For $s\in\mathbb{R}$, the Sobolev spaces H^s on $\mathbb{R}^d$ can be defined as

$$H^s(\mathbb{R}^d):=\left\{f\in L^2(\mathbb{R}^d):\|f\|^2_{H^s(\mathbb{R}^d)}=\int_{\mathbb{R}^d}\left|\widehat{D^sf}(\xi)\right|^2\,d\xi<+\infty\right\}$$

with inner product

$$(f,g)_{H^s}=(D^sf,D^sg)_{L^2},$$

where the operator $D^s=(I-\Delta)^{s/2}$ is defined by

$$\widehat{D^sf}(\xi)=(1+|\xi|^2)^{s/2}\widehat{f}(\xi).$$

Then it is clear that $(I-\Delta)^{-1}$ is a bounded operator from H^s to H^{s+2}. When the function spaces are defined on $\mathbb{R}^d$ and if there is no ambiguity, we drop $\mathbb{R}^d$ for brevity.

We denote the commutator between linear operators A and B by $[A,B]$, i.e., $[A,B]=AB-BA$. For a set E, $\mathbf{1}_E(x)$ is the indicator function on E, i.e., it is equal to 1 when $x\in E$, and zero otherwise. We will use $\lesssim$ to denote an inequality that holds up to some constants, which may be different from line to line.

We next briefly recall some background on the theory of infinite dimensional stochastic analysis which we use below (see [15, 22, 30] for more details). Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, where $\mathbb{P}$ is a probability measure on Ω, $\mathcal{F}$ is a sigma-algebra. We endow the probability space $(\Omega,\mathcal{F},\mathbb{P})$ with an increasing filtration $\{\mathcal{F}_t\}_{t\geq0}$, which is a right-continuous filtration on $(\Omega,\mathcal{F})$ such that $\{\mathcal{F}_0\}$ contains all the $\mathbb{P}$-negligible subsets. For $t>0$, $\sigma\{X(\tau),Y(\tau)\}_{\tau\in[0,t]}$ stands for the completion of the union σ-algebra generated by $(X(\tau),Y(\tau))$ with $\tau\in[0,t]$. All stochastic integrals are defined in the sense of Itô and $\mathbb{E}Y$ is the mathematical expectation of the stochastic process $Y=Y(\omega,t)$ with respect to $\mathbb{P}$. For any separable complete metric space $\mathcal{X}$, we use the symbol $\mathcal{B}(\mathcal{X})$ to denote its Borel sigma-field and let $\mathcal{P}(X)$ be the collection of Borel probability measures on $\mathcal{X}$.

Let $\mathcal{W}(t)=\mathcal{W}(\omega,t),\omega\in\Omega$ be a cylindrical Wiener process, which is adapted to $\{\mathcal{F}_t\}_{t\geq0}$ and takes values on an auxiliary separable Hilbert spaces $\mathbb{U}$. More specific, if $\{e_k\}$ is a complete orthonormal basis of $\mathbb{U}$ and $\{W_k\}_{k\geq1}$ is a sequence of mutually independent standard one-dimensional Brownian motions, then we may formally define the cylindrical Wiener process $\mathcal{W}$ as

$$\mathcal{W}=\sum_{k=1}^{\infty}e_kW_k\quad\mathbb{P}-a.s.$$

However, the above formal summation is not convergent on $\mathbb{U}$. Therefore we consider a larger separable Hilbert space $\mathbb{U}_0$ such that the canonical embedding $\mathbb{U}\hookrightarrow\mathbb{U}_0$ is Hilbert–Schmidt. Then we have that for any $T>0$, cf. [15, 22, 31],

$$\mathcal{W}=\sum_{k=1}^{\infty}e_kW_k\in C([0,T];\mathbb{U}_0)\quad\mathbb{P}-a.s.$$

From now on, we call $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq0},\mathcal{W})$ a stochastic basis and let $\mathcal{L}_2(\mathbb{U};\mathcal{X})$ be the collection of Hilbert–Schmidt operators from U to some separable Hilbert space $\mathcal{X}$, i.e.,

$$G\in\mathcal{L}_2(\mathbb{U};\mathcal{X})\Leftrightarrow\|G\|^2_{\mathcal{L}_2(\mathbb{U};\mathcal{X})}=\sum_{k=1}^{\infty}\|Ge_k\|^2_{\mathcal{X}}<\infty.$$

As in [15, 44], for $\mathcal{X}$-valued predictable process $G\in L^2\left(\Omega;L^2_{\rm loc}\left([0,\infty);\mathcal{L}_2(\mathbb{U};\mathcal{X})\right)\right)$, one can define the Itô stochastic integral

$$\int_0^t G\,d\mathcal{W}=\sum_{k=1}^{\infty}\int_0^t Ge_k\,dW_k,$$

where "predictable" is given in the following definition:

Definition 2.1. *For a given stochastic basis $\mathcal{S}$, let $\Phi=\Omega\times[0,\infty)$ and take $\mathcal{G}$ to be the σ -algebra generated by sets of the form $(s,t]\times F$ with $0\leq s<t<\infty,F\in\mathcal{F}_s$ and $\{0\}\times F$ with $F\in\mathcal{F}_0$. $u:\Omega\times[0,\infty)\to\mathcal{X}$ is an $\mathcal{X}$-valued process. Then u is called predictable (with respect to the stochastic basis $\mathcal{S}$) if it is $(\Phi,\mathcal{G})-(\mathcal{X},\mathcal{B}(\mathcal{X}))$ measurable.*

Here we remark that the stochastic integral $\int_0^t G\,\mathrm{d}\mathcal{W}$ is an $\mathcal{X}$-valued square integrable martingale, and it does not depend on the choice of the space U_0, cf. [15, 44]. For example, U_0 can be defined as

$$U_0=\left\{v=\sum_{k=1}^{\infty}a_ke_k:\sum_{k=1}^{\infty}\frac{a_k^2}{k^2}<\infty\right\},\quad \|v\|_{U_0}=\sum_{k=1}^{\infty}\frac{a_k^2}{k^2}.$$

Moreover, we have that for all almost surely bounded stopping times τ,

$$\left(\int_0^\tau G\,\mathrm{d}\mathcal{W},v\right)_{\mathcal{X}}=\sum_{k=1}^{\infty}\int_0^\tau (Ge_k,v)_{\mathcal{X}}\,\mathrm{d}W_k\quad \mathbb{P}-a.s.$$

In particular the Burkholder-Davis-Gundy (BDG) inequality in the present context reads as

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t G\,\mathrm{d}\mathcal{W}\right\|_{\mathcal{X}}^p\right)\le C\mathbb{E}\left(\int_0^T\|G\|^2_{\mathcal{L}_2(\mathbb{U};\mathcal{X})}\,\mathrm{d}t\right)^{\frac{p}{2}},\quad p\ge 1,$$

or in terms of the coefficients,

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\sum_{k=1}^{\infty}\int_0^t Ge_k\,\mathrm{d}W_k\right\|_{\mathcal{X}}^p\right)\le C\mathbb{E}\left(\int_0^T\sum_{k=1}^{\infty}\|Ge_k\|^2_{\mathcal{X}}\,\mathrm{d}t\right)^{\frac{p}{2}},\quad p\ge 1.$$

2.2. **Hypotheses and definitions.** We first prescribe some conditions on the noise coefficient σ.

Hypothesis $\mathbf{H_1}$. Throughout this paper, we assume that $\sigma:[0,\infty)\times H^s\ni(t,u)\mapsto\sigma(t,u)\in\mathcal{L}_2(\mathbb{U};H^s)$ for $u\in H^s$ with $s\ge 0$ such that σ is continuous in (t,u). Furthermore, we assume the following:

(1) There is a non-decreasing function $f(\cdot):[0,+\infty)\to[0,+\infty)$, which is locally bounded and $f(0)=0$, such that for any $t>0$ and $s\ge 0$,

$$\|\sigma(t,u)\|_{\mathcal{L}_2(\mathbb{U};H^s)}\le f(\|u\|_{W^{1,\infty}})(1+\|u\|_{H^s}).$$

(2) There is a locally bounded non-decreasing function $g(\cdot):[0,+\infty)\to[0,+\infty)$, such that for any $K>0$ and $s\ge 0$,

$$\sup_{t\ge 0,\|u\|_{H^s}\vee\|v\|_{H^s}\le K}\|\sigma(t,u)-\sigma(t,v)\|_{\mathcal{L}_2(\mathbb{U};H^s)}\le g(K)\|u-v\|_{H^s}.$$

Hypothesis $\mathbf{H_2}$. When the non-negativity of solutions is considered, we assume that there is a $C>0$ such that for any $t>0$,

$$\|\sigma(t,u)\|^2_{\mathcal{L}_2(\mathbb{U};L^2)}-2\|\nabla u\|^2_{L^2}\le C\|u\|^2_{L^2},\ \forall u\in H^1.$$

Hypothesis $\mathbf{H_3}$. When (1.5) with non-autonomous linear noise $\beta(t)u\,\mathrm{d}W$ is considered, we assume that:

(1) $\beta(t)\in C([0,\infty))$;
(2) There are β^* and β_* such that $0<\beta_*\le\beta^2(t)\le\beta^*$ for all $t\ge 0$.

We remark here that if Hypothesis $\mathrm{H_3}$ is satisfied, then Hypotheses $\mathrm{H_1}$ and $\mathrm{H_2}$ are also verified for $\sigma(t,u)=\beta(t)u$. This fact will be used in Section 5.1.

Before we formulate our main results, we give the definitions for the pathwise solutions to the problem (1.4).

Definition 2.2 (Pathwise solutions)**.** *Let $s>\frac{d}{2}+4$. Fix a stochastic basis $\mathcal{S}$ and assume $\sigma(t,\cdot):H^s\ni u\mapsto\sigma(t,u)\in\mathcal{L}_2(\mathbb{U};H^s)$. Let u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable. A local pathwise solution to (1.4) is a pair (u,τ), where τ is a stopping time satisfying $\mathbb{P}\{\tau>0\}=1$ and $u:\Omega\times[0,\infty)\to H^s$ is an $\mathcal{F}_t$ predictable process satisfying*

$$u(\cdot\wedge\tau)\in C\left([0,\infty);H^s\right)\ \mathbb{P}-a.s.,\tag{2.1}$$

and the following equation holds true almost surely:

$$u(t\wedge\tau)-u(0)+\int_0^{t\wedge\tau}(-\Delta u+\chi F(u))\,\mathrm{d}t'=\int_0^{t\wedge\tau}\sigma(t',u)\,\mathrm{d}\mathcal{W},\quad t\ge 0.\tag{2.2}$$

Definition 2.3 (Pathwise uniqueness). *The local solutions to* (1.4) *are said to be pathwise unique, if any two pairs of local solutions* (u_1,τ_1) *and* (u_2,τ_2)*, which are defined on the same stochastic basis* $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{W})$*, satisfy that* $\mathbb{P}\{u_1(0)=u_2(0)\}=1$ *can imply*

$$\mathbb{P}\{u_1(t,x)=u_2(t,x),\ \forall t\in[0,\tau_1\wedge\tau_2)\}=1.$$

Definition 2.4 (Maximal solution). *Let the conditions be exactly as in Definition 2.2 above. A maximal pathwise solution to* (1.4) *is a triple* $(u,\{\tau_n\}_{n\geq 1},\tau^*)$ *such that*

(1) For any $n\in\mathbb{N}$*,* (u,τ_n) *is a pathwise solution;*
(2) $\tau_n\to\tau^*$ *increasingly and*

$$\sup_{t\in[0,\tau_n]}\|u\|_{H^s}\geq n,\ \textit{on the set}\ \{\tau^*<\infty\}.$$

If $\tau^*=\infty$ $\mathbb{P}-a.s.$*, then such a solution is called global.*

When it is clear from the context, we just write (u,τ^*) instead of $(u,\{\tau_n\}_{n\geq 1},\tau^*)$ for simplicity.

2.3. Main results and remarks.

Theorem 2.1 (Local existence, blow-up criterion and non-negativity). *Let* $d\geq 2$*,* $\chi\in\mathbb{R}\setminus\{0\}$ *and* $s>d/2+4$*. Given a stochastic basis* $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{W})$*, if* u_0 *is an* H^s*-valued* $\mathcal{F}_0$ *measurable random variable such that* $\mathbb{E}\|u_0\|^2_{H^s}<\infty$ *and* $\sigma(t,u)$ *satisfies Hypothesis* H_1*, then we have the following for* (1.4)*:*

(1) There is a unique pathwise solution (u,τ) *to* (1.4) *in the sense of Definitions 2.2–2.3. Moreover,* u *satisfies*

$$u(\cdot\wedge\tau)\in L^2\left(\Omega;C\left([0,\infty);H^s\right)\right), \tag{2.3}$$

and it can be extended to a maximal solution (u,τ^*) *in the sense of Definition 2.4.*

(2) u *blows up at* $\tau^*<\infty$ *almost surely if and only if*

$$\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\}}=\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^\gamma}=\infty\}}\quad \mathbb{P}-a.s.$$

where $\gamma\in(\frac{d}{2}+1,s]$.

(3) If Hypothesis H_2 *is also satisfied and* $u_0\geq 0$ $\mathbb{P}-a.s.$*, then*

$$\mathbb{P}\{u\geq 0,\ t\in[0,\tau^*)\}=1.$$

Remark 2.1. Now we give a remark to discuss the differences among Theorem 2.1 and the exiting works, the main difficulties encountered in the proof and the main strategies we used. We first notice that the target model is not monotone (see (3.7), H^{s+1}-norm appears when we estimate $(F(u)-F(v),u-v)_{H^s}$) in the sense of [44] so that the Galerkin approximation under a Gelfand triple developed for quasi-linear SPDEs can not be used directly in our case.

- (Mollifying and cut-off) The starting point of our analysis is to consider (1.4) as an SDE in H^s, which can be achieved by mollifying the equation. Then we have a sequence of approximation solution $\{u_\varepsilon\}\in C([0,T_\varepsilon);H^s)$ for some $T_\varepsilon>0$. The first difficulty arises in the *a priori* estimate since the estimate on $\mathbb{E}\|u_\varepsilon\|^2_{H^s}$ involves $\mathbb{E}\|u_\varepsilon\|_{H^r}\|u_\varepsilon\|^2_{H^s}$ for some $r>\frac{d}{2}+1$ (see the estimate for the nonlinear term in Lemma 3.4 and Remark 3.1), which can not be split, and hence prevents one from closing the estimates. We will add a *cut-off* function $\theta_R(\|\cdot\|_{H^r})$ to cut the H^r-norm with $r>\frac{d}{2}+1$ to deal with this difficulty. See (4.1) for the construction of $\theta_R(\|\cdot\|_{H^r})$. Though the cut-off technique is insufficient to make the equation global Lipschitz in H^s, it still provides linear growth in H^s, and hence $T_\varepsilon=\infty$ almost surely. Otherwise we have to show $\inf_\varepsilon T_\varepsilon>0$ $\mathbb{P}-a.s.$ However, how to find such quantitative lower bound is generally not clear.
- (Convergence of the approximation solutions) To obtain a pathwise solution, one need to take limit in the mollified problem. Our method is different from the martingale approach (by first establishing martingale solutions and then obtaining pathwise solution via pathwise uniqueness) used in many previous works. For example, we refer to [7, 8] for different examples in unbounded domains with linear growing noise. However, the method used in [7, 8] are not applicable in our case. This is because, as mentioned above, in our case we need a *cut-off* function $\theta_R(\|\cdot\|_{H^r})$, which is a global object. Even though one can actually establish the probabilistic compactness $L^2(\Omega;H^s_{\rm loc})$"$\hookrightarrow\hookrightarrow$"$L^2(\Omega;H^{s-2}_{\rm loc})$ (cf. Prokhorov's Theorem and Skorokhod's Theorem) and obtain

the converging in $H^{s-2}_{\rm loc}$ (cf. Skorokhod's Theorem), we can *not* pass the limit because $\|\cdot\|_{H^r}$ is a global object involving all $x\in\mathbb{R}^d$ and it can *not* be controlled by the $H^{s-2}_{\rm loc}$-topology even in the case $s-2>r$. In this paper, inspired by [36], we will show that there is a subsequence of the approximation solutions converging in $C([0,T];H^{s-3})$ almost surely (see Lemma 4.3 below) directly. We outline that the convergence holds true on $[0,T]$, which *a priori* may look surprising because a sequence of stopping times is usually needed to estimate nonlinear terms, and as is mentioned above, the lower bound of such stopping times are difficult to obtain in stochastic setting. To take limit in $\theta_R(\|\cdot\|_{H^r})$, we choose $r\le s-3$. This and the previous condition $r>\frac{d}{2}+1$ imply that $s>\frac{d}{2}+4$.

- (Removal of cut-off) For almost surely bounded initial variable u_0, one can introduce a stopping time $\tau=\tau(u_0,R)$ (as in (4.27)) to remove the cut-off. Inspired by [25, 27], we use a cutting-combining argument to remove all the additional conditions on u_0 and guarantee that τ is positive almost surely. We also remark that the assumption $f(0)=0$ in Hypothesis H_1 is used for the cutting-combining argument: If $\{\Omega_k\}$ is mutually disjoint, $\sum_k\mathbf{1}_{\Omega_k}=\mathbf{1}$ and $u_0=\sum_k\mathbf{1}_{\Omega_k}u_0$ almost surely, and u_k is the solution to (1.4) with initial data $\mathbf{1}_{\Omega_k}u_0$, then $\sum_k\mathbf{1}_{\Omega_k}u_k$ is a solution to u_0.
- (Non-negativity) The idea of showing the almost surely non-negativity is to show that $u^-=0$ $\mathbb{P}-a.s.$, where u^- is the negative part of u. In the deterministic PDEs, this can be achieved by showing that $\|u^-(t)\|_{L^2}=0$ for all $t>0$. Now, we prove it by showing that $\mathbb{E}\|u^-\|_{L^2}=0$ for all $t>0$. To this end, we need to consider Itô formula for $\|u^-\|^2_{L^2}$, where the main difficulty is that the nonlinear functional $\|\cdot^-\|^2_{L^2}:H^s\ni u\mapsto\|u^-\|^2_{L^2}\in\mathbb{R}$, is not C^2. To overcome this, we consider smooth C^2-approximations of $\|u^-\|_{L^2}$ motivated by [11–13].

Then we consider (1.5), where $\sigma(t,u){\rm d}\mathcal{W}=\beta(t)u{\rm d}W$. This particular structure only requires a single standard 1-D Brownian motion W rather than a cylindrical Wiener process $\mathcal{W}$. Hence for (1.5), the stochastic basis becomes $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$, where W is a standard 1-D Brownian motion.

Theorem 2.2 (Decay of L^∞-norm in $\mathbb{R}^2$ almost surely)**.** *Let $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be a fixed stochastic basis, where W is a standard 1-D Brownian motion. Let $d=2$, $\chi>0$ and $s>5$. Assume Hypothesis H_3 holds true. Then*

$$A=A(\omega)=\sup_{t>0}{\rm e}^{\int_0^t\beta(t')\,{\rm d}W_{t'}-\int_0^t\frac{\beta^2(t')}{2}\,{\rm d}t'}. \tag{2.4}$$

Then $A<\infty$ $\mathbb{P}-a.s.$ Moreover, let u_0 be an $H^s\cap L^1$-valued $\mathcal{F}_0$ measurable random variable satisfying $u_0\ge0$ $\mathbb{P}-a.s.$ and $\mathbb{E}\|u_0\|^2_{H^s}<\infty$. If for some large constant $C>0$,

$$\mathbb{P}\left\{\|u_0\|_{L^1}\le\frac{1}{4C\chi A}\right\}=1, \tag{2.5}$$

then there is a random variable $0<K=K(\omega)<\infty$ $\mathbb{P}-a.s.$ such that the solution to (1.5) satisfies and

$$\mathbb{P}\left\{\|u(t)\|_{L^\infty}\le CK\max\{\|u_0\|_{L^1},\|u_0\|_{L^\infty}\}\,{\rm e}^{\int_0^t\beta(t')\,{\rm d}W_{t'}-\int_0^t\frac{\beta^2(t')}{2}\,{\rm d}t'},\ \forall t>0\right\}=1. \tag{2.6}$$

That is to say, $\mathbb{P}\left\{\|u(t)\|_{L^\infty}\text{ decays with (least) rate }{\rm e}^{\int_0^t\beta(t')\,{\rm d}W_{t'}-\int_0^t\frac{\beta^2(t')}{2}\,{\rm d}t'}\right\}=1.$

Theorem 2.3 (Decay of H^s-norm in $\mathbb{R}^d$ with high probability)**.** *Let $\chi\in\mathbb{R}\setminus\{0\}$, $d\ge2$ and $s>d/2+4$. Let $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be a fixed stochastic basis, where W is a standard 1-D Brownian motion. Assume Hypothesis H_3 holds true. Let u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable. For any $R>1$, if for some $C=C(s)>0$, $\|u_0\|_{H^s}\le\frac{\beta_*}{2C|\chi|R}$ $\mathbb{P}-a.s.$, then (1.5) has a maximal solution (u,τ^*) satisfying that for any $\lambda_1>2,\lambda_2>\frac{2\lambda_1}{\lambda_1-2}$,*

$$\mathbb{P}\left\{\|u(t)\|_{H^s}<\frac{\beta_*}{C\lambda_1|\chi|}{\rm e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t\beta^2(t')\,{\rm d}t'},\ \forall t>0\right\}\ge1-\left(\frac{1}{R}\right)^{2/\lambda_2}.$$

which means, $\mathbb{P}\left\{\|u(t)\|_{H^s}\text{ decays with (least) rate }{\rm e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t\beta^2(t')\,{\rm d}t'}\right\}\ge1-\left(\frac{1}{R}\right)^{2/\lambda_2}.$

Theorem 2.4 (Strong noise prevents blow-up almost surely)**.** *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ be a fixed stochastic basis. Let $d \geq 2$, $s > \frac{d}{2} + 4$, $\chi \in \mathbb{R} \setminus \{0\}$ and $u_0 \in H^s$ be an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. If q and a satisfy*

$$\begin{cases} a \in \mathbb{R} \setminus \{0\}, & \text{if } q > \frac{1}{2}, \\ a^2 > 2D|\chi|, & \text{if } q = \frac{1}{2}, \end{cases} \tag{2.7}$$

where D is the constant given in Lemma 3.4, then (1.6) has a unique global solution starting from u_0.

Remark 2.2. We give the following remarks concerning Theorems 2.2, 2.3 and 2.4.

- The proof for Theorems 2.2 and 2.3 involves a Girsanov type transformation (we transfer (1.5) to (5.2) or (5.3)). Although the stochastic integral is absent in (5.2) or (5.3), to extend the deterministic results to the stochastic setting, we need to overcome a few technical difficulties since the system is not only random but also non-autonomous. We manage to gain some uniform-in-time estimates of solutions and asymptotic limits of some stochastic processes (e.g., see (5.15), (5.18) and Lemma 3.6), which enable us to extend the deterministic ideas pathwisely (namely for a.e. $\omega \in \Omega$).
- In the proof of Theorem 2.2, the well-known Moser-Alikakos iteration technique and decay estimate of Girsanov type process are used (for the Moser iteration involving expectation, we refer to [26]). Theorem 2.2 entails that if the initial mass is small, then the solution of (1.5) is bounded globally and decays to zero. In contrast to the deterministic counterpart of (1.5), where the decay rate of $\|u\|_{L^\infty}$ is only algebraic (cf. [29]), Theorem 2.2 shows that the multiplicative noise $\beta(t)u\,\mathrm{d}W$ brings more dissipation in the sense that $\|u\|_{L^\infty}$ decays exponentially with (least) rate $\mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'} - \int_0^t \frac{\beta^2(t')}{2}\,\mathrm{d}t'}$.
- The proof for Theorem 2.3 involves extracting a damping part from the transformation (5.1) (see (5.14)) and using some estimates for the exit times of Girsanov type process. In Theorem 2.3, for fixed $\lambda_1 > 2$ and $\lambda_2 > \frac{2\lambda_1}{\lambda_1 - 2}$, if we let $R \gg 1, \beta_* \gg 1$ such that $1/R^{2/\lambda_2}$ is small enough but $\beta_* \gg 2C|\chi|R$ is large, then the H^s norm of initial data can be large. Moreover, the H^s norm of the solution starting from this large initial data decays exponentially with high probability.
- Theorem 2.4 is proved by using a Lyapunov function $\log(1 + \|u\|^2_{H^s})$, cf. [45, 48], and the result means that if the nonlinear noise is strong enough, i.e., (2.7) is satisfied, then the global existence holds almost surely without any smallness assumption on initial data.
- Without noise, it is well-known that the solution to (1.5) may blow up in two dimensions with a critical mass and three (or higher) dimensions for small mass (cf. [16, 37, 39]). The results of Theorems 2.3 and 2.4 indicate that large multiplicative noise can provide some "regularization" effect allowing the global boundedness of the Keller-Segel model (1.2) (In Theorem 2.2, decay of L^∞ norm becomes faster; In Theorem 2.3, H^s norm decays exponentially with high probability, and in Theorem 2.4, solution exists globally without any kind smallness conditions on the initial data). For deterministic aggregation-diffusion equations, different mechanisms have been proposed to ensure the pattern formation without blow-up. Theorems 2.2, 2.3 and 2.4 show that certain noise can also induce some dissipation/regularization effect to aggregation-diffusion model.

3. Some preliminary results

Now we gather some necessary results from analysis. For any $\varepsilon \in (0,1)$, J_ε is the Friedrichs mollifier defined by

$$J_\varepsilon f(x) = j_\varepsilon * f(x). \tag{3.1}$$

where $*$ stands for the convolution, $j_\varepsilon(x) = \frac{1}{\varepsilon^d} j(\frac{x}{\varepsilon})$ and $j(x)$ is a Schwartz function satisfying $0 \leq \widehat{j}(\xi) \leq 1$ for $\xi \in \mathbb{R}^d$ and $\widehat{j}(\xi) = 1$ for all $\xi \in \mathbb{R}^d$ with $|\xi| \leq 1$. From the construction of j_ε, we see that $\widehat{j}_\varepsilon(\xi) = \widehat{j}(\varepsilon\xi)$ and for any $u \in H^s$, it holds that (see for example [36])

$$\|I - J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \lesssim \varepsilon^{s-r}, \quad r < s, \tag{3.2}$$

$$\|J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \lesssim O(\varepsilon^{s-r}), \quad r > s. \tag{3.3}$$

In addition, for the operator $D^s = (I-\Delta)^{s/2}$ defined in previous section, we have the following properties and we will use them without further notice in the sequel,

$$D^s J_\varepsilon = J_\varepsilon D^s,$$
$$(J_\varepsilon f, g)_{L^2} = (f, J_\varepsilon g)_{L^2},$$
$$\|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s},$$

We also notice the following commutator estimate for J_ε.

Lemma 3.1 ([52]). *Let $d \ge 1$. Let $f, g : \mathbb{R}^d \to \mathbb{R}^d$ such that $g \in W^{1,\infty}$ and $f \in L^2$. Then for some $C > 0$,*

$$\|[J_\varepsilon, (g\cdot\nabla)]f\|_{L^2} \le C\|\nabla g\|_{L^\infty}\|f\|_{L^2}.$$

Lemma 3.2 ([32]). *If $f, g \in H^s \cap W^{1,\infty}$ for $s > 0$, then*

$$\|\,[D^s, (f\cdot\nabla)]\,g\|_{L^2} \le C_s(\|D^s f\|_{L^2}\|\nabla g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|D^{s-1}\nabla g\|_{L^2}).$$

If $f, g \in H^s \cap L^\infty$, then

$$\|fg\|_{H^s} \le C_s(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}).$$

Lemma 3.3. *Let $F(\cdot)$ be defined in (1.3). For any $u, v \in H^{s+1}$ with $s > d/2+1$ and $a > d/2$, we have*

$$\|F(v)\|_{H^s} \lesssim \|v\|_{H^a}\left(\|v\|_{H^s} + \|v\|_{H^{s+1}}\right), \tag{3.4}$$
$$\|F(u) - F(v)\|_{H^s} \lesssim \|u\|_{H^{s+1}}\|w\|_{H^s} + \|v\|_{H^s}\|w\|_{H^{s+1}}, \tag{3.5}$$
$$|(F(u)-F(v), u-v)_{L^2}| \lesssim \left(\|u\|_{W^{1,\infty}} + \|v\|_{H^a}\right)\|u-v\|_{L^2}^2, \tag{3.6}$$
$$|(F(u)-F(v), u-v)_{H^s}| \lesssim \left(\|u\|_{H^{s+1}} + \|v\|_{H^s}\right)\|u-v\|_{H^s}^2. \tag{3.7}$$

Proof. Using Lemma 3.2, $H^a \hookrightarrow L^\infty$ with noticing that $(I-\Delta)^{-1}$ is bounded from H^s to H^{s+2}, we have

$$\begin{aligned}\|F(v)\|_{H^s} \lesssim& \|vQ(v)\|_{H^{s+1}} \lesssim \|v\|_{L^\infty}\|Q(v)\|_{H^{s+1}} + \|v\|_{H^{s+1}}\|Q(v)\|_{L^\infty}\\ \lesssim& \|v\|_{H^a}\left(\|v\|_{H^s} + \|v\|_{H^{s+1}}\right),\end{aligned}$$

which is (3.4). Set $w = u - v$. Then we have

$$\begin{aligned}&\|F(u)-F(v)\|_{H^s}\\ \lesssim& \|\mathrm{div}(uQ(w))\|_{H^s} + \|\mathrm{div}(wQ(v))\|_{H^s}\\ \lesssim& \|u\|_{H^{s+1}}\|Q(w)\|_{H^{s+1}} + \|w\|_{H^{s+1}}\|Q(v)\|_{H^{s+1}},\end{aligned}$$

which implies (3.5). Similarly, it follows that

$$\begin{aligned}&|(F(u)-F(v), w)_{L^2}|\\ \lesssim& |(\nabla u\cdot Q(w), w)_{L^2}| + |(u\mathrm{div}Q(w), w)_{L^2}| + |(\nabla w\cdot Q(v), w)_{L^2}| + |(w\mathrm{div}Q(v), w)_{L^2}|\\ \lesssim& \|\nabla u\|_{L^\infty}\|Q(w)\|_{L^2}\|w\|_{L^2} + \|u\|_{L^\infty}\|\mathrm{div}Q(w)\|_{L^2}\|w\|_{L^2} + \|\mathrm{div}Q(v)\|_{L^\infty}\|w\|_{L^2}^2.\end{aligned}$$

Since $\mathrm{div}Q(v) = \Delta(I-\Delta)^{-1}v = (I-\Delta)^{-1}v - v$, we have $\|\mathrm{div}Q(v)\|_{L^\infty} \lesssim \|v\|_{H^a}$ and therefore

$$|(F(u)-F(v), w)_{L^2}| \lesssim \|u\|_{W^{1,\infty}}\|w\|_{L^2}^2 + \|v\|_{H^a}\|w\|_{L^2}^2,$$

which is (3.6). As for (3.7), we first notice that $H^s \hookrightarrow W^{1,\infty}$ and hence

$$\|\mathrm{div}Q(v)\|_{L^\infty} \lesssim \|\nabla Q(v)\|_{L^\infty} \lesssim \|Q(v)\|_{H^s} \lesssim \|v\|_{H^{s-1}}.$$

Then we use Lemma 3.2 and integration by parts to deduce that

$$\begin{aligned}|(F(u)-F(v), w)_{H^s}| \lesssim& |(D^s\mathrm{div}(uQ(w)), D^s w)_{L^2}| + |([D^s, (Q(v)\cdot\nabla)]w, D^s w)_{L^2}|\\ &+ |(Q(v)\cdot\nabla D^s w, D^s w)_{L^2}| + |(D^s(w\mathrm{div}Q(v)), D^s w)_{L^2}|\\ \lesssim& \|uQ(w)\|_{H^{s+1}}\|w\|_{H^s} + \|v\|_{H^s}\|w\|_{H^s}^2 + \|w\mathrm{div}Q(v)\|_{H^s}\|w\|_{H^s}\\ \lesssim& \|u\|_{H^{s+1}}\|w\|_{H^s}^2 + \|v\|_{H^s}\|w\|_{H^s}^2,\end{aligned}$$

which yields (3.7). □

Lemma 3.4. *Let $F(\cdot)$ be defined in (1.3), J_ε be defined in (3.1) and let $b > \frac{d}{2}+1$. For all $u \in H^s$ with $s > \frac{d}{2}+4$, there is a constant $D = D(s) > 0$ such that for all $\varepsilon > 0$, we have*

$$|(J_\varepsilon F(u), J_\varepsilon u)_{H^s}| \le D\|u\|_{H^b}\|u\|_{H^s}^2.$$

Proof. Using Lemmas 3.1 and 3.2, integration by parts, $\nabla(I-\Delta)^{-1}$ is bounded from H^s to H^{s+1} and the embedding $H^b \hookrightarrow W^{1,\infty}$, we obtain that for some $D > 0$,

$$\begin{aligned}
&(D^s J_\varepsilon F(u), D^s J_\varepsilon u)_{L^2}\\
=&(D^s J_\varepsilon\left[(Q(u)\cdot\nabla)u\right], D^s J_\varepsilon u)_{L^2} + (D^s J_\varepsilon\left[u\mathrm{div}Q(u)\right], D^s J_\varepsilon u_\varepsilon)_{L^2}\\
=&\left([D^s,(Q(u)\cdot\nabla)]\,u, D^s J_\varepsilon^2 u\right)_{L^2} + ([J_\varepsilon,(Q(u)\cdot\nabla)]D^s u, D^s J_\varepsilon u)_{L^2}\\
&+((Q(u)\cdot\nabla)D^s J_\varepsilon u, D^s J_\varepsilon u)_{L^2} + (D^s J_\varepsilon\left[u\mathrm{div}Q(u)\right], D^s J_\varepsilon u)_{L^2}\\
\lesssim&\|Q(u)\|_{H^s}\|\nabla u\|_{L^\infty}\|u\|_{H^s} + \|\nabla Q(u)\|_{L^\infty}\|u\|_{H^s}^2 + \|u\mathrm{div}Q(u)\|_{H^s}\|u\|_{H^s}\\
\lesssim&\|u\|_{H^b}\|u\|_{H^s}^2 + \left(\|u\|_{L^\infty}\|\mathrm{div}Q(u)\|_{H^s} + \|u\|_{H^s}\|\mathrm{div}Q(u)\|_{L^\infty}\right)\|u\|_{H^s}\\
\lesssim&\|u\|_{H^b}\|u\|_{H^s}^2 + \left(\|u\|_{H^b}\|u\|_{H^s} + \|u\|_{H^s}\|Q(u)\|_{H^b}\right)\|u\|_{H^s}\\
\le&D\|u\|_{H^b}\|u\|_{H^s}^2,
\end{aligned}$$

which gives the desired result. □

Remark 3.1. We remark that if $u \in H^{s+1}$, we can omit the mollifier J_ε in the proof for Lemma 3.4 to deduce that

$$|(F(u), u)_{H^s}| \le D\|u\|_{H^b}\|u\|_{H^s}^2.$$

However, in applications, sometimes we can only know $u \in H^s$, and hence $(F(u), u)_{H^s}$ is *not* well-defined because $F(u) \in H^{s-1}$ (cf. Lemma 3.3). This means that we can not apply Itô formula for $\|u(t)\|_{H^s}^2$ directly (cf. (4.22) below). In this case, we need Lemma 3.4, where the constant D does not depend on ε.

Now we introduce some functions which will be used in the study of the non-negativity of the solutions. Let $\rho(z) = -\mathbf{1}_{\{z<0\}}z$ be the negative part of $z \in \mathbb{R}$ and let $k(\cdot) = \rho^2(\cdot)$. We define the following functions:

$$a(x) = \begin{cases} 0, & x \ge 0,\\ 1, & x < 0,\end{cases} \quad \text{and} \quad k_\varepsilon(x) = \begin{cases} x^2 - \dfrac{\varepsilon^2}{6}, & x < -\varepsilon,\\ \dfrac{-x^3}{\varepsilon}\left(\dfrac{x}{2\varepsilon} + 4/3\right), & -\varepsilon \le x < 0,\\ 0, & x \ge 0.\end{cases}$$

Then the following results hold (cf. [12, Lemma 3.1]).

Lemma 3.5. *For $k(x), \rho(x), k_\varepsilon(x)$ and $a(x)$ defined in above, we have*

- *$k(x) = 0$ if $x \ge 0$ and $k(x) = x^2$ if $x < 0$;*
- *$k_\varepsilon'(x)$ and $k_\varepsilon''(x)$ are continuous;*
- *$k_\varepsilon' \le 0$, $k_\varepsilon'' \ge 0$, $k_\varepsilon'(x) = 0$ if $x \ge 0$;*
- *$k_\varepsilon(x) \to k(x)$, $k_\varepsilon'(x) \to -2\rho(x)$ and $k_\varepsilon''(x) \to 2a(x)$ uniformly on $\mathbb{R}$.*

Lemma 3.6 ([36, 47])**.** *Assume Hypothesis H_3 holds true and $\rho(t) \in C([0,\infty))$ is a bounded function. Let*

$$X(t) = \mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'} + \int_0^t \rho(t') - \frac{\beta^2(t')}{2}\,\mathrm{d}t'}, \quad t \ge 0.$$

Then we have the following properties:

(1) Let $\phi(t) := \int_0^t \beta^2(t')\,\mathrm{d}t'$ and $\phi^{-1}(t)$ be the inverse function of ϕ. If

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}}\left(\int_0^{\phi^{-1}(t)} \rho(t')\,\mathrm{d}t' - \frac{t}{2}\right) < -1,$$

then $\lim_{t\to\infty} X(t) = 0$ $\mathbb{P}-a.s.$ And if

$$\liminf_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}}\left(\int_0^{\phi^{-1}(t)} \rho(t')\,\mathrm{d}t' - \frac{t}{2}\right) > 1,$$

then $\lim_{t\to\infty} X(t) = +\infty\ \mathbb{P}-a.s.$

(2) Let $\rho(t) = \lambda\beta^2(t)$ *with* $\lambda < \frac{1}{2}$ *and* $\tau_R = \inf\{t \geq 0 : X(t) > R\}$ *with* $R > 1$*, then*

$$\mathbb{P}(\tau_R = \infty) \geq 1 - \left(\frac{1}{R}\right)^{1-2\lambda}.$$

4. Proof of Theorem 2.1

For clarity, we complete the proof of Theorem 2.1 in the following several subsections/steps.

4.1. **Approximation scheme and associated estimates.** We will construct the approximation scheme as follows.

Cut-off. Let $s > \frac{d}{2} + 4$ and $r \in (\frac{d}{2} + 1, s - 3]$. For any $R > 0$, we let $\theta_R(x) : [0, \infty) \to [0, 1]$ be a C^∞ function such that $\theta_R(x) = 1$ for $|x| \in [0, R]$ and $\theta_R(x) = 0$ for $|x| > 2R$. Then we consider the problem by cutting the nonlinearities in (1.4) as follows,

$$\begin{cases} \mathrm{d}u + [-\Delta u + \chi\theta_R(\|u\|_{H^r})F(u)]\,\mathrm{d}t = \theta_R(\|u\|_{H^r})\sigma(t,u)\,\mathrm{d}\mathcal{W}, \quad x \in \mathbb{R}^d,\ t > 0, \\ u(\omega, 0, x) = u_0(\omega, x) \in H^s. \end{cases} \tag{4.1}$$

Mollifying. Recall that J_ε is the Friedrichs mollifier defined in the previous section. Then we mollify (4.1) and consider the following approximate problem:

$$\begin{cases} \mathrm{d}u + G_{1,\varepsilon}(u)\,\mathrm{d}t = G_2(t,u)\,\mathrm{d}\mathcal{W}, \\ \qquad G_{1,\varepsilon}(u) = -J_\varepsilon^2\Delta u + \chi\theta_R(\|u\|_{H^r})J_\varepsilon F(J_\varepsilon u), \\ \qquad G_2(t,u) = \theta_R(\|u\|_{H^r})\sigma(t,u), \\ \qquad u(\omega,0,x) = u_0(\omega,x). \end{cases} \tag{4.2}$$

Lemma 4.1. *Let* $\chi \in \mathbb{R} \setminus \{0\}$, $s > \frac{d}{2} + 4$ *and* $r \in (\frac{d}{2} + 1, s - 3]$. *Fix a stochastic basis* $\mathcal{S}$ *and let* $u_0 \in L^2(\Omega; H^s)$ *be an* H^s*-valued* $\mathcal{F}_0$ *measurable random variable. Assume* σ *satisfies Hypothesis* H_1. *For any* $R > 1$ *and* $\varepsilon \in (0, 1)$, (4.2) *admits a unique solution* $u_\varepsilon \in C([0, \infty); H^s)$ $\mathbb{P}-a.s.$ *Moreover, for any* $T > 0$, $\{u_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^2(\Omega; C([0, T]; H^s))$ *is bounded uniformly in* ε. *More precisely, there is a constant* $C = C(\chi, R, T, u_0) > 0$ *such that*

$$\sup_{\varepsilon>0} \mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|^2_{H^s} \leq C, \tag{4.3}$$

Proof. Using Hypothesis H_1, (3.3) and Lemma 3.3, it is easy to obtain that for any $T > 0$ and $R > 1$, there exist $l_1 = l_1(R, \varepsilon, \chi)$ and $l_2 = l_2(R)$ such that for all $u \in C([0, T]; H^\rho)$ with $\rho > \frac{d}{2} + 1$,

$$\|G_{1,\varepsilon}(u)\|_{H^\rho} \leq l_1(1 + \|u\|_{H^\rho}), \quad \|G_2(t,u)\|_{\mathcal{L}_2(\mathbb{U};H^\rho)} \leq l_2(1 + \|u\|_{H^\rho}), \quad \forall t \in [0,T]. \tag{4.4}$$

For any $R > 1$, $s > \frac{d}{2} + 4$ and $\varepsilon \in (0, 1)$, (4.4) implies that (4.2) defines an SDE in H^s with linear growth condition. Similarly, we can infer from Hypothesis H_1 and Lemma 3.3 that for any $t > 0$, $G_{1,\varepsilon}(u)$ and $G_2(t, u)$ are locally Lipschitz in $u \in H^s$. Then the theory of SDE in Hilbert space (see for example [44, Theorem 4.2.4 with Example 4.1.3]) shows that (4.2) admits a unique solution $u_\varepsilon \in C([0, \infty); H^s)$ almost surely.

Now we establish the uniform-in-ε estimate for (4.2). By Itô formula, we have

$$\begin{aligned} \mathrm{d}\|u_\varepsilon\|^2_{H^s} + 2\|\nabla J_\varepsilon u_\varepsilon\|^2_{H^s}\,\mathrm{d}t =& 2\theta_R(\|u_\varepsilon\|_{H^r})\left(\sigma(t,u_\varepsilon)\,\mathrm{d}\mathcal{W}, u_\varepsilon\right)_{H^s} \\ & - 2\chi\theta_R(\|u_\varepsilon\|_{H^r})\left(D^sJ_\varepsilon\left[(Q(J_\varepsilon u_\varepsilon)\cdot\nabla)J_\varepsilon u_\varepsilon\right], D^s u_\varepsilon\right)_{L^2}\,\mathrm{d}t \\ & - 2\chi\theta_R(\|u_\varepsilon\|_{H^r})\left(D^sJ_\varepsilon\left[J_\varepsilon u_\varepsilon \mathrm{div}Q(J_\varepsilon u_\varepsilon)\right], D^s u_\varepsilon\right)_{L^2}\,\mathrm{d}t \\ & + \theta_R^2(\|u_\varepsilon\|_{H^r})\|\sigma(t,u_\varepsilon)\|^2_{\mathcal{L}_2(\mathbb{U};H^s)}\,\mathrm{d}t \\ =& A_1 + \sum_{i=2}^4 A_i\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

Let $T>0$. Integrating the above equation, taking a supremum for $t\in[0,T]$ and using the BDG inequality yield

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2\\ \le&\mathbb{E}\|u_0\|_{H^s}^2+C\mathbb{E}\left(\int_0^T\|u_\varepsilon\|_{H^s}^2\theta_R^2(\|u_\varepsilon\|_{H^r})\|\sigma(t,u_\varepsilon)\|_{\mathcal{L}_2(\mathbb{U};H^s)}^2\,\mathrm{d}t\right)^{\frac12}+\sum_{i=2}^4\mathbb{E}\int_0^T|A_i|\,\mathrm{d}t\\ \le&\mathbb{E}\|u_0\|_{H^s}^2+\sum_{i=2}^4\mathbb{E}\int_0^T|A_i|\,\mathrm{d}t+C\mathbb{E}\left(\sup_{t\in[0,T]}\|u_\varepsilon\|_{H^s}^2\int_0^T\theta_R^2(\|u_\varepsilon\|_{H^r})\|\sigma(t,u_\varepsilon)\|_{\mathcal{L}_2(\mathbb{U};H^s)}^2\,\mathrm{d}t\right)^{\frac12}\\ \le&\mathbb{E}\|u_0\|_{H^s}^2+\sum_{i=2}^4\int_0^T\mathbb{E}\sup_{t'\in[0,t]}|A_i(t')|\,\mathrm{d}t+\frac12\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon\|_{H^s}^2+Cf^2(2R)\int_0^T\left(1+\mathbb{E}\sup_{t'\in[0,t]}\|u_\varepsilon(t')\|_{H^s}^2\right)\mathrm{d}t.\end{aligned}$$

Recalling J_ε is self-adjoint. In the same way as in Lemma 3.4, we derive that

$$|A_2|+|A_3|\le C|\chi|R\|u_\varepsilon\|_{H^s}^2.$$

Therefore we arrive at

$$\int_0^T\mathbb{E}\sup_{t'\in[0,t]}\left(|A_2(t')|+|A_3(t')|\right)\,\mathrm{d}t\le C|\chi|R\int_0^T\mathbb{E}\sup_{t'\in[0,t]}\|u_\varepsilon(t')\|_{H^s}^2\,\mathrm{d}t.$$

Similarly, we use $H^r\hookrightarrow W^{1,\infty}$ and Hypothesis H_1 to deduce that

$$|A_4|\le\theta_R^2(\|u_\varepsilon\|_{H^r})f^2(\|u_\varepsilon\|_{H^r})(1+\|u_\varepsilon\|_{H^s})^2.$$

Consequently, we have

$$\sum_{k=2}^4\int_0^T\mathbb{E}\sup_{t'\in[0,t]}|J_k(t')|\,\mathrm{d}t\le\left(C_\chi R+Cf^2(2R)\right)\int_0^T\left(1+\mathbb{E}\sup_{t'\in[0,t]}\|u_\varepsilon(t')\|_{H^s}^2\right)\mathrm{d}t'.$$

Combining the above estimates,, we see that u_ε satisfies

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2\le2\mathbb{E}\|u_0\|_{H^s}^2+C_{\chi,R}\int_0^T\left(1+\mathbb{E}\sup_{t'\in[0,t]}\|u_\varepsilon(t')\|_{H^s}^2\right)\mathrm{d}t. \tag{4.6}$$

Thanks to the Grönwall's inequality, we obtain that

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2<C(R,T,\chi,u_0),$$

for some constant $C(R,T,\chi,u_0)>0$. □

Lemma 4.2. *Assume the conditions in Lemma 4.1 hold true. For any $T>0$ and $K>0$, we define*

$$\tau_{\varepsilon,K}^T=\inf\{t\ge0:\|u_\varepsilon(t)\|_{H^s}\ge K\}\wedge T, \tag{4.7}$$

and

$$\tau_{\varepsilon,\eta,K}^T=\tau_{\varepsilon,K}^T\wedge\tau_{\eta,K}^T. \tag{4.8}$$

Then we have

$$\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]}\|u_\varepsilon-u_\eta\|_{H^{s-3}}=0,\quad K>1. \tag{4.9}$$

Proof. For the solutions u_ε and u_η to (4.2), we consider the following problem for $v_{\varepsilon,\eta}=u_\varepsilon-u_\eta$,

$$\mathrm{d}v_{\varepsilon,\eta}+[G_{1,\varepsilon}(u_\varepsilon)-G_{1,\eta}(u_\eta)]\,\mathrm{d}t=[G_2(t,u_\varepsilon)-G_2(t,u_\eta)]\,\mathrm{d}\mathcal{W},\quad v_{\varepsilon,\eta}=0. \tag{4.10}$$

We notice that

$$\begin{aligned}
&G_{1,\varepsilon}(u_\varepsilon)-G_{1,\eta}(u_\eta)\\
=&-J_\varepsilon^2\Delta u_\varepsilon+J_\eta^2\Delta u_\eta+\chi\theta_R(\|u_\varepsilon\|_{H^r})J_\varepsilon F(J_\varepsilon u_\varepsilon)-\chi\theta_R(\|u_\eta\|_{H^r})J_\eta F(J_\eta u_\eta)\\
=&\left(J_\eta^2-J_\varepsilon^2\right)\Delta u_\varepsilon+J_\eta^2\Delta\left(u_\eta-u_\varepsilon\right)\\
&+\chi\left[\theta_R(\|u_\varepsilon\|_{H^r})-\theta_R(\|u_\eta\|_{H^r})\right]J_\varepsilon F(J_\varepsilon u_\varepsilon)+\chi\theta_R(\|u_\eta\|_{H^r})(J_\varepsilon-J_\eta)\left[F(J_\varepsilon u_\varepsilon)\right]\\
&+\chi\theta_R(\|u_\eta\|_{H^r})J_\eta\left[F(J_\varepsilon u_\varepsilon)-F(J_\eta u_\varepsilon)\right]+\chi\theta_R(\|u_\eta\|_{H^r})J_\eta\left[F(J_\eta u_\varepsilon)-F(J_\eta u_\eta)\right]\\
=&\sum_{i=1}^{6}\mathcal{R}_i,
\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}
&G_2(t,u_\varepsilon)-G_2(t,u_\eta)\\
=&\theta_R(\|u_\varepsilon\|_{H^r})\sigma(t,u_\varepsilon)-\theta_R(\|u_\eta\|_{H^r})\sigma(t,u_\eta)\\
=&\left[\theta_R(\|u_\varepsilon\|_{H^r})-\theta_R(\|u_\eta\|_{H^r})\right]\sigma(t,u_\varepsilon)+\theta_R(\|u_\eta\|_{H^r})[\sigma(t,u_\varepsilon)-\sigma(t,u_\eta)]\\
=&\sum_{i=7}^{8}\mathcal{R}_i.
\end{aligned}\tag{4.12}$$

Then we use the Itô formula with noticing (4.11) and (4.12) to find that for any $t>0$,

$$\|v_{\varepsilon,\eta}(t)\|_{H^{s-3}}^2+2\int_0^t\left(\mathcal{R}_2,v_{\varepsilon,\eta}\right)_{H^{s-3}}\,\mathrm{d}t'=R_1-\int_0^t R_2\,\mathrm{d}t'+\int_0^t R_3\,\mathrm{d}t',\tag{4.13}$$

where

$$\begin{cases}
R_1=2\displaystyle\int_0^t\left([G_2(t,u_\varepsilon)-G_2(t,u_\eta)]\,\mathrm{d}\mathcal{W},v_{\varepsilon,\eta}\right)_{H^{s-3}},\\
R_2=2\displaystyle\sum_{i\in\{1,3,4,5,6\}}\left(\mathcal{R}_i,v_{\varepsilon,\eta}\right)_{H^{s-3}},\\
R_3=\left\|\displaystyle\sum_{i=7}^{8}\mathcal{R}_i\right\|_{\mathcal{L}_2(\mathbb{U};H^{s-3})}^2.
\end{cases}\tag{4.14}$$

Obviously, integration by parts and the fact that J_η is self-adjoint imply

$$(\mathcal{R}_2,v_{\varepsilon,\eta})_{H^{s-3}}=-\left(J_\eta^2\Delta v_{\varepsilon,\eta},v_{\varepsilon,\eta}\right)_{H^{s-3}}=\|\nabla J_\eta v_{\varepsilon,\eta}\|_{H^{s-3}}^2\geq 0,$$

Then (4.13) yields

$$\|v_{\varepsilon,\eta}(t)\|_{H^{s-3}}^2\leq|R_1|+\int_0^t|R_2|\,\mathrm{d}t'+\int_0^t|R_3|\,\mathrm{d}t'\quad\mathbb{P}-a.s.\tag{4.15}$$

Using the BDG inequality to (4.15) with noticing Hypothesis H_1, we derive

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]}\|v_{\varepsilon,\eta}(t)\|_{H^{s-3}}^2\\
\leq&C\mathbb{E}\left(\int_0^{\tau_{\varepsilon,\eta,K}^T}\|v_{\varepsilon,\eta}\|_{H^{s-3}}^2|R_3|\,\mathrm{d}t\right)^{\frac12}+\sum_{i=2}^{3}\mathbb{E}\int_0^{T_{\varepsilon,\eta,K}}|R_i|\,\mathrm{d}t\\
\leq&\frac12\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]}\|v_{\varepsilon,\eta}\|_{H^{s-3}}^2+C\mathbb{E}\int_0^{T_{\varepsilon,\eta,K}}|R_3|\,\mathrm{d}t+\mathbb{E}\int_0^{T_{\varepsilon,\eta,K}}|R_2|\,\mathrm{d}t.
\end{aligned}$$

By (4.8), the mean value theorem for $\theta_R(\cdot)$ and Hypothesis H_1, we find

$$\|\mathcal{R}_7\|_{\mathcal{L}_2(\mathbb{U};H^{s-3})}\leq C\|v_{\varepsilon,\eta}\|_{H^{s-3}}f(\|u_\varepsilon\|_{H^s})\|u_\varepsilon\|_{H^s}.$$

and

$$\|\mathcal{R}_8\|_{\mathcal{L}_2(\mathbb{U};H^{s-3})}\leq Cg(K)\|v_{\varepsilon,\eta}\|_{H^{s-3}},\quad t\in[0,\tau^T_{\varepsilon,\eta,K}]\ \ \mathbb{P}-a.s.,$$

where $\mathcal{R}_7$ and $\mathcal{R}_8$ are given in (4.12) and g is given in Hypothesis H_1. Consequently, we can find a constant $C(K)>0$ such that

$$\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3}}\leq C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t')\|^2_{H^{s-3}}\,\mathrm{d}t+2\mathbb{E}\int_0^{T_{\varepsilon,\eta,K}}|R_2|\,\mathrm{d}t. \tag{4.16}$$

Claim: R_2 satisfies

$$|R_2|\lesssim\max\{\varepsilon^2,\eta^2\}\left(\|u_\varepsilon\|^4_{H^s}+\|u_\varepsilon\|^2_{H^s}\right)+\left(1+\|u_\varepsilon\|^2_{H^s}+\|u_\eta\|^2_{H^s}\right)\|v_{\varepsilon,\eta}\|^2_{H^{s-3}}. \tag{4.17}$$

To show this, we first notice that $J_\varepsilon J_\eta=J_\eta J_\varepsilon$, which means $J_\varepsilon^2-J_\eta^2=(J_\varepsilon+J_\eta)(J_\varepsilon-J_\eta)$, and then we have,

$$\begin{aligned}\left|(\mathcal{R}_1,v_{\varepsilon,\eta})_{H^{s-3}}\right|\leq&\|\left(J_\varepsilon^2-J_\eta^2\right)u_\varepsilon\|_{H^{s-1}}\|v_{\varepsilon,\eta}\|_{H^{s-3}}\\ \lesssim&\max\{\varepsilon,\eta\}\|u_\varepsilon\|_{H^s}\|v_{\varepsilon,\eta}\|_{H^{s-3}}\lesssim\max\{\varepsilon^2,\eta^2\}\|u_\varepsilon\|^2_{H^s}+\|v_{\varepsilon,\eta}\|^2_{H^{s-3}}.\end{aligned}$$

Since $H^{s-3}\hookrightarrow H^r$, we use the mean value theorem for $\theta_R(\cdot)$, (3.2) and Lemma 3.3 to derive

$$\begin{aligned}\left|(\mathcal{R}_3,v_{\varepsilon,\eta})_{H^{s-3}}\right|\leq&|\chi|\left[\theta_R(\|u_\varepsilon\|_{H^r})-\theta_R(\|u_\eta\|_{H^r})\right]\|J_\varepsilon F(J_\varepsilon u_\varepsilon)\|_{H^{s-3}}\|v_{\varepsilon,\eta}\|_{H^{s-3}}\\ \leq&C|\chi|\|v_{\varepsilon,\eta}\|^2_{H^{s-3}}\|F(J_\varepsilon u_\varepsilon)\|_{H^{s-3}}\\ \leq&C|\chi|\|v_{\varepsilon,\eta}\|^2_{H^{s-3}}\|u_\varepsilon\|_{H^{s-3}}\left(\|u_\varepsilon\|_{H^{s-3}}+\|u_\varepsilon\|_{H^{s-2}}\right)\\ \leq&C|\chi|\|v_{\varepsilon,\eta}\|^2_{H^{s-3}}\|u_\varepsilon\|^2_{H^s}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|(\mathcal{R}_4,v_{\varepsilon,\eta})_{H^{s-3}}\right|\leq&|\chi|\left\|(J_\varepsilon-J_\eta)\left[F(J_\varepsilon u_\varepsilon)\right]\right\|_{H^{s-3}}\|v_{\varepsilon,\eta}\|_{H^{s-3}}\\ \leq&C|\chi|\max\{\varepsilon,\eta\}\|u_\varepsilon\|^2_{H^s}\|v_{\varepsilon,\eta}\|_{H^{s-3}}\lesssim\max\{\varepsilon^2,\eta^2\}\|u_\varepsilon\|^4_{H^s}+\|v_{\varepsilon,\eta}\|^2_{H^{s-3}},\end{aligned}$$

$$\begin{aligned}\left|(\mathcal{R}_5,v_{\varepsilon,\eta})_{H^{s-3}}\right|\leq&|\chi|\|F(J_\varepsilon u_\varepsilon)-F(J_\eta u_\varepsilon)\|_{H^{s-3}}\|v_{\varepsilon,\eta}\|_{H^s}\\ \leq&C|\chi|\left(\|u_\varepsilon\|_{H^{s-2}}\|J_\varepsilon u_\varepsilon-J_\eta u_\varepsilon\|_{H^{s-3}}+\|u_\varepsilon\|_{H^{s-3}}\|J_\varepsilon u_\varepsilon-J_\eta u_\varepsilon\|_{H^{s-2}}\right)\|v_{\varepsilon,\eta}\|_{H^s}\\ \lesssim&\max\{\varepsilon,\eta\}\|u_\varepsilon\|^2_{H^s}\|v_{\varepsilon,\eta}\|_{H^{s-3}}\lesssim\max\{\varepsilon^2,\eta^2\}\|u_\varepsilon\|^4_{H^s}+\|v_{\varepsilon,\eta}\|^2_{H^{s-3}},\end{aligned}$$

and

$$\begin{aligned}\left|(\mathcal{R}_6,v_{\varepsilon,\eta})_{H^{s-3}}\right|\leq&|\chi|\left|(F(J_\eta u_\varepsilon)-F(J_\eta u_\eta),J_\eta v_{\varepsilon,\eta})_{H^{s-3}}\right|\\ =&C|\chi|\left|(F(J_\eta u_\varepsilon)-F(J_\eta u_\eta),J_\eta u_\varepsilon-J_\eta u_\eta)_{H^{s-3}}\right|\\ \lesssim&\left(\|u_\varepsilon\|_{H^{s-2}}+\|u_\eta\|_{H^{s-3}}\right)\|v_{\varepsilon,\eta}\|^2_{H^{s-3}}.\end{aligned}$$

Summarizing the above estimates for $(\mathcal{R}_i,v_{\varepsilon,\eta})_{H^{s-3}}$ with $i=1,3,4,5,6$, we obtain (4.17).

Combining (4.16) and (4.17), we have

$$\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3}}\leq C(K,\chi)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3}}\,\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\}, \tag{4.18}$$

which means that

$$\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-3}}\leq C(K,T,\chi)\max\{\varepsilon,\eta\}, \tag{4.19}$$

and hence (4.9) holds true. □

We first let that $\varepsilon=\varepsilon_n$ be countable set in Lemma 4.3 to guarantee that for all n, u_{ε_n} can be defined on the same set $\widetilde{\Omega}$ with $\mathbb{P}\{\widetilde{\Omega}\}=1$. Then we can find a subsequence converging almost surely. Precisely, we have

Lemma 4.3. *Let $s > \frac{d}{2} + 4$, $T > 0$ and $\{u_\varepsilon\}$ be given in Lemma 4.1. There is an $\{\mathcal{F}_t\}_{t\geq 0}$ progressive measurable H^s-valued process*

$$u \in L^2\left(\Omega; L^\infty\left(0, T; H^s\right)\right) \tag{4.20}$$

and a countable subsequence of $\{u_\varepsilon\}$ (still denoted as $\{u_\varepsilon\}$) such that

$$u_\varepsilon \xrightarrow{\varepsilon\to 0} u \text{ in } C([0,T]; H^{s-3}) \quad \mathbb{P}-a.s. \tag{4.21}$$

Proof. Recall (4.7) and (4.8). For any $\epsilon > 0$, by using Lemmas 4.1 and 4.2 and Chebyshev's inequality, we see that

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,T]} \|u_\varepsilon - u_\eta\|_{H^{s-3}} > \epsilon\right\}\\
=&\mathbb{P}\left\{\left(\left\{\tau^T_{\varepsilon,\eta,K} < T\right\} \cup \left\{\tau^T_{\varepsilon,\eta,K} = T\right\}\right) \cap \left\{\sup_{t\in[0,T]} \|u_\varepsilon - u_\eta\|_{H^{s-3}} > \epsilon\right\}\right\}\\
\leq&\mathbb{P}\left\{\tau^T_{\varepsilon,K} < T\right\} + \mathbb{P}\left\{\tau^T_{\eta,K} < T\right\} + \mathbb{P}\left\{\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]} \|u_\varepsilon - u_\eta\|_{H^{s-3}} > \epsilon\right\}\\
\leq&\frac{2C(R,T,u_0)}{K^2} + \mathbb{P}\left\{\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]} \|u_\varepsilon - u_\eta\|_{H^{s-3}} > \epsilon\right\}.
\end{aligned}$$

Now (4.9) clearly forces

$$\lim_{\varepsilon\to 0}\sup_{\eta\leq\varepsilon} \mathbb{P}\left\{\sup_{t\in[0,T]} \|u_\varepsilon - u_\eta\|_{H^{s-3}} > \epsilon\right\} \leq \frac{2C(R,T,u_0)}{K^2}, \quad K > 1.$$

Letting $K \to \infty$, we see that u_ε converges in probability in $C([0,T]; H^{s-3})$. Therefore, up to a further subsequence, (4.21) hold true. Now we prove (4.20). Since $H^s \hookrightarrow H^{s-3}$ is continuous, there exist continuous maps $\phi_m : H^{s-3} \to H^s (m \geq 1)$ such that

$$\|\phi_m u\|_{H^s} \leq \|u\|_{H^s} \quad \text{and} \quad \lim_{m\to\infty} \|\phi_m u\|_{H^s} = \|u\|_{H^s}, \quad u \in H^{s-3},$$

where $\|u\|_{H^s} := \infty$ if $u \notin H^s$. For example, one may take ϕ_m as the standard mollifier. Then it follows from Lemma 4.1 and Fatou's lemma that

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} \|u(t)\|^2_{H^s} \leq& \liminf_{m\to\infty} \mathbb{E}\sup_{t\in[0,T]} \|\phi_m u(t)\|^2_{H^s}\\
\leq& \liminf_{m\to\infty} \liminf_{\varepsilon\to 0} \mathbb{E}\sup_{t\in[0,T]} \|\phi_m u_\varepsilon(t)\|^2_{H^s}\\
\leq& \liminf_{m\to\infty} \liminf_{\varepsilon\to 0} \mathbb{E}\sup_{t\in[0,T]} \|u_\varepsilon(t)\|^2_{H^s} < C(R, u_0, T).
\end{aligned}$$

Hence (4.20) holds true. □

4.2. **Solving the cut-off problem.** With the help of Lemma 4.3, we can obtain the existence of a solution to (4.1).

Proposition 4.1. *Let $\chi \in \mathbb{R} \setminus \{0\}$, $s > \frac{d}{2} + 4$ and $r \in (\frac{d}{2} + 1, s - 3]$. Fix a stochastic basis $\mathcal{S}$ and let $u_0 \in L^2(\Omega; H^s)$ be an H^s-valued $\mathcal{F}_0$ measurable random variable. Assume σ satisfies Hypothesis H_1. For any $R > 1$ and $T > 0$, (4.1) has a solution $u \in L^2\left(\Omega; C\left([0,T]; H^s\right)\right)$.*

Proof. Since for each $\varepsilon \in (0,1)$, u_ε is $\{\mathcal{F}_t\}_{t\geq 0}$ progressive measurable, so is u. By Lemma 4.3 and the embedding $H^{s-3} \hookrightarrow H^r$, we can send $\varepsilon \to 0$ in (4.2) to conclude that u solves (4.1). To finish the proof, due to (4.20), we only need to prove that $u \in C\left([0,T]; H^s\right)$ almost surely. We first notice that Lemma 4.3 means $u \in C([0,T]; H^{s-3}) \cap L^\infty\left(0, T; H^s\right)$ almost surely. Since H^s is dense in H^{s-3}, we know that (cf. [53, page 263, Lemma 1.4]) $u \in C_w\left([0,T]; H^s\right)$, where $C_w\left([0,T]; H^s\right)$ is the space of weakly continuous functions with values in H^s. Therefore we only need to prove the continuity of $[0,T] \ni t \mapsto \|u(t)\|_{H^s}$.

In order to use the Itô formula in a Hilbert space, we recall the mollifier J_ε defined in Section 3, and then we consider the Itô formula for $\|J_\varepsilon u(t)\|_{H^s}^2$ rather than $\|u(t)\|_{H^s}^2$ (cf. Remark 3.1). Then we arrive at

$$\begin{aligned} \mathrm{d}\|J_\varepsilon u(t)\|_{H^s}^2 + 2\|\nabla J_\varepsilon u\|_{H^s}\,\mathrm{d}t =& 2\theta_R(\|u\|_{H^r})\left(J_\varepsilon\sigma(t,u)\,\mathrm{d}\mathcal{W}, J_\varepsilon u\right)_{H^s} \\ &- 2\chi\theta_R(\|u\|_{H^r})\left(J_\varepsilon F(u), J_\varepsilon u\right)_{H^s}\,\mathrm{d}t \\ &+ \theta_R^2(\|u\|_{H^r})\|J_\varepsilon\sigma(t,u)\|_{\mathcal{L}_2(\mathbb{U};H^s)}^2\,\mathrm{d}t. \end{aligned} \tag{4.22}$$

By (4.20),

$$\tau_N = \inf\{t \ge 0 : \|u(t)\|_{H^s} > N\} \to \infty \text{ as } N \to \infty \quad \mathbb{P}-a.s. \tag{4.23}$$

Then we only need to prove the continuity up to time $\tau_N \wedge T$ for each $N \ge 1$. For any $[t_2, t_1] \subset [0, T]$ with $t_1 - t_2 < 1$, we use Lemma 3.4, the BDG inequality and Hypothesis H_1 and (4.23) to find

$$\mathbb{E}\left[\left(\|J_\varepsilon u(t_1 \wedge \tau_N)\|_{H^s}^2 - \|J_\varepsilon u(t_2 \wedge \tau_N)\|_{H^s}^2\right)^4\right] \le C(N,T)|t_1 - t_2|^2.$$

Using Fatou's lemma, we arrive at

$$\mathbb{E}\left[\left(\|u(t_1 \wedge \tau_N)\|_{H^s}^2 - \|u(t_2 \wedge \tau_N)\|_{H^s}^2\right)^4\right] \le C(N,T)|t_1 - t_2|^2.$$

This and Kolmogorov's continuity theorem ensure the continuity of $t \mapsto \|u(t \wedge \tau_N)\|_{H^s}$. We complete the proof. □

Remark 4.1. Since the convergence is in H^{s-3}, and the cut-off involves H^r-norm, we have to let $r \le s-3$ to guarantee the validity of taking limit in (4.2).

4.3. **Solving the original problem.** Now we are going to prove (1) in Theorem 2.1. We first consider the pathwise uniqueness for the original problem (1.4) since we need similar estimate later.

Lemma 4.4. *Let $\chi \in \mathbb{R}\setminus\{0\}$, $s > \frac{d}{2} + 4$ and let Hypothesis H_1 be verified. Suppose that u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable satisfying $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. If $(\mathcal{S}, u_1, \tau_1)$ and $(\mathcal{S}, u_2, \tau_2)$ are two local solutions to (1.4), defined on the same basis $\mathcal{S}$ such that $\mathbb{E}\left(\sum_{i=1}^2 \sup_{t\in[0,\tau_i]}\|u_i(t)\|_{H^s}^2\right) < \infty$ and*

$$\mathbb{P}\{u_1(0) = u_2(0) = u_0(x)\} = 1,$$

then

$$\mathbb{P}\left\{u_1(t,x) = u_2(t,x),\ \forall (t,x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{R}^d\right\} = 1.$$

Proof. We first define the stopping time

$$\tau_K := \inf\{t \ge 0 : \|u_1(t)\|_{H^s} + \|u_2(t)\|_{H^s} > K\}. \tag{4.24}$$

For $i = 1, 2$, since $\mathbb{E}\left(\sum_{i=1}^2 \sup_{t\in[0,\tau_i]}\|u_i(t)\|_{H^s}^2\right) < \infty$, we have

$$\mathbb{P}\left\{\liminf_{K\to\infty} \tau_K > \tau_1 \wedge \tau_2\right\} = 1. \tag{4.25}$$

For any $T > 0$, we denote

$$\tau_K^T = \tau_K \wedge T. \tag{4.26}$$

Let $w = u_1 - u_2$. We have

$$\mathrm{d}w + \left[-\Delta w + \chi\left(F(u_1) - F(u_2)\right)\right]\mathrm{d}t = \left[\sigma(t,u_1) - \sigma(t,u_2)\right]\mathrm{d}\mathcal{W}.$$

Then we use the Itô formula and neglect the positive term $\int_0^t \|\nabla w\|_{L^2}^2 \,\mathrm{d}t'$ on the left hand side of the resulting equation to find that

$$\begin{aligned}\|w(t)\|_{L^2}^2 \leq & 2\left|\left(\int_0^t (\sigma(t,u_1)-\sigma(t,u_2))\,\mathrm{d}\mathcal{W}, w\right)_{L^2}\right| \\ & + 2|\chi|\int_0^t |(F(u_1)-F(u_2),w)_{L^2}|\,\mathrm{d}t' \\ & + \int_0^t \|\sigma(t',u_1)-\sigma(t',u_2)\|_{\mathcal{L}_2(\mathbb{U};L^2)}^2\,\mathrm{d}t' \\ = & H_1 + \int_0^t H_2\,\mathrm{d}t' + \int_0^t H_3\,\mathrm{d}t'.\end{aligned}$$

Taking a supremum over $t \in [0, \tau_K^T]$ and using Hypothesis H_1, BDG inequality, (4.24), (4.26) and Cauchy–Schwarz inequality yield a constant $C = C(K) > 0$ such that

$$\begin{aligned} & \mathbb{E}\sup_{t\in[0,\tau_K^T]}\|w(t)\|_{L^2}^2 \\ \leq & C\mathbb{E}\left(\int_0^{\tau_K^T}\|\sigma(t,u_1)-\sigma(t,u_2)\|_{\mathcal{L}_2(\mathbb{U};L^2)}^2\|w\|_{L^2}^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \sum_{k=2}^3\mathbb{E}\int_0^{\tau_K^T}|H_k|\,\mathrm{d}t \\ \leq & Cg^2(K)\mathbb{E}\left(\sup_{t\in[0,\tau_K^T]}\|w\|_{L^2}^2\cdot\int_0^{\tau_K^T}\|w\|_{L^2}^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \sum_{k=2}^3\mathbb{E}\int_0^{\tau_K^T}|H_k|\,\mathrm{d}t \\ \leq & \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_K^T]}\|w\|_{L^2}^2 + C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_K^t]}\|w(t')\|_{L^2}^2\,\mathrm{d}t + \sum_{k=2}^3\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_K^t]}|H_k(t')|\,\mathrm{d}t.\end{aligned}$$

Using Lemma 3.3, Hypothesis H_1, (4.24) and (4.26), we get

$$\sum_{i=2}^3\mathbb{E}\int_0^{\tau_K^T}H_i\,\mathrm{d}t \leq C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_K^t]}\|w(t')\|_{L^2}^2\,\mathrm{d}t, \quad C = C(K,\chi).$$

As a result, we find that for some $C = C(K,\chi)$,

$$\mathbb{E}\sup_{t\in[0,\tau_k^T]}\|w(t)\|_{L^2}^2 \leq C\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_K^t]}\|w(t')\|_{L^2}^2\,\mathrm{d}t.$$

Hence $\mathbb{E}\sup_{t\in[0,\tau_K^T]}\|w(t)\|_{L^2}^2 = 0$. Sending $K, T \to \infty$ and using the monotone convergence theorem and (4.25) yield

$$\mathbb{P}\left\{u_1(t,x) = u_2(t,x),\ \forall (t,x) \in [0, \tau_1\wedge\tau_2)\times\mathbb{R}^d\right\} = 1,$$

which is the desired result. □

According to Proposition 4.1 and Lemma 4.4, to prove (1) in Theorem 2.1, we only need to remove the cut-off function. The method used here is inspired by the works [25, 27, 36].

Proof of (1) *in Theorem 2.1.* For $u_0(\omega,x) \in L^2(\Omega; H^s)$ with $s > \frac{d}{2}+4$, we consider

$$\Omega_k = \{k-1 \leq \|u_0\|_{H^s} < k\},\ k\in\mathbb{N},\ k\geq 1.$$

Since $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we have

$$u_0(\omega,x) = \sum_{k\geq 1}u_{0,k}(\omega,x) = \sum_{k\geq 1}u_0(\omega,x)\mathbf{1}_{k-1\leq\|u_0\|_{H^s}<k}\ \ \mathbb{P}-a.s.$$

On account of Proposition 4.1, we let $u_{k,R}$ be the pathwise global solution to the cut-off problem (4.1) with initial value $u_{0,k}$ and cut-off function $\theta_R(\cdot)$. Define

$$\tau_{k,R} = \inf\left\{ t > 0 : \sup_{t' \in [0,t]} \|u_{k,R}(t')\|_{H^s}^2 > \|u_{0,k}\|_{H^s}^2 + 2 \right\}. \tag{4.27}$$

Then for any $R > 0$ and $k \in \mathbb{N}$, we have $\mathbb{P}\{\tau_{k,R} > 0\} = 1$. Assign $R = R_k > \sqrt{k^2+2}$ to be discrete for each $k \geq 1$ and then denote $(u_k, \tau_k) = (u_{k,R_k}, \tau_{k,R_k})$. Obviously, $\mathbb{P}\{\tau_k > 0,\ \forall k \geq 1\} = 1$. Then it follows from the inequality $\|\cdot\|_{H^r} \leq \|\cdot\|_{H^s}$ that

$$\mathbb{P}\left\{ \|u_k\|_{H^r}^2 \leq \|u_k\|_{H^s}^2 \leq \|u_{0,k}\|_{H^s}^2 + 2 < R_k^2,\ \forall t \in [0,\tau_k],\ \forall k \geq 1 \right\} = 1.$$

This, together with the definition of $\theta_R(\cdot)$ implies (u_k, τ_k) is the pathwise solution to (1.4) with initial value $u_{0,k}$. As a result, we find that

$$\begin{aligned} &\mathbf{1}_{\Omega_k} u_k(t \wedge \tau_k) - \mathbf{1}_{\Omega_k} u_{0,k} \\ =& \int_0^{t \wedge \mathbf{1}_{\Omega_k} \tau_k} \mathbf{1}_{\Omega_k} \left[\Delta u_k - \chi F(u_k) \right] \, \mathrm{d}t' + \int_0^{t \wedge \mathbf{1}_{\Omega_k} \tau_k} \mathbf{1}_{\Omega_k} \sigma(t, u_k) \, \mathrm{d}\mathcal{W}. \end{aligned}$$

Besides, it has that

$$\mathbf{1}_{\Omega_k} \sigma(t, u_k) = \sigma(t, \mathbf{1}_{\Omega_k} u_k) - \mathbf{1}_{\Omega_k^C} \sigma(t, 0),$$

Since $\|\sigma(t, \mathbf{0})\|_{\mathcal{L}_2(\mathbb{U}; H^s)} < \infty$ (cf. Hypothesis H_1), we have

$$\int_0^{t \wedge \mathbf{1}_{\Omega_k} \tau_k} \mathbf{1}_{\Omega_k} \sigma(t, u_k) \, \mathrm{d}\mathcal{W} = \int_0^{t \wedge \mathbf{1}_{\Omega_k} \tau_k} \sigma(t, \mathbf{1}_{\Omega_k} u_k) \, \mathrm{d}\mathcal{W}.$$

Similarly, $\mathbf{1}_{\Omega_k} \left[\Delta u_k - \chi F(u_k) \right] = \left[\Delta(\mathbf{1}_{\Omega_k} u_k) - \chi F(\mathbf{1}_{\Omega_k} u_k) \right]$, and hence

$$\begin{aligned} &\mathbf{1}_{\Omega_k} u_k(t \wedge \tau_k) - \mathbf{1}_{\Omega_k} u_{0,k} \\ =& \mathbf{1}_{\Omega_k} u_k(t \wedge \mathbf{1}_{\Omega_k} \tau_k) - u_{0,k} \\ =& \int_0^{t \wedge \mathbf{1}_{\Omega_k} \tau_k} \left[\Delta(\mathbf{1}_{\Omega_k} u_k) - \chi F(\mathbf{1}_{\Omega_k} u_k) \right] \, \mathrm{d}t' + \int_0^{t \wedge \mathbf{1}_{\Omega_k} \tau_k} \sigma(t, \mathbf{1}_{\Omega_k} u_k) \, \mathrm{d}\mathcal{W}. \end{aligned}$$

which implies that $(\mathbf{1}_{\Omega_k} u_k, \mathbf{1}_{\Omega_k} \tau_k)$ is a solution to (1.4) with initial data $u_{0,k}$. Since $\Omega_k \bigcap \Omega_{k'} = \emptyset$ for $k \neq k'$ and $\bigcup_{k \geq 1} \Omega_k$ is a set of full measure, we see that

$$\left(u = \sum_{k \geq 1} \mathbf{1}_{k-1 \leq \|u_0\|_{H^s} < k} u_k, \quad \tau = \sum_{k \geq 1} \mathbf{1}_{k-1 \leq \|u_0\|_{H^s} < k} \tau_k \right)$$

is a pathwise solution to (1.4) corresponding to the initial condition u_0. Besides, using (4.27), we have

$$\begin{aligned} \sup_{t \in [0,\tau]} \|u\|_{H^s}^2 =& \sum_{k \geq 1} \mathbf{1}_{k-1 \leq \|u_0\|_{H^s} < k} \sup_{t \in [0,\tau_k]} \|u_k\|_{H^s}^2 \\ \leq& \sum_{k \geq 1} \mathbf{1}_{k-1 \leq \|u_0\|_{H^s} < k} \left(\|u_{0,k}\|_{H^s}^2 + 2 \right) \leq 2\|u_0\|_{H^s}^2 + 4. \end{aligned}$$

Taking expectation gives rise to (2.3). Uniqueness comes from Lemma 4.4. The extension from a local solution to the maximal solution can be obtained as in [25, 46], here we omit the details. □

4.4. **Blow-up criterion.** Now we prove (2) in Theorem 2.1. We first establish the following lemma, which asserts the relationship between the explosion time of $\|u(t)\|_{H^s}$ and that of $\|u(t)\|_{H^r}$, is the key step to obtain the blow-up criterion. It is motivated by the recent work for the stochastic Euler equation [14].

Proof of (2) in Theorem 2.1. Let (u, τ) be the pathwise solution to (1.4) guaranteed by (1) in Theorem 2.1. Recall that $\gamma \in (\frac{d}{2}+1, s]$. Define

$$\tau_{1,m} = \inf\left\{ t \geq 0 : \|u(t)\|_{H^s} \geq m \right\}, \quad \tau_{2,n} = \inf\left\{ t \geq 0 : \|u(t)\|_{H^\gamma} \geq n \right\},$$

and let $\tau_1 = \lim_{m\to\infty} \tau_{1,m}$ and $\tau_2 = \lim_{n\to\infty} \tau_{2,n}$. To prove (2) in Theorem 2.1, we only need to show

$$\tau_1 = \tau_2 \ \mathbb{P} - a.s.$$

Since $u(\cdot \wedge \tau) \in C([0,\infty); H^s)$, we have $[u(t)]^{-1}(Y) = [u(t)]^{-1}(H^s \cap Y)$, $\forall Y \in \mathcal{B}(H^r)$. Therefore $u(t)$, as an H^γ-valued process, is also $\mathcal{F}_t$ adapted. Moreover,

$$\sup_{t\in[0,\tau_{1,m}]} \|u(t)\|_{H^\gamma} \le \sup_{t\in[0,\tau_{1,m}]} \|u(t)\|_{H^s} \le m,$$

which implies $\tau_{1,m} \le \tau_{2,m} \le \tau_2$ $\mathbb{P} - a.s.$ Therefore we have

$$\tau_1 \le \tau_2 \ \ \mathbb{P} - a.s. \tag{4.28}$$

Now we prove the converse inequality. We first notice that for all $n, k \in \mathbb{Z}^+$,

$$\left\{ \sup_{t\in[0,\tau_{2,n}\wedge k]} \|u(t)\|_{H^s} < \infty \right\} \subset \bigcup_{m\in\mathbb{Z}^+} \{\tau_{2,n} \wedge k \le \tau_{1,m}\} \subset \{\tau_{2,n} \wedge k \le \tau_1\}.$$

If we can show

$$\mathbb{P}\left\{ \sup_{t\in[0,\tau_{2,n}\wedge k]} \|u(t)\|_{H^s} < \infty \right\} = 1, \quad \forall n, k \in \mathbb{Z}^+, \tag{4.29}$$

then for all $n, k \in \mathbb{Z}^+$, $\mathbb{P}\{\tau_{2,n} \wedge k \le \tau_1\} = 1$ and

$$\mathbb{P}\{\tau_2 \le \tau_1\} = \mathbb{P}\left\{ \bigcap_{n\in\mathbb{Z}^+} \{\tau_{2,n} \le \tau_1\} \right\} = \mathbb{P}\left\{ \bigcap_{n,k\in\mathbb{Z}^+} \{\tau_{2,n} \wedge k \le \tau_1\} \right\} = 1. \tag{4.30}$$

Notice that (4.28) and (4.30) imply the desired result. Since (4.30) requires the assumption (4.29), the proof is completed by proving (4.29). As is mentioned in Proposition 4.1 (see also Remark 3.1), we will first consider the Itô formula for $\|J_\varepsilon u\|^2_{H^s}$ instead of $\|u\|^2_{H^s}$. Then for any $t > 0$, we have

$$\begin{aligned} & \mathrm{d}\|J_\varepsilon u(t)\|^2_{H^s} + 2\|\nabla J_\varepsilon u(t)\|^2_{H^s}\,\mathrm{d}t \\ =& 2\left(J_\varepsilon \sigma(t,u)\,\mathrm{d}\mathcal{W}, J_\varepsilon u\right)_{H^s} - 2\chi\left(J_\varepsilon F(u), J_\varepsilon u\right)_{H^s}\,\mathrm{d}t \\ & + \|J_\varepsilon \sigma(t,u)\|^2_{\mathcal{L}_2(\mathbb{U};H^s)}\,\mathrm{d}t. \end{aligned} \tag{4.31}$$

Therefore we have

$$\begin{aligned} & \|J_\varepsilon u(t)\|^2_{H^s} - \|J_\varepsilon u(0)\|^2_{H^s} \\ \le & 2\left(\int_0^t J_\varepsilon \sigma(t',u)\,\mathrm{d}\mathcal{W}, J_\varepsilon u\right)_{H^s} - 2\chi\int_0^t \left(J_\varepsilon F(u), J_\varepsilon u\right)_{H^s}\,\mathrm{d}t' \\ & + \int_0^t \|J_\varepsilon \sigma(t',u)\|^2_{\mathcal{L}_2(\mathbb{U};H^s)}\,\mathrm{d}t' \\ =& L_1 + \sum_{j=2}^3 \int_0^t L_j\,\mathrm{d}t'. \end{aligned}$$

It follows from the properties of J_ε, the BDG inequality, $H^\gamma \hookrightarrow W^{1,\infty}$ and Hypothesis H_1 that

$$\mathbb{E}\left(\sup_{t\in[0,\tau_{2,n}\wedge k]} |L_1(t)| \right) \le \frac{1}{2}\mathbb{E} \sup_{t\in[0,\tau_{2,n}\wedge k]} \|J_\varepsilon u\|^2_{H^s} + Cf^2(n)\int_0^k \left(1 + \mathbb{E}\|u\|^2_{H^s}\right)\,\mathrm{d}t.$$

We use Lemma 3.4 to find

$$\mathbb{E}\int_0^{\tau_{2,n}\wedge k} |L_2|\,\mathrm{d}t \le C|\chi|\mathbb{E}\int_0^{\tau_{2,n}\wedge k} \|u\|_{H^\gamma}\|u\|^2_{H^s}\,\mathrm{d}t \le C|\chi|n\int_0^k \mathbb{E}\|u\|^2_{H^s}\,\mathrm{d}t$$

Similarly, Hypothesis H_1 yields

$$\mathbb{E}\int_0^{\tau_{2,n}\wedge k} |L_3|\,\mathrm{d}t \le Cf^2(n)\int_0^k \left(1 + \mathbb{E}\|u\|^2_{H^s}\right)\,\mathrm{d}t.$$

Therefore we combine the above estimates to find

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|J_\varepsilon u(t)\|_{H^s}^2\leq 2\mathbb{E}\|u_0\|_{H^s}^2+C\int_0^k\left(1+\mathbb{E}\sup_{t'\in[0,t\wedge\tau_{2,n}]}\|u(t')\|_{H^s}^2\right){\rm d}t,$$

where C depends on χ and n. Notice that the right hand side of the above estimate does not depend on ε and J_ε tends to u for all $u\in C([0,T];H^s)$ with any $T>0$. We send $\varepsilon\to 0$ and then use the Grönwall's inequality to obtain that for each $n,k\in\mathbb{Z}^+$, there is a $C(\chi,n,k,u_0)>0$ such that

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|u(t)\|_{H^s}^2<C(\chi,n,k,u_0),$$

which gives (4.29). The proof is completed. □

4.5. **Non-negativity of solutions.**

Proof of (3) *in Theorem 2.1.* We first consider the non-negativity of solutions for the cut-off problem (4.1) such that global existence holds. We recall the definitions of k,k_ε and ρ in Lemma 3.5 and consider the nonlinear functional: $\Phi_\varepsilon(\cdot)=\int_{\mathbb{R}^d}k_\varepsilon(\cdot)\,{\rm d}x:L^2\mapsto\mathbb{R}$. Let $R>1$. Using the Itô formula (see Theorem 2.10 in [22]) for $\Phi_\varepsilon(u)=\int_{\mathbb{R}^d}k_\varepsilon(u)\,{\rm d}x$ yields that for any $t>0$,

$$\begin{aligned}\Phi_\varepsilon(u(t))=&\Phi_\varepsilon(u(0))+\int_0^t\left(k_\varepsilon'(u),\Delta u\right)_{L^2}{\rm d}t'\\&-\chi\int_0^t\theta_R(\|u\|_{H^r})\left(k_\varepsilon'(u),F(u)\right)_{L^2}{\rm d}t'\\&+\left(\int_0^t\theta_R(\|u\|_{H^r})\sigma(t',u)\,{\rm d}\mathcal{W},k_\varepsilon'(u)\right)_{L^2}\\&+\frac12\int_0^t\sum_{j=1}^\infty\theta_R^2(\|u\|_{H^r})\left(k_\varepsilon''(u)\left(\sigma(t,u)[\sigma(t,u)]^*y_j\right),y_j\right)_{L^2}{\rm d}t',\end{aligned}$$

where $\{y_j\}$ is a complete orthonormal basis of L^2. Taking expectation over the above equality yields

$$\begin{aligned}&\mathbb{E}\Phi_\varepsilon(u(t))-\mathbb{E}\Phi_\varepsilon(u(0))\\=&-\mathbb{E}\int_0^t\left(k_\varepsilon''(u),|\nabla u|^2\right)_{L^2}{\rm d}t'-\chi\mathbb{E}\int_0^t\theta_R(\|u\|_{H^r})\left(k_\varepsilon'(u),F(u)\right)_{L^2}{\rm d}t'\\&+\frac12\mathbb{E}\int_0^t\sum_{j=1}^\infty\theta_R^2(\|u\|_{H^r})\left(k_\varepsilon''(u)\left(\sigma(t,u)[\sigma(t,u)]^*y_j\right),y_j\right)_{L^2}{\rm d}t'.\end{aligned}$$

We notice that $\nabla(\mathbf{1}_{\{u<0\}}u)=\nabla u$ when $u<0$ (see Lemma 7.6 in [24]). Noticing the uniformly convergence in Lemma 3.5, $\theta_R^2(\|u\|_{H^r})\leq 1$, one can send $\varepsilon\to 0$ to find

$$\begin{aligned}&\mathbb{E}\|\rho(u(t))\|_{L^2}^2\\=&\mathbb{E}\|\rho(u_0)\|_{L^2}^2+2\chi\mathbb{E}\int_0^t\theta_R(\|u\|_{H^r})\left(\rho(u),F(u)\right)_{L^2}{\rm d}t'\\&+\mathbb{E}\int_0^t a(u)\theta_R^2(\|u\|_{H^r})\sum_{j=1}^\infty\left(\sigma(t,u)[\sigma(t,u)]^*y_j,y_j\right)_{L^2}{\rm d}t'-2\mathbb{E}\int_0^t a(u)\|\nabla u\|_{L^2}^2\,{\rm d}t'\\=&2\chi\mathbb{E}\int_0^t\theta_R(\|u\|_{H^r})\left(\rho(u),F(u)\right)_{L^2}{\rm d}t'\\&+\mathbb{E}\int_0^t a(u)\theta_R^2(\|u\|_{H^r})\|\sigma(t,u)\|_{\mathcal{L}_2(\mathbb{U};L^2)}^2\,{\rm d}t'-2\mathbb{E}\int_0^t a(u)\|\nabla u\|_{L^2}^2\,{\rm d}t'.\end{aligned}\tag{4.32}$$

It is easily seen from (1.3) that

$$\|F(u)\|_{L^2}\leq\|Q(u)\|_{L^2}\|\nabla u\|_{L^\infty}+\|u\|_{L^2}\|{\rm div}Q(u)\|_{L^\infty}\leq\|u\|_{L^2}\|u\|_{H^r}$$

and hence

$$|\theta_R(\|u\|_{H^r})\,(\rho(u),F(u))_{L^2}| = \left|\theta_R(\|u\|_{H^r})\mathbf{1}_{\{u<0\}}\,(u,F(u))_{L^2}\right| \le 2R\|\rho(u)\|_{L^2}^2.$$

Combining Hypothesis H_2 and the above observation, we have

$$\begin{aligned}\mathbb{E}\|\rho(u(t))\|_{L^2}^2 \le& 4|\chi|R\mathbb{E}\int_0^t \|\rho(u)\|_{L^2}^2\,\mathrm{d}t' + C\mathbb{E}\int_0^t a(u)\|u\|_{L^2}^2\,\mathrm{d}t'\\ \le& 4|\chi|R\mathbb{E}\int_0^t \|\rho(u)\|_{L^2}^2\,\mathrm{d}t' + C\mathbb{E}\int_0^t \|-\rho(u)\|_{L^2}^2\,\mathrm{d}t'\\ \le& C(\chi,R)\int_0^t \mathbb{E}\|\rho(u)\|_{L^2}^2\,\mathrm{d}t',\end{aligned}$$

which implies that for any $R>1$,

$$\mathbb{E}\|\rho(u(t))\|_{L^2}^2 = 0, \quad \forall t\in[0,\infty).$$

Hence $u\ge 0$ almost surely for all $t\ge 0$. Because u has continuous path, then (3) in Theorem 2.1 holds true. Now we are left with the task of removing the cut-off function, which can be obtained by using the decomposition technique as in subsection 4.3 again, which is omit here for brevity. □

5. Regularization effect of multiplicative noise

5.1. **Non-autonomous linear case.** Now we consider (1.5). This case covers the linear noise case $\beta u\,\mathrm{d}W$ in [25, 35, 46, 50]. We introduce the following Girsanov type transform

$$z=\frac{1}{\eta(\omega,t)}u, \quad \eta(\omega,t)=\mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'}-\int_0^t \frac{\beta^2(t')}{2}\,\mathrm{d}t'}, \tag{5.1}$$

Then the following result can be established.

Proposition 5.1. *Let $s>\frac{d}{2}+4$ and $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be a fixed stochastic basis. Let $u_0(\omega,x)$ be an H^s-valued $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2<\infty$ and let Hypothesis H_3 be verified. Then (1.5) admits a unique maximal solution (u,τ^*). For $t\in[0,\tau^*)$, z defined by (5.1) solves the following problem almost surely,*

$$\begin{cases} z_t-\Delta z+\chi\eta F(z)=0, & x\in\mathbb{R}^d,\ t>0,\\ z(\omega,0,x)=u_0(\omega,x), \end{cases} \tag{5.2}$$

where $F(\cdot)$ and η are given by (1.3) and (5.1), respectively. Equivalently, z and $b=(I-\Delta)^{-1}z=\frac{1}{\eta}(I-\Delta)^{-1}u$ satisfy

$$\begin{cases} z_t-\Delta z+\chi\eta\mathrm{div}(z\nabla b)=0, & x\in\mathbb{R}^d,\ t>0,\\ -\Delta b=z-b, & x\in\mathbb{R}^d,\ t>0,\\ z(\omega,0,x)=u_0(\omega,x). \end{cases} \tag{5.3}$$

Moreover, $z\in C([0,\tau^*);H^s)\cap C^1([0,\tau^*);H^{s-2})$ $\mathbb{P}-a.s.$ Furthermore, if $u_0\ge0$ $\mathbb{P}-a.s.$, then $z\ge0$ $\mathbb{P}-a.s.$ and $\|z\|_{L^1}=\|u_0\|_{L^1}$ $\mathbb{P}-a.s.$

Proof. Since $\beta(t)$ satisfies Hypothesis H_3, $\sigma(t,u)=\beta(t)u$ satisfies Hypotheses H_1 and H_2. Consequently, (1) in Theorem 2.1 implies that (1.5) has a unique maximal solution (u,τ^*). Direct computation with Itô formula yields

$$\mathrm{d}\frac{1}{\eta}=-\beta(t)\frac{1}{\eta}\,\mathrm{d}W+\beta^2(t)\frac{1}{\eta}\,\mathrm{d}t.$$

Then it immediately follows from (5.1) that

$$\mathrm{d}z=\Delta z\,\mathrm{d}t-\chi\eta F(z)\,\mathrm{d}t.$$

Since $z(0)=u_0(\omega,x)$, we see that z solves (5.2), $\mathbb{P}-a.s.$ Using $b=(I-\Delta)^{-1}z=\frac{1}{\eta}(I-\Delta)^{-1}u$, we see that (5.2) is equivalent to (5.3). Moreover, (1) in Theorem 2.1 implies the regularity of u, and therefore the regularity of z. If $u_0\ge0$ $\mathbb{P}-a.s.$, then (3) in Theorem 2.1 gives that $z\ge0$ $\mathbb{P}-a.s.$ □

5.1.1. *Decay of L^∞ norm in $\mathbb{R}^2$ almost surely.*

Proof of Theorem 2.2. We first recall (2.4). It follows from (1) in Lemma 3.6 that

$$A(\omega)=\sup_{t>0}\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}-\int_0^t\frac{\beta^2(t')}{2}\,\mathrm{d}t'}<\infty\ \mathbb{P}-a.s. \tag{5.4}$$

Since $z\geq 0$ almost surely, b is also non-negative almost surely due to the maximum principle for $(5.3)_2$.

Step 1: The estimate of $\|z\|_{L^p}$ for $1<p<\infty$. We multiply the first equation of (5.3) by z^{p-1} $(1<p<\infty)$ with noticing $z,b\geq 0$ to find that for $t\in[0,\tau^*)$,

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}z^p\,\mathrm{d}x=&\int_{\mathbb{R}^d}z^{p-1}\Delta z\,\mathrm{d}x-\chi\eta\int_{\mathbb{R}^d}z^{p-1}\mathrm{div}(z\nabla b)\,\mathrm{d}x\\ =&-(p-1)\int_{\mathbb{R}^d}z^{p-2}|\nabla z|^2\,\mathrm{d}x+\chi(p-1)\eta\int_{\mathbb{R}^d}(z^{p-1}\nabla z\cdot\nabla b)\,\mathrm{d}x\\ =&-\frac{4(p-1)}{p^2}\int_{\mathbb{R}^d}\left|\nabla z^{\frac{p}{2}}\right|^2\mathrm{d}x+\chi\frac{p-1}{p}\eta\int_{\mathbb{R}^d}z^p(z-b)\,\mathrm{d}x\\ \leq&-\frac{4(p-1)}{p^2}\int_{\mathbb{R}^d}\left|\nabla z^{\frac{p}{2}}\right|^2\mathrm{d}x+\chi\frac{p-1}{p}\eta\int_{\mathbb{R}^d}z^{p+1}\,\mathrm{d}x\quad\mathbb{P}-a.s.\end{aligned}\tag{5.5}$$

When $d=2$, we use the Gagliardo-Nirenberg-Sobolev inequality to find that

$$\begin{aligned}\eta(t)\int_{\mathbb{R}^2}z^{p+1}\,\mathrm{d}x=&\eta(t)\|z^{\frac{p+1}{2}}\|^2_{L^2(\mathbb{R}^2)}\\ \leq&C\eta(t)\left\|\nabla z^{\frac{p+1}{2}}\right\|^2_{L^1(\mathbb{R}^2)}\\ =&C\eta(t)\left(\frac{p+1}{p}\right)^2\left(\int_{\mathbb{R}^2}z^{\frac12}\left|\nabla z^{p/2}\right|\mathrm{d}x\right)^2\\ \leq&\left\|\nabla z^{p/2}\right\|^2_{L^2(\mathbb{R}^2)}\left(C\eta(t)\left(\frac{p+1}{p}\right)^2\|z\|_{L^1(\mathbb{R}^2)}\right)\\ \leq&\left\|\nabla z^{p/2}\right\|^2_{L^2(\mathbb{R}^2)}C\left(\frac{p+1}{p}\right)^2A\|u_0\|_{L^1(\mathbb{R}^2)}\ \ \mathbb{P}-a.s.,\end{aligned}$$

where A is defined by (2.4) and satisfies (5.4). The above estimate and (5.5) give rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}z^p\,\mathrm{d}x\leq(p-1)\left\|\nabla z^{p/2}\right\|^2_{L^2}\left[\frac{-4}{p}+\chi C\left(\frac{p+1}{p}\right)^2A\|u_0\|_{L^1}\right],\quad 1<p<\infty. \tag{5.6}$$

If (2.5) holds true, then

$$\mathbb{P}\left\{\|u_0\|_{L^1}\leq\frac{4}{\chi CpA}\left(\frac{p}{p+1}\right)^2,\ 2\leq p\leq 4\right\}=1,$$

which together with (5.6) gives that

$$\mathbb{P}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}z^p\,\mathrm{d}x\leq 0,\quad 2\leq p\leq 4,\quad t\in[0,\tau^*)\right\}=1.$$

That is to say,

$$\mathbb{P}\left\{\|z\|_{L^p}\leq\|u_0\|_{L^p},\quad 2\leq p\leq 4,\quad t\in[0,\tau^*)\right\}=1. \tag{5.7}$$

Besides, by $(5.3)_2$, $W^{2,r}\hookrightarrow W^{1,\infty}$ with $r>2$ and (5.7), we have

$$\mathbb{P}\left\{\|\nabla b\|_{L^\infty}\lesssim\|u_0\|_{L^p},\quad 2\leq p\leq 4,\quad t\in[0,\tau^*)\right\}=1.$$

For a.e. $\omega \in \Omega$ and for $1 < p < \infty$, using (5.7), Gagliardo-Nirenberg interpolation inequality and Young's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^2} z^{p+1}\,\mathrm{d}x \le& \|z\|_{L^2}\left\|z^{\frac{p}{2}}\right\|_{L^4}^2\\ \le& C\|u_0\|_{L^2}\left\|z^{\frac{p}{2}}\right\|_{L^2}\left\|\nabla z^{\frac{p}{2}}\right\|_{L^2}\\ \le& \frac{C^2\|u_0\|_{L^2}^2 p\chi A}{8}\|z\|_{L^p}^p + \frac{2}{p\chi A}\left\|\nabla z^{\frac{p}{2}}\right\|_{L^2}^2.\end{aligned}$$

Combining the above estimate and (5.5), we have that almost surely,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^p\,\mathrm{d}x\\ \le& -\frac{4(p-1)}{p}\int_{\mathbb{R}^d}\left|\nabla z^{\frac{p}{2}}\right|^2\mathrm{d}x + \chi(p-1)\eta\int_{\mathbb{R}^2} z^{p+1}\,\mathrm{d}x\\ \le& -\frac{4(p-1)}{p}\int_{\mathbb{R}^d}\left|\nabla z^{\frac{p}{2}}\right|^2\mathrm{d}x + \chi(p-1)\eta\left(\frac{C^2\|u_0\|_{L^2}^2 p\chi A}{8}\|z\|_{L^p}^p + \frac{2}{p\chi A}\left\|\nabla z^{\frac{p}{2}}\right\|_{L^2}^2\right)\\ \le& -\frac{2(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla z^{\frac{p}{2}}\right|^2\mathrm{d}x + C^2\|u_0\|_{L^2}^2\chi^2A^2p(p-1)\|z\|_{L^p}^p, \quad t\in[0,\tau^*), \quad 1<p<\infty.\end{aligned}$$

Let $M_1(\omega) = C^2\|u_0\|_{L^2}^2\chi^2A^2$. Since $A(\omega) < \infty$ almost surely (by (5.4)) and $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we have $0 < M_1(\omega) < \infty$ almost surely. Then we have almost surely that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^p\,\mathrm{d}x + \frac{2(p-1)}{p}\int_{\mathbb{R}^2}\left|\nabla z^{\frac{p}{2}}\right|^2\mathrm{d}x \le M_1p(p-1)\int_{\mathbb{R}^2} z^p\,\mathrm{d}x, \quad t\in[0,\tau^*), \quad 1<p<\infty. \tag{5.8}$$

It follows from the Grönwall's inequality and the above inequality that for any $1 < p < \infty$,

$$\int_{\mathbb{R}^2} z^p\,\mathrm{d}x \le \|u_0\|_{L^p}^p \mathrm{e}^{M_1p(p-1)t}, \quad t\in[0,\tau^*) \quad \mathbb{P}-a.s.$$

However, this L^p bound depends on t. Now we need to establish a pathwise uniform-in-time L^∞ estimate for z.

Step 2: The estimate of $\|z\|_{L^\infty}$: the Moser-Alikakos iteration. With (5.4) in mind, we will extend the well-known Moser-Alikakos iteration technique (see [1, 51]) to this non-autonomous random system to obtain the pathwise uniform-in-time L^∞ estimate for z. For $k \in \mathbb{N}^+$, we let $a_k = 2^k$. Then almost surely it has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x + 2\frac{a_k-1}{a_k}\int_{\mathbb{R}^2}\left|\nabla z^{a_{k-1}}\right|^2\mathrm{d}x \le M_1a_k^2\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x, \quad k\in\mathbb{N}^+. \tag{5.9}$$

Using the Gagliardo-Nirenberg interpolation inequality and Young's inequality, we arrive at

$$\begin{aligned}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x = \|z^{a_{k-1}}\|_{L^2}^2 \le& C\|z^{a_{k-1}}\|_{L^1}\|\nabla z^{a_{k-1}}\|_{L^2}\\ \le& C\left(\frac{1}{4\varepsilon}\|z^{a_{k-1}}\|_{L^1}^2 + \varepsilon\|\nabla z^{a_{k-1}}\|_{L^2}^2\right), \ \varepsilon > 0.\end{aligned}$$

Therefore we add both sides of (5.9) by $\varepsilon\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x$ and then use the above estimate to find that for $k \ge 1$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x + 2\frac{a_k-1}{a_k}\int_{\mathbb{R}^2}\left|\nabla z^{a_{k-1}}\right|^2\mathrm{d}x + \varepsilon\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x\\ \le& \left(M_1a_k^2+\varepsilon\right)\left(C\varepsilon\int_{\mathbb{R}^2}\left|\nabla z^{a_{k-1}}\right|^2\mathrm{d}x\right) + \left(M_1a_k^2+\varepsilon\right)\left(C\frac{1}{4\varepsilon}\|z^{a_{k-1}}\|_{L^1}^2\right) \quad \mathbb{P}-a.s.\end{aligned}$$

Remember that $M_1 < \infty$ almost surely. For a.e. $\omega \in \Omega$ and for each $k \in \mathbb{N}^+$, we pick up $\varepsilon = \varepsilon_k(\omega) > 0$ sufficiently small such that

$$\left(M_1 a_k^2 + \varepsilon_k\right) C\varepsilon_k \leq 2\frac{a_k - 1}{a_k}. \tag{5.10}$$

Let

$$b_k = \left(M_1 a_k^2 + \varepsilon_k\right) \frac{C}{4\varepsilon_k},$$

and then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x + \varepsilon_k \int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x \leq b_k \left(\sup_{t\in[0,\tau^*)}\int_{\mathbb{R}^2} z^{a_{k-1}}\,\mathrm{d}x\right)^2, \quad k \in \mathbb{N}^+ \quad \mathbb{P}-a.s.,$$

which gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\varepsilon_k t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x\right) \leq \mathrm{e}^{\varepsilon_k t} b_k \left(\sup_{t\in[0,\tau^*)}\int_{\mathbb{R}^2} z^{a_{k-1}}\,\mathrm{d}x\right)^2, \quad k \in \mathbb{N}^+ \quad \mathbb{P}-a.s.$$

Then we have

$$\int_{\mathbb{R}^2} z^{a_k}(t)\,\mathrm{d}x \leq \max\left\{\int_{\mathbb{R}^2} u_0^{a_k}\,\mathrm{d}x, \delta_k \left(\sup_{t\in[0,\tau^*)}\int_{\mathbb{R}^2} z^{a_{k-1}}\,\mathrm{d}x\right)^2\right\}, \quad k \in \mathbb{N}^+ \quad \mathbb{P}-a.s., \tag{5.11}$$

where

$$\delta_k = \frac{b_k}{\varepsilon_k} = \left(M_1 a_k^2 + \varepsilon_k\right) \frac{C}{4\varepsilon_k^2}. \tag{5.12}$$

Moreover, we let

$$I = \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\right\}, \quad A_k = \sup_{t\in[0,\tau^*)}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x.$$

It follows from (5.10) that for a.e. $\omega \in \Omega$ and for each $k \in \mathbb{N}^+$, ε_k is of order $\frac{1}{a_k^2}$. Hence $\delta_k \geq 1$. Then (5.11) now reads as $A_k \leq \max\left\{I^{a_k}, \delta_k A_{k-1}^2\right\}$, $k \in \mathbb{N}^+$ $\mathbb{P}-a.s.$, from which we obtain recursively

$$\begin{aligned}
A_k \leq & \max\left\{I^{a_k}, \delta_k \max\left\{I^{a_{k-1}}, \delta_{k-1}A_{k-2}^2\right\}^2\right\} \\
= & \max\left\{I^{a_k}, \max\left\{\delta_k I^{a_k}, \delta_k\delta_{k-1}^2 A_{k-2}^4\right\}\right\} \\
= & \max\left\{\delta_k I^{a_k}, \delta_k\delta_{k-1}^{a_1} A_{k-2}^{a_2}\right\} \\
\leq & \max\left\{\delta_k\delta_{k-1}^{a_1} I^{a_k}, \delta_k\delta_{k-1}^{a_1}\delta_{k-2}^{a_2} A_{k-3}^{a_3}\right\} \\
\leq & \cdots \\
\leq & \max\left\{\delta_k\delta_{k-1}^{a_1}\cdots\delta_2^{a_{k-2}} I^{a_k}, \delta_k\delta_{k-1}^{a_1}\cdots\delta_2^{a_{k-2}}\delta_1^{a_{k-1}} A_0^{a_k}\right\} \\
\leq & \delta_k\delta_{k-1}^{a_1}\cdots\delta_2^{a_{k-2}}\delta_1^{a_{k-1}} I^{a_k} \quad \mathbb{P}-a.s.
\end{aligned}$$

By (5.12) and the fact that ε_k is of order $\frac{1}{a_k^2}$ almost surely, we can find a random variable $K = K(\omega) > 0$ such that $0 < K < \infty$ almost surely and

$$\mathbb{P}\left\{\frac{\delta_k}{a_k^6} \leq K, \ \forall k \in \mathbb{N}^+\right\} = 1,$$

$$A_k^{\frac{1}{a_k}} \leq \delta_k^{\frac{1}{a_k}}\delta_{k-1}^{\frac{a_1}{a_k}}\cdots\delta_2^{\frac{a_{k-2}}{a_k}}\delta_1^{\frac{a_{k-1}}{a_k}} I \leq \prod_{i=1}^{k}\left(\left(K a_i^6\right)^{\frac{1}{a_i}}\right) I = K^{\left(1-\frac{1}{2^k}\right)} 2^{3\sum_{i=1}^k \frac{i}{2^{i-1}}} I \leq 2^{15} K I \quad \mathbb{P}-a.s.$$

Sending $k \to \infty$, we finally conclude that

$$\sup_{t\in[0,\tau^*)} \|z(t)\|_{L^\infty} \leq CK \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\right\} \quad \mathbb{P}-a.s. \tag{5.13}$$

As a result, even though the H^s-norm of z may blow up at τ^*, (5.13) implies that the L^∞-norm of z will survice for all the time becuause the right hand side of (5.13) does not depend on time. Then we obtain

$$\sup_{t\geq 0}\|z(t)\|_{L^\infty}\leq CK\max\{\|u_0\|_{L^1},\|u_0\|_{L^\infty}\}\quad \mathbb{P}-a.s.$$

Step 3: Completing the proof. By (5.1), we have

$$\|u(t)\|_{L^\infty}\leq CK\max\{\|u_0\|_{L^1},\|u_0\|_{L^\infty}\}\,\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}-\int_0^t\frac{\beta^2(t')}{2}\,\mathrm{d}t'},\quad \forall t>0\ \ \mathbb{P}-a.s.,$$

which gives (2.6). From (1) in Lemma 3.6, we see that $\|u\|_{L^\infty}$ decays exponentially. □

5.1.2. *Decay of H^s norm in $\mathbb{R}^d$ with high probability.*

Proof of Theorem 2.3. Using similar estimates as in the proof of (4.31), we can conclude that there is a constant $C=C(s)$ such that for a.e. $\omega\in\Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{H^s}^2\leq C|\chi|\eta(t)\|z(t)\|_{H^{s-2}}\|z(t)\|_{H^s}^2,\quad \eta(t)=\eta(\omega,t)=\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}-\int_0^t\frac{\beta^2(t')}{2}\,\mathrm{d}t'}.$$

From the above estimate, for $w=\mathrm{e}^{-\int_0^t\beta(t')\,\mathrm{d}W_{t'}}u=\mathrm{e}^{-\int_0^t\frac{\beta^2(t')}{2}\,\mathrm{d}t'}z$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s}+\frac{\beta^2(t)}{2}\|w(t)\|_{H^s}\leq C|\chi|\alpha(\omega,t)\|w(t)\|_{H^{s-2}}\|w(t)\|_{H^s},\tag{5.14}$$

where $\alpha(\omega,t)=\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}}$. For any $R>1$, we fix R, let $\lambda_1>2$ and define $K(\lambda_1,\beta_*,\chi,R)=\frac{\beta_*}{C\lambda_1|\chi|\cdot R}$ and then define the process

$$\tau_1(\omega)=\inf\left\{t>0:\alpha(\omega,t)\|w\|_{H^{s-2}}=\|u\|_{H^{s-2}}>\frac{\beta^2(t)}{C\lambda_1|\chi|}\right\}.\tag{5.15}$$

Assume that $\|u_0\|_{H^s}<K(\lambda_1,\beta_*,\chi,R)<\frac{\beta_*}{C\lambda_1|\chi|}$ $\mathbb{P}-a.s.$, then $\mathbb{P}\{\tau_1>0\}=1$, and for $t\in[0,\tau_1]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s}+\frac{(\lambda_1-2)\beta^2(t)}{2\lambda_1}\|w(t)\|_{H^s}\leq 0.$$

The above inequality and $w=\mathrm{e}^{-\int_0^t\beta(t')\,\mathrm{d}W_{t'}}u$ imply that for a.e. $\omega\in\Omega$, for any $\lambda_2>\frac{2\lambda_1}{\lambda_1-2}$ and for $t\in[0,\tau_1]$,

$$\begin{aligned}\|u(t)\|_{H^s}=&\|w(t)\|_{H^s}\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}}\\ \leq&\|w_0\|_{H^s}\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}-\int_0^t\frac{(\lambda_1-2)\beta^2(t')}{2\lambda_1}\,\mathrm{d}t'}\\ =&\|u_0\|_{H^s}\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}-\int_0^t\frac{\beta^2(t')}{\lambda_2}\,\mathrm{d}t'}\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t\beta^2(t')\,\mathrm{d}t'}\end{aligned}\tag{5.16}$$

Define the stopping time

$$\tau_2(\omega)=\inf\left\{t>0:\mathrm{e}^{\int_0^t\beta(t')\,\mathrm{d}W_{t'}-\int_0^t\frac{\beta^2(t')}{\lambda_2}\,\mathrm{d}t'}>R\right\}.$$

Notice that $\mathbb{P}\{\tau_2>0\}=1$. From (5.16), we have

$$\begin{aligned}\|u(t)\|_{H^s}\leq&RK(\lambda_1,\beta_*,\chi,R)\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t\beta^2(t')\,\mathrm{d}t'}\\ <&\frac{\beta_*}{C\lambda_1|\chi|}\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t\beta^2(t')\,\mathrm{d}t'}<\frac{\beta_*}{C\lambda_1|\chi|},\quad t\in[0,\tau_1\wedge\tau_2).\end{aligned}\tag{5.17}$$

Combining (5.17) and (5.15), we find that

$$\mathbb{P}\{\tau_1\geq\tau_2\}=1.\tag{5.18}$$

Therefore it follows from (5.17) that

$$\mathbb{P}\left\{\|u(t)\|_{H^s}<\frac{\beta_*}{C\lambda_1|\chi|}\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t\beta^2(t')\,\mathrm{d}t'}\ \text{ for all } t>0\right\}\geq\mathbb{P}\{\tau_2=+\infty\}.$$

Then we apply (2) in Lemma 3.6 with $\rho(t)=\left(\frac{1}{2}-\frac{1}{\lambda_2}\right)\beta^2(t)$ to find that

$$\mathbb{P}\{\tau_2=+\infty\}>1-\left(\frac{1}{R}\right)^{2/\lambda_2},$$

which completes the proof. □

5.2. **Nonlinear case.** Following [45, 48], now we prove Theorem 2.4.

Proof for Theorem 2.4. Let $s>\frac{d}{2}+4$ and $r\in(\frac{d}{2}+1,s-3]$. Let $h(u)=a\left(1+\|u\|_{H^r}\right)^q u$. By mean value theorem for $(1+\cdot)^q$ with $q\geq\frac{1}{2}$, we have that for any $u,v\in H^s$,

$$\|h(u)-h(v)\|_{H^s}\leq|a|g\left(\|u\|_{H^s}+\|v\|_{H^s}\right)\|u-v\|_{H^s}$$

for some increasing function $g:[0,\infty)\mapsto[0,\infty)$, which means $h(\cdot)$ is locally Lipschitz in H^s. Besides, one can follow the steps as in Section 4 to obtain a unique pathwise solution u in H^s.

Then we apply the Itô formula to $\|u(t)\|^2_{H^s}$ (if necessary, consider $\|J_\varepsilon u(t)\|^2_{H^s}$ and then let $\varepsilon\to 0$ as in subsection 4.4) to find

$$\begin{aligned}\mathrm{d}\|u\|^2_{H^s}=&2a(1+\|u\|_{H^r})^q\|u\|^2_{H^s}\,\mathrm{d}W-2\left(\nabla D^s u,\nabla D^s u\right)_{L^2}\,\mathrm{d}t\\&-2\chi\left(D^sF(u),D^su\right)_{L^2}\,\mathrm{d}t+a^2(1+\|u\|_{H^r})^{2q}\|u\|^2_{H^s}\,\mathrm{d}t.\end{aligned}$$

Now we consider the Lyapunov function $\log(1+x^2)$, use the Itô formula to $\log(1+\|u\|^2_{H^s})$ and neglect the positive term $\frac{\|\nabla u\|^2_{H^s}}{1+\|u\|^2_{H^s}}$ in the left hand side of the equation to find

$$\begin{aligned}&\mathrm{d}\log(1+\|u\|^2_{H^s})\\\leq&\frac{2a(1+\|u\|_{H^r})^q}{1+\|u\|^2_{H^s}}\|u\|^2_{H^s}\,\mathrm{d}W-\frac{1}{1+\|u\|^2_{H^s}}2\chi\left(F(u),u\right)_{H^s}\,\mathrm{d}t\\&+\frac{1}{1+\|u\|^2_{H^s}}a^2(1+\|u\|_{H^r})^{2q}\|u\|^2_{H^s}\,\mathrm{d}t-2\frac{1}{(1+\|u\|^2_{H^s})^2}a^2(1+\|u\|_{H^r})^{2q}\|u\|^4_{H^s}\,\mathrm{d}t.\end{aligned}$$

Let

$$\tau_m=\inf\left\{t\geq 0:\|u(t)\|_{H^s}\geq m\right\}.$$

Using Lemma 3.4, we find that there is a $D>0$ such that for any $t>0$,

$$\begin{aligned}&\mathbb{E}\log(1+\|u(t\wedge\tau_m)\|^2_{H^s})-\mathbb{E}\log(1+\|u_0\|^2_{H^s})\\\leq&\mathbb{E}\int_0^{t\wedge\tau_m}\frac{1}{1+\|u\|^2_{H^s}}\left(-2\chi\left(F(u),u\right)_{H^s}+a^2(1+\|u\|_{H^r})^{2q}\|u\|^2_{H^s}\right)-2a^2\frac{(1+\|u\|_{H^r})^{2q}\|u\|^4_{H^s}}{(1+\|u\|^2_{H^s})^2}\,\mathrm{d}t'\\\leq&\mathbb{E}\int_0^{t\wedge\tau_m}\frac{2D|\chi|\|u\|_{H^r}\|u\|^2_{H^s}+a^2(1+\|u\|_{H^r})^{2q}\|u\|^2_{H^s}}{1+\|u\|^2_{H^s}}-\frac{2a^2(1+\|u\|_{H^r})^{2q}\|u\|^4_{H^s}}{(1+\|u\|^2_{H^s})^2}\,\mathrm{d}t'.\end{aligned}$$

Since q and a satisfy (2.7), it is straightforward to see that there are constants $K_1,K_2>0$ such that

$$\frac{2|\chi|Dxy^2+a^2(1+x)^{2q}y^2}{1+y^2}-\frac{2a^2(1+x)^{2q}y^4}{(1+y^2)^2}+\frac{a^2K_2(1+x)^{2q}y^4}{(1+y^2)^2(1+\log(1+y^2))}\leq K_1,\quad 0<x\leq y<\infty.\tag{5.19}$$

Indeed, this is because

$$\begin{aligned}&\frac{2|\chi|Dxy^2+a^2(1+x)^{2q}y^2}{1+y^2}-\frac{2a^2(1+x)^{2q}y^4}{(1+y^2)^2}+\frac{a^2K_2(1+x)^{2q}y^4}{(1+y^2)^2(1+\log(1+y^2))}\\\leq&a^2(1+x)^{2q}\left(\frac{2|\chi|Dx}{a^2(1+x)^{2q}}+1-\frac{2x^4}{(1+x^2)^2}+\frac{K_2}{(1+\log(1+x^2))}\right).\end{aligned}$$

When (2.7) is satisfied,

$$\limsup_{x\to\infty}\left(\frac{2|\chi|Dx}{a^2(1+x)^{2q}}+1-\frac{2x^4}{(1+x^2)^2}+\frac{K_2}{(1+\log(1+x^2))}\right)<0,$$

which means (5.19) holds true. Consequently, we arrive at

$$\begin{aligned}&\mathbb{E}\log(1+\|u(t\wedge\tau_m)\|_{H^s}^2)-\mathbb{E}\log(1+\|u_0\|_{H^s}^2)\\ \le&K_1t-\mathbb{E}\int_0^{t\wedge\tau_m}K_2\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t',\end{aligned}$$

which means for some constant $C=C(u_0,K_1,K_2,t)>0$,

$$\mathbb{E}\int_0^{t\wedge\tau_m}\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t'\le C(u_0,K_1,K_2,t)<\infty. \tag{5.20}$$

Therefore, for any $T>0$, it follows from the BDG inequality that

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\log(1+\|u\|_{H^s}^2)-\mathbb{E}\log(1+\|u_0\|_{H^s}^2)\\ \le&C\mathbb{E}\left(\int_0^{T\wedge\tau_m}\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2}\,\mathrm{d}t\right)^{\frac12}\\ &+\mathbb{E}\int_0^{T\wedge\tau_m}\left|K_1-K_2\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\right|\,\mathrm{d}t\\ \le&\frac12\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\left(1+\log(1+\|u\|_{H^s}^2)\right)+C\mathbb{E}\int_0^{T\wedge\tau_m}\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t\\ &+K_1T+\mathbb{E}\int_0^{T\wedge\tau_m}K_2\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t.\end{aligned}$$

Thus we use (5.20) to obtain

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\log(1+\|u\|_{H^s}^2)\\ \le&2\mathbb{E}\left(1+\log(1+\|u_0\|_{H^s}^2)\right)+C\mathbb{E}\int_0^{T\wedge\tau_m}\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t\\ &+K_1T+\mathbb{E}\int_0^{T\wedge\tau_m}K_2\frac{a^2(1+\|u\|_{H^r})^{2q}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t\\ \le&C(u_0,K_1,K_2,T).\end{aligned}$$

Let τ^* be the maximal existence time of u in H^s. Then for any $T>0$,

$$\mathbb{P}\{\tau^*<T\}\le\mathbb{P}\{\tau_m<T\}\le\mathbb{P}\left\{\sup_{t\in[0,T\wedge\tau_m]}\log(1+\|u\|_{H^s}^2)\ge\log(1+m^2)\right\}\le\frac{C(u_0,K_1,K_2,T)}{\log(1+m^2)}.$$

Sending $m\to\infty$ clearly forces that $\mathbb{P}\{\tau^*<T\}=0$ for any $T>0$, which means $\mathbb{P}\{\tau^*=\infty\}=1$. □

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