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A closed-form solution to a viscoelastically supported Timoshenko beam under harmonic line load

W.L. Luo[1], Y. Xia[1*] and X.Q. Zhou[2]

[1]*Department of Civil and Environmental Engineering, The Hong Kong Polytechnic University, Kowloon, Hong Kong, China*

*Corresponding author, email: ceyxia@polyu.edu.hk

[2]*College of Civil Engineering, Shenzhen University, China*

Abstract:

This study aims to formulate a closed-form solution to a viscoelastically supported Timoshenko beam under a harmonic line load. The differential governing equations of motion are converted into algebraic equations by assuming the deflection and rotation of the beam in harmonic forms with respect to time and space. The characteristic equation is biquadratic and thus contains 14 explicit roots. These roots are then substituted into Cauchy's residue theorem; consequently, five forms of the closed-form solution are generated. The present solution is consistent with that of an Euler–Bernoulli beam on a Winkler foundation, which is a special case of the present problem. The current solution is also verified through numerical examples.

1. Introduction

The closed-form solution to beams on elastic or viscoelastic foundations serves as a benchmark for many soil-structure interaction problems [1-3]. Obtaining the solution generally consists of three steps [4]: (a) converting the differential governing equations of motion into algebraic equations, (b) identifying the roots of the characteristic polynomials at degree four/five, and (c) solving beam deflection with the identified roots.

However, only a few closed-form solutions are reported in literature. The first was presented by Kenny (1954) for the vibration of an Euler–Bernoulli (EB) beam on a Winkler foundation due to a moving load [5]. This solution was further extended by Mathew (1958) to a similar model subjected to an oscillating load [6] and by Frýba (1999) to a model on a Kelvin foundation [4]. Sun (2001) also derived a closed-form solution for an EB beam on a viscoelastic foundation under harmonic line loads [7]. Subsequently, this researcher also formulated a closed-form solution for a beam on viscoelastic subgrade under moving loads [8] and an explicit representation of the steady-state response of a beam on an elastic foundation under moving harmonic line loads [9]. EB beams were considered in these cases.

The closed-form solution for the Timoshenko beam is rarely obtained because the roots of the characteristic equation cannot always be expressed with elementary functions [10]. Although analytical solutions are usually presented in a general form of Cauchy’s residue theorem [11-14], a closed-form solution is unavailable and numerical algorithms must be applied to find roots. Examples include the steady-state solution for a Timoshenko beam on a Pasternak foundation under a moving harmonic load [12], the steady-state solution for a Timoshenko beam made of a laminated composite and located on a Pasternak viscoelastic foundation [13], and the transient response of an EB beam on a viscoelastic foundation that is subjected to arbitrary dynamic loads [14]. Semi-analytical approaches have also been developed that directly solve the integral representation of beam deflection using numerical algorithms. For instance, the inverse fast Fourier transform (IFFT) was employed to investigate the line load-induced vibrations of a hysteretically supported Timoshenko beam [10].

The present study intends to formulate a closed-formed solution for a Timoshenko beam on a viscoelastic foundation under a harmonic line load. In this case, the characteristic equation is biquadratic, and thus generates roots in explicit forms. The Timoshenko beam is chosen instead of the EB beam in this work because the former can account for shear deformation and rotational inertia. Therefore, the Timoshenko beam is suitable for describing the behavior of thick understructures, such as deep foundations and multilayered pavements. Furthermore, this beam is generic and can be simplified into the EB beam by setting the shear rigidity to infinity and the radius of gyration to zero. The formulation begins with the differential governing equations of the motion of a viscoelastically supported Timoshenko beam together with convergence conditions at infinity. These equations are subsequently converted into algebraic ones on the basis of the assumption that beam deflection and rotation are in the harmonic forms of time and space. The integral representation of beam deflection is generated through the inverse Fourier transform. The roots of the characteristic equation are categorized into 14 cases. The closed-form solution is finally obtained by combining Cauchy's residue theorem with the determined roots. The present closed-form solutions are compared with those of an EB beam on a Winkler foundation and the numerical solutions of a viscoelastically supported shear or Timoshenko beam.

2. Problem formulation

Figure 1 illustrates a Timoshenko beam on a viscoelastic foundation that is subjected to a harmonic line load. $w(x, t)$ and $\theta(x, t)$ are defined as the vertical deflection and rotation of the beam, respectively.

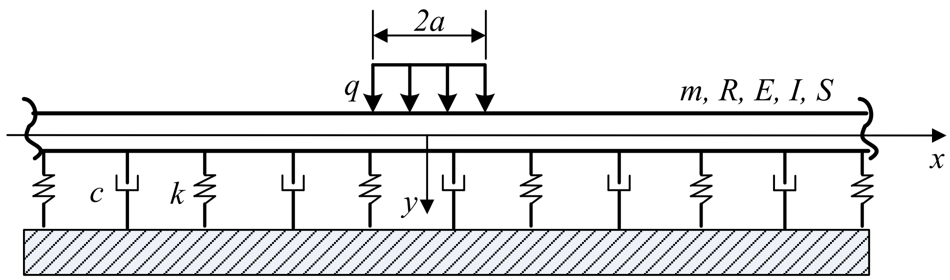


Figure 1: Timoshenko beam on a viscoelastic foundation under a harmonic line load.

The governing equations of motion for this beam-foundation system are given by [10]:

$$m\ddot{w}(x,t)+S\left[\theta'(x,t)-w''(x,t)\right]+kw(x,t)+c\dot{w}(x,t)=Q(x,t) \tag{1}$$

$$EI\theta''(x,t)+S\left[w'(x,t)-\theta(x,t)\right]=mR^2\ddot{\theta}(x,t) \tag{2}$$

with convergence conditions at infinity, that is:

$$\lim_{x\to\pm\infty} w(x,t)=0,\ \lim_{x\to\pm\infty} w'(x,t)=0,\ \lim_{x\to\pm\infty} \theta(x,t)=0,\ \lim_{x\to\pm\infty} \theta'(x,t)=0\,, \tag{3}$$

where m is the mass of the beam per unit length; E is the Young's modulus; I is the cross-sectional moment of inertia; S and R are the shear rigidity and the radius of gyration of the beam, respectively; the dot over a variable denotes a differentiation with respect to time; the prime of a variable represents a differentiation with respect to space; and k and c are foundation stiffness and foundation damping per unit length, respectively. $Q(x, t)$ is the harmonic line load defined as:

$$Q(x,t) = q\left[H(x-a) - H(x+a)\right]\exp(i\omega t), \tag{4}$$

where $i = \sqrt{-1}$, H is the Heaviside function; q is the amplitude of the load per unit of length; and ω is circular frequency of the load.

The resulting beam deflection and rotation are also harmonic with respect to time. Presumably, $w(x,t) = \overline{w}\exp(i\omega t)\exp(i\xi x)$ and $\theta(x,t) = \overline{\theta}\exp(i\omega t)\exp(i\xi x)$, where $\overline{w}$ and $\overline{\theta}$ are amplitudes of deflection and rotation, respectively, ξ is the wavenumber with respect to space x, and the bar over a variable denotes the variable in the frequency–wavenumber (ξ–ω) domain. Substituting $w(x, t)$ and $\theta(x, t)$ into Eqs. (1) and (2) generates the following two algebraic equations [10]:

$$-\omega^2 m\overline{w} + S\left[i\xi\overline{\theta} + \xi^2\overline{w}\right] + k\overline{w} + i\omega c\overline{w} = 2q\sin(a\xi)/\xi \tag{5}$$

$$-\xi^2 EI\overline{\theta} + S\left[i\xi\overline{w} - \overline{\theta}\right] = -\omega^2 mR^2\overline{\theta}\,. \tag{6}$$

Eqs. (5) and (6) are two linear equations in terms of $\overline{\theta}$ and $\overline{w}$. Eliminating $\overline{\theta}$ from the two equations yields the following equation regarding the beam deflection in ξ–ω domain:

$$\overline{w}(\xi,\omega) = \frac{2q\sin(a\xi)\left[EI\xi^2 + \left(S - mR^2\omega^2\right)\right]}{EIS\xi\left[\xi^4 + (A+Bi)\xi^2 + (C+Di)\right]}, \tag{7}$$

where A, B, C, and D are real and are defined as $A = [EI(\Omega^2 - \omega^2) - R^2\omega^2 S]m/EIS$, $B = \omega c/S$, $C = (S - mR^2\omega^2)(\Omega^2 - \omega^2)m/EIS$, and $D = (S - mR^2\omega^2)\omega c/EIS$. $\Omega = \sqrt{k/m}$ is the reference critical frequency of the beam with stiffness k and mass m.

Applying the inverse Fourier transform to Eq. (7) yields the integral representation of beam deflection:

$$w(x,t)=\frac{\exp(i\omega t)}{2\pi}\int_{-\infty}^{\infty}\frac{2q\sin(a\xi)\left[EI\xi^2+\left(S-mR^2\omega^2\right)\right]}{EIS\xi\left[\xi^4+(A+Bi)\xi^2+(C+Di)\right]}\exp(i\xi x)\,d\xi\,. \tag{8}$$

The denominator of Eq. (8) is the characteristic equation of the present beam-foundation system. In the following sections, the roots of this characteristic equation are identified first, followed by the closed-form solution and verification examples.

3. Roots of the characteristic equation

Before evaluating the integral of Eq. (8) further, the roots of the characteristic equation are solved. The trivial root is $\xi_0 = 0$; the others are solutions of the following fourth-order equation:

$$\xi^4+(A+Bi)\xi^2+(C+Di)=0\,. \tag{9}$$

Suppose all parameters of the present system are positive except for $c \geq 0$. The shear rigidity is assumed to be considerably large such that $S > mr^2\omega^2$. Given these assumptions, A and C may be positive, zero, or negative, whereas B and D are nonnegative.

Let $z = \xi^2$, Eq. (9) becomes a quadratic equation of z:

$$z^2+(A+Bi)z+(C+Di)=0\,. \tag{10}$$

The solutions of Eq. (10) are:

$$z_{1,2}=\frac{-\left(A+Bi\right)\pm\sqrt{\Delta}}{2},\tag{11}$$

where Δ is the determinant and is defined as $\Delta = g + hi$ with $g = A^2 - B^2 - 4C$ and $h = 2AB - 4D$. Therefore, all four roots of Eq. (9) are expressed as follows:

$$\xi_{1,2}=\pm\sqrt{z_1},\quad \xi_{3,4}=\pm\sqrt{z_2}\,.\tag{12}$$

The explicit forms of the roots can be categorized into the following three cases according to the conditions of viscous damping and loading frequency.

3.1. When damping and load frequency are considered ($c > 0$ *and* $\omega > 0$)

The conditions $c > 0$ and $\omega > 0$ imply that B and D are positive and Eq. (10) is a quadratic equation with complex coefficients. Two roots of this equation are non-conjugate complex, and the proof is presented in Appendix A. Two major cases can be categorized based on the value of h; in each major case, two sub-cases are analyzed.

(a) $h \geq 0$. The square roots of determinant Δ are $r + is$ and $-r - is$. Both r and s are either positive or zero and are defined as $r=\sqrt{\left(\sqrt{g^2+h^2}+g\right)/2}$ and $s=\sqrt{\left(\sqrt{g^2+h^2}-g\right)/2}$. Therefore, a pair of non-conjugate complex roots of Eq. (11) can be rewritten as $z_1 = g_1 + ih_1$ and $z_2 = g_2 + ih_2$ with $g_1 = (r - A)/2$, $h_1 = (s - B)/2$, $g_2 = -(r + A)/2$, and $h_2 = -(s + B)/2$. h_1 can be positive, zero, or negative, whereas h_2 is negative.

(a.1) If $h_1 \geq 0$, that is, $s \geq B$, then the two roots in Eq. (12) are $\xi_1 = r_1 + is_1$ and $\xi_2 = -r_1 - is_1$. Both r_1 and s_1 are either positive or zero and are defined as $r_1=\sqrt{\left(\sqrt{g_1^2+h_1^2}+g_1\right)/2}$

and $s_1 = \sqrt{\left(\sqrt{g_1^2 + h_1^2} - g_1\right)/2}$. The other two roots are $\xi_3 = r_2 - is_2$ and $\xi_4 = -r_2 + is_2$; both r_2 and s_2 are either positive or zero and are defined as $r_2 = \sqrt{\left(\sqrt{g_2^2 + h_2^2} + g_2\right)/2}$ and $s_2 = \sqrt{\left(\sqrt{g_2^2 + h_2^2} - g_2\right)/2}$.

(a.2) If $h_1 < 0$, that is, $s < B$, then the two roots in Eq. (12) are $\xi_1 = r_1 - is_1$ and $\xi_2 = -r_1 + is_1$. The other two roots are $\xi_3 = r_2 - is_2$ and $\xi_4 = -r_2 + is_2$.

(b) $h < 0$. The square roots of determinant Δ are $r - is$ and $-r + is$. The pair of non-conjugate complex roots of Eq. (11) can be rewritten as $z_1 = g_1 + ih_1$ and $z_2 = g_2 + ih_2$ with $g_1 = (r - A)/2$, $h_1 = -(s + B)/2$, $g_2 = -(r + A)/2$, and $h_2 = (s - B)/2$. h_1 is negative, whereas h_2 can be positive, zero, or negative.

(b.1) When $h_2 \geq 0$, that is, $s \geq B$, the two roots in Eq. (12) are $\xi_1 = r_1 - is_1$ and $\xi_2 = -r_1 + is_1$. The other two roots are $\xi_3 = r_2 + is_2$ and $\xi_4 = -r_2 - is_2$.

(b.2) When $h_2 < 0$, that is, $s < B$, the two roots in Eq. (12) are $\xi_1 = r_1 - is_1$ and $\xi_2 = -r_1 + is_1$. The other two roots are $\xi_3 = r_2 - is_2$ and $\xi_4 = -r_2 + is_2$.

Table 1 summarizes four roots under various conditions when the viscous damping of the foundation and loading frequency are considered.

Table 1 Four roots at various conditions when $c > 0$ and $\omega > 0$

Conditions	g_1, h_1, g_2, h_2	Sub-condition	Roots	Case
$h \geq 0$	$g_1 = (r - A)/2$ $h_1 = (s - B)/2$	$h_1 \geq 0$	$\xi_1 = r_1 + is_1$, $\xi_2 = -r_1 - is_1$ $\xi_3 = r_2 - is_2$, $\xi_4 = -r_2 + is_2$	1
	$g_2 = -(r + A)/2$ $h_2 = -(s + B)/2$	$h_1 < 0$	$\xi_1 = r_1 - is_1$, $\xi_2 = -r_1 + is_1$ $\xi_3 = r_2 - is_2$, $\xi_4 = -r_2 + is_2$	2
$h < 0$	$g_1 = (r - A)/2$	$h_2 \geq 0$	$\xi_1 = r_1 - is_1$, $\xi_2 = -r_1 + is_1$	3

	$h_1 = -(s+B)/2$ $g_2 = -(r+A)/2$ $h_2 = (s-B)/2$		$\xi_3 = r_2 + is_2$, $\xi_4 = -r_2 - is_2$	
		$h_2 < 0$	$\xi_1 = r_1 - is_1$, $\xi_2 = -r_1 + is_1$ $\xi_3 = r_2 - is_2$, $\xi_4 = -r_2 + is_2$	4

Note: $r = \sqrt{\left(\sqrt{g^2+h^2}+g\right)/2}$, $s = \sqrt{\left(\sqrt{g^2+h^2}-g\right)/2}$, $r_{1,2} = \sqrt{\left(\sqrt{g_{1,2}^2+h_{1,2}^2}+g_{1,2}\right)/2}$, and $s_{1,2} = \sqrt{\left(\sqrt{g_{1,2}^2+h_{1,2}^2}-g_{1,2}\right)/2}$.

3.2. Without damping ($c = 0$ *and* $\omega > 0$)

If viscous damping is not considered ($c = 0$), then $B = D = 0$. The characteristic equation is written as follows with real coefficients:

$$\xi^4 + A\xi^2 + C = 0 \cdot \tag{13}$$

Eq. (13) is further simplified into a quadratic equation with real coefficients:

$$z^2 + Az + C = 0 \cdot \tag{14}$$

The determinant $\Delta = A^2 - 4C$ is also real. In the following paragraphs, three sub-cases are obtained according to the determinant.

(a) When $\Delta > 0$, Eq. (14) has two real roots: $z_{1,2} = \left(-A \pm \sqrt{\Delta}\right)/2$.

(a.1) If $z_1 > 0$, then $\xi_{1,2} = \pm\sqrt{\left(-A+\sqrt{\Delta}\right)/2}$ is real. Specifically, if $A \geq 0$, then $\sqrt{\Delta} > A$, thus implying $C < 0$, that is, $\omega > \Omega$. In this case, $z_2 < 0$, that is, $\xi_{3,4} = \pm i\sqrt{\left(A+\sqrt{\Delta}\right)/2}$. If $A <$ 0, then $\sqrt{\Delta} > A$ holds for any C: when $C < 0$ ($\omega > \Omega$), $z_2 < 0$, then

$\xi_{3,4} = \pm i\sqrt{\left(A+\sqrt{\Delta}\right)/2}$; when $C = 0$ ($\omega = \Omega$), $z_2 = 0$, then $\xi_{3,4} = 0$; and when $C > 0$ ($\omega < \Omega$), $z_2 > 0$, then $\xi_{3,4} = \pm\sqrt{\left(-A-\sqrt{\Delta}\right)/2}$.

(a.2) If $z_1 = 0$, $\xi_{1,2} = 0$, then $C = 0$ ($\omega = \Omega$). This outcome implies that $A > 0$. Then $z_2 = -A$, that is, $\xi_{3,4} = \pm i\sqrt{A}$.

(a.3) If $z_1 < 0$, then $\xi_{1,2} = \pm i\sqrt{\left(A-\sqrt{\Delta}\right)/2}$ are two conjugate complex roots. This result also implies $A > 0$ and $C < 0$ ($\omega > \Omega$). Under this condition, $z_2 < 0$, that is, $\xi_{3,4} = \pm i\sqrt{\left(A+\sqrt{\Delta}\right)/2}$.

(b) When $\Delta = 0$, Eq. (14) has double real roots: $z_{1,2} = -A/2$.

This condition implies $C \geq 0$, that is, $\omega \leq \Omega$. If $A > 0$, then Eq. (13) has two double conjugate complex roots, namely, $\xi_{1,3} = i\sqrt{A/2}$ and $\xi_{2,4} = -i\sqrt{A/2}$. If $A = 0$, then Eq. (13) has four zeros $\xi_{1,2,3,4} = 0$. If $A < 0$, then Eq. (13) has two double real roots $\xi_{1,3} = \sqrt{-A/2}$ and $\xi_{2,4} = -\sqrt{-A/2}$.

(c) When $\Delta < 0$, Eq. (14) has two conjugate complex roots: $z_{1,2} = \left(-A \pm i\sqrt{-\Delta}\right)/2$.

This condition implies that $C > 0$, that is, $\omega < \Omega$. The four complex roots of Eq. (13) are $\xi_1 = \sqrt{\left(\sqrt{C}-A/2\right)/2} + i\sqrt{\left(\sqrt{C}+A/2\right)/2}$, $\xi_2 = -\xi_1$, $\xi_3 = \sqrt{\left(\sqrt{C}-A/2\right)/2} - i\sqrt{\left(\sqrt{C}+A/2\right)/2}$, and $\xi_4 = -\xi_3$. ξ_1 and ξ_3 constitute a pair of complex conjugates, and ξ_2 and ξ_4 comprise the other pair.

Table 2 lists four roots under various conditions when viscous damping is not considered.

Table 2 Roots under various conditions when $c = 0$ and $\omega > 0$

Conditions				Roots	Case
$\Delta > 0$	$z_1 > 0$	$A \geq 0, \omega > \Omega$		$\xi_{1,2} = \pm\sqrt{\left(-A+\sqrt{\Delta}\right)/2}$, $\xi_{3,4} = \pm i\sqrt{\left(A+\sqrt{\Delta}\right)/2}$	5
		$A < 0$	$\omega > \Omega$	$\xi_{1,2} = \pm\sqrt{\left(-A+\sqrt{\Delta}\right)/2}$, $\xi_{3,4} = \pm i\sqrt{\left(A+\sqrt{\Delta}\right)/2}$	6
			$\omega = \Omega$	$\xi_{1,2} = \pm\sqrt{\left(-A+\sqrt{\Delta}\right)/2}$, $\xi_{3,4} = 0$	7
			$\omega < \Omega$	$\xi_{1,2} = \pm\sqrt{\left(-A+\sqrt{\Delta}\right)/2}$, $\xi_{3,4} = \pm\sqrt{\left(-A-\sqrt{\Delta}\right)/2}$	8
	$z_1 = 0$	$A > 0, \omega = \Omega$		$\xi_{1,2} = 0,\ \xi_{3,4} = \pm i\sqrt{A}$	9
	$z_1 < 0$	$A > 0, \omega > \Omega$		$\xi_{1,2} = \pm i\sqrt{\left(A-\sqrt{\Delta}\right)/2}$, $\xi_{3,4} = \pm i\sqrt{\left(A+\sqrt{\Delta}\right)/2}$	10
$\Delta = 0$	$A > 0$	$\omega \leq \Omega$		$\xi_{1,3} = i\sqrt{A/2}, \quad \xi_{2,4} = -i\sqrt{A/2}$	11
	$A = 0$			$\xi_{1,2,3,4} = 0$	12
	$A < 0$			$\xi_{1,3} = \sqrt{-A/2}, \quad \xi_{2,4} = -\sqrt{-A/2}$	13
$\Delta < 0$	-	$\omega < \Omega$		$\xi_1 = \sqrt{\frac{\sqrt{C}-A/2}{2}} + i\sqrt{\frac{\sqrt{C}+A/2}{2}}$, $\xi_2 = -\xi_1$ $\xi_3 = \sqrt{\frac{\sqrt{C}-A/2}{2}} - i\sqrt{\frac{\sqrt{C}+A/2}{2}}$, $\xi_4 = -\xi_3$	14

3.3. Static case ($\omega = 0$)

If a static load is considered, then $B = D = 0$, $A = k/S > 0$, and $C = k/EI > 0$. If the determinant is positive, then $z_{1,2}$ are negative and the four roots are the same as those in Case 10. If the determinant is zero, then $z_{1,2}$ are also negative and the four roots are identical to those in Case 11. If the determinant is negative, then $z_{1,2}$ are complex and the four roots are the same as those in Case 14.

4. Closed-form solution

The beam-foundation system is symmetric with x = 0; therefore, the defection at positive x is identical to that at its negative x counterpart. Without loss of generality, only the case of $x \geq 0$ is considered. Apart from $\xi_0 = 0$, the other two roots for $x \geq 0$ are chosen such that they possess positive imaginary parts or are positive and real. The physical meaning is that only the solution with finite deflection is accepted, and waves without attenuation must propagate away from the source [15]. Therefore, the general solution can be expressed in a form of Cauchy's residue theorem:

$$w(x,t) = \left\{ i \sum_{\mathrm{Im}\xi>0} \mathrm{Res}\left[W(\xi)\right] + \frac{i}{2} \sum_{\mathrm{Im}\xi=0} \mathrm{Res}\left[W(\xi)\right] \right\} \exp(i\omega t), x \geq 0, \tag{15}$$

where $W(\xi) = \overline{w}(\xi,\omega)\exp(i\xi x)$, Res[.] represents Cauchy's residue of the function inside the parenthesis; Imξ = 0 denotes real roots; and Imξ > 0 corresponds to the complex roots in the upper half of the complex plane. The first summation is dropped if all roots involved are real, and the second one is unnecessary if all the roots involved are imaginary. If Res[$W(\xi)$] has a single order of root ξ_j, then the residue component can be evaluated by [4]:

$$\mathrm{Res}\left[W(\xi)\right]\Big|_{\xi=\xi_j} = \lim_{\xi\to\xi_j} (\xi-\xi_j) \frac{2q\sin(a\xi)\left[EI\xi^2 + (S - mr^2\omega^2)\right]\exp(i\xi x)}{\xi(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)EIS}. \tag{16}$$

If Res[$W(\xi)$] has a root of order m at ξ_j, the residue component can be evaluated by [4]:

$$\mathrm{Res}\left[W(\xi)\right]\Big|_{\xi=\xi_j} = \frac{1}{(m-1)!} \lim_{\xi\to\xi_j} \frac{\mathrm{d}^{m-1}}{\mathrm{d}\xi^{m-1}} \left[(\xi-\xi_j)^m W(\xi)\right]. \tag{17}$$

The numerator of Res[.] is zero when $\xi = 0$. In other words, this root does not contribute to the integral of Eq. (8) [7].

According to the characteristics of the roots, Eq. (15) is elaborated further into the following five solution forms.

(a) If two roots are unequal and complex with positive imaginary parts, then only the first part of Eq. (15) is considered. The solution is expressed as:

$$w(x,t)=\sum_{j=1}^{2}\frac{i\exp(i\omega t)}{EIS\left(G_j+iH_j\right)}2q\sin\left(a\varepsilon_j\right)\left[EI\varepsilon_j^2+\left(S-mR^2\omega^2\right)\right]\exp\left(i\varepsilon_j x\right). \tag{18}$$

(b) If two roots are double complex with positive imaginary parts, then the residue is calculated with Eq. (17). The solution is written as:

$$w(x,t)=\frac{2qi\exp(i\omega t)}{EIS\varepsilon_1^2\left(\varepsilon_1-\varepsilon_2\right)^3}\left(G+iH\right), \tag{19}$$

where ε_1 is the double complex root for the beam deflection at $x \geq 0$, whereas ε_2 is not.

(c) If two roots are unequal, real, and positive, then only the second part of Eq. (15) is considered. The solution is expressed as:

$$w(x,t)=\sum_{j=1}^{2}\frac{i\exp(i\omega t)}{EISG_j}q\sin\left(a\varepsilon_j\right)\left[EI\varepsilon_j^2+\left(S-mR^2\omega^2\right)\right]\exp\left(i\varepsilon_j x\right). \tag{20}$$

(d) If two roots are double positive and real, then the residue is calculated by Eq. (17). The solution is written as:

$$w(x,t)=\frac{qi\exp(i\omega t)}{EIS\varepsilon_1^2(\varepsilon_1-\varepsilon_2)^3}[G+iH], \tag{21}$$

where ε_1 is the double real root for beam deflection at $x \geq 0$, whereas ε_2 is not.

(e) If one root is complex with a positive imaginary part and the other is positive and real, then both parts of Eq. (15) are considered. The solution is expressed as:

$$\begin{aligned} w(x,t)=&\frac{i\exp(i\omega t)}{EIS(G_c+iH_c)}2q\sin(a\varepsilon_c)\left[EI\varepsilon_c^2+\left(S-mR^2\omega^2\right)\right]\exp(i\varepsilon_c x)\\ &+\frac{i\exp(i\omega t)}{EIS(G_r+iH_r)}q\sin(a\varepsilon_r)\left[EI\varepsilon_r^2+\left(S-mR^2\omega^2\right)\right]\exp(i\varepsilon_r x) \end{aligned}. \tag{22}$$

Table 3 summarizes the closed-form solution of all cases at the position $x \geq 0$.

Table 3 Closed-form solution of a viscoelastically supported Timoshenko beam that is subjected to a harmonic line load ($x \geq 0$)

Case	Solution form	Roots	Coefficients
1	Eq. (18)	$\varepsilon_1=\xi_1, \varepsilon_2=\xi_4$	$G_1=R_1, H_1=R_2, G_2=R_3, H_2=R_4$
2 and 4	Eq. (18)	$\varepsilon_1=\xi_2, \varepsilon_2=\xi_4$	$G_1=R_5, H_1=R_6, G_2=R_7, H_2=R_8$
3	Eq. (18)	$\varepsilon_1=\xi_2, \varepsilon_2=\xi_3$	$G_1=R_1, H_1=-R_2, G_2=R_3, H_2=-R_4$
5 and 6	Eq. (22)	$\varepsilon_r=\xi_1, \varepsilon_c=\xi_3$	$G_r=G_c=A^2-2A\sqrt{\Delta}+\Delta, H_r=H_c=0$
7	Eq. (20)	$\varepsilon_r=\xi_1$	$G_1=\left(A^2-2A\sqrt{\Delta}+\Delta\right)/2$
8	Eq. (20)	$\varepsilon_1=\xi_1, \varepsilon_2=\xi_3$	$G_1=-A\sqrt{\Delta}+\Delta,\ G_2=A\sqrt{\Delta}+\Delta$
9	Eq. (22)	$\varepsilon_c=\xi_3$	$G_c=2A^2,\ H_c=0$
10	Eq. (18)	$\varepsilon_1=\xi_1, \varepsilon_2=\xi_3$	$G_1=-A\sqrt{\Delta}+\Delta, G_2=A\sqrt{\Delta}+\Delta, H_1=H_2=0$
11	Eq. (19)	$\varepsilon_1=\xi_1, \varepsilon_2=\xi_3$	$G=R_9, H=R_{10}$

12	Non-existent	-	-
13	Eq. (21)	$\varepsilon_1 = \xi_1, \varepsilon_2 = \xi_3$	$G = R_9, H = R_{10}$
14	Eq. (18)	$\varepsilon_1 = \xi_1, \varepsilon_2 = \xi_4$	$G_1 = G_2 = A^2 - 4C, H_1 = -A\sqrt{-G_1}$, $H_2 = -H_1$

Note: R_1 to R_{10} are given in Appendix B.

5. Verification examples

5.1. Comparison with the closed-form solutions for the EB beam on a Winkler foundation

When S = infinity, R = 0, and c = 0, the present beam-foundation system is simplified into an EB beam on a Winkler foundation, in which with the characteristic equation is:

$$\xi^4 + C = 0, \tag{23}$$

where $C = (\Omega^2 - \omega^2)m/EI$.

If $\omega < \Omega$, that is, $C > 0$, then $\Delta = -4C < 0$. This case belongs to Case 14. As per Table 2, the four roots of the characteristic equation are $\xi_1 = \left(1/\sqrt{2} + i/\sqrt{2}\right)\sqrt[4]{C}$, $\xi_2 = -\xi_1$, $\xi_3 = \left(1/\sqrt{2} - i/\sqrt{2}\right)\sqrt[4]{C}$, and $\xi_4 = -\xi_3$. These findings are identical to the results in reference [7]. Figure 2(a) illustrates these roots and the trivial root $\xi_0 = 0$ in the complex plane. The coefficients are $H_1 = H_2 = 0$ and $G_1 = G_2 = -4C$; thus, the response is given by:

$$w(x,t) = \frac{iq\exp(i\omega t)}{-2EIC}\sum_{j=1}^{2}\sin(a\varepsilon_j)\exp(i\varepsilon_j x), \tag{24}$$

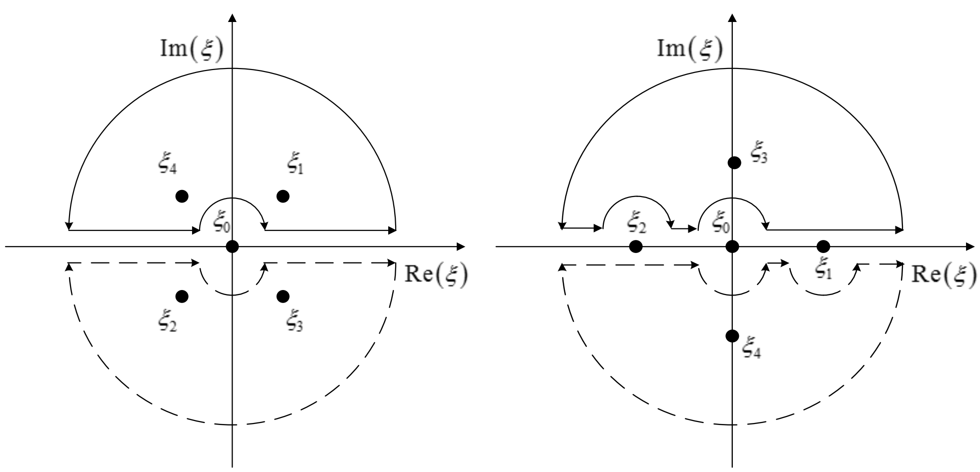
where $\varepsilon_1 = \xi_1$ and $\varepsilon_2 = \xi_4$. This solution is consistent with that in reference [7] although a minus sign is dropped.

If $\omega > \Omega$, that is, $C < 0$, then $\Delta = -4C > 0$ and $z_1 > 0$. This case belongs to Case 5. According to Table 2, the four roots of the characteristic equation are $\xi_1 = \sqrt[4]{-C}$, $\xi_2 = -\sqrt[4]{-C}$, $\xi_3 = i\sqrt[4]{-C}$, and $\xi_4 = -i\sqrt[4]{-C}$; these outcomes are identical to the results in reference [7]. Figure 2(b) illustrates these roots and the trivial root $\xi_0 = 0$ in the complex plane. The coefficients are $G_r = G_c = -4C$ and $H_r = H_c = 0$; therefore, the response is given by:

$$w(x,t) = \frac{i\exp(i\omega t)}{-4EIC}\left[2q\sin(a\varepsilon_c)\exp(i\varepsilon_c x) + q\sin(a\varepsilon_r)\exp(i\varepsilon_r x)\right], \quad (25)$$

where $\varepsilon_r = \xi_1$ and $\varepsilon_c = \xi_3$. The same formula can be obtained from Eq. (28) in reference [7].

If $\omega = \Omega$, that is, $C = 0$, then $\Delta = 0$. This case belongs to Case 12, and all roots are zeros. The solution is non-existent, which is again consistent with reference [7].



(a) (b)

Figure 2: Contour for the EBE beam on a Winkler foundation. (a) $\omega < \Omega$, and (b) $\omega > \Omega$.

5.2. Comparison with the numerical examples of a shear beam

When $R = 0$ and $c \neq 0$, the present beam-foundation system is simplified into a shear beam on a viscoelastic foundation. This beam was numerically studied in reference [16] with IFFT. In this section, the numerical results are compared with the present solutions. The properties of the beam are EI = 363.35 kNm^2, m = 297.5 kg/m, and S = 100 MN; those of the foundation are k = 77.17 MPa and c = 10 kNs/m^2. The loads are $2a$ = 0.1524 m and q = 70 kN/m. The critical frequency of this system is 81.06 Hz, which is defined as $v_{cr} = \sqrt[4]{4EIk/m^2}$ [4]. Frequencies of 70 and 100 Hz are applied; the former is lower than the critical frequency of the system, whereas the latter is higher than this critical frequency.

When the load frequencies are 70 and 100 Hz, h = –48.40 and –69.22 and h_2 = 7.37 and 0.79, respectively. In other words, both cases belong to Case 3. Figure 3 depicts the agreement between the present closed-form and the numerical solutions at both frequencies. The distribution of beam deflection along with the positive x at 70 Hz is similar that of the static case, and the deflection is maximized at the center of the load. In the case of high loading frequency (f = 100 Hz), however, the distribution of beam deflection is much wider than the load and deflection is no longer maximized at the loading center.

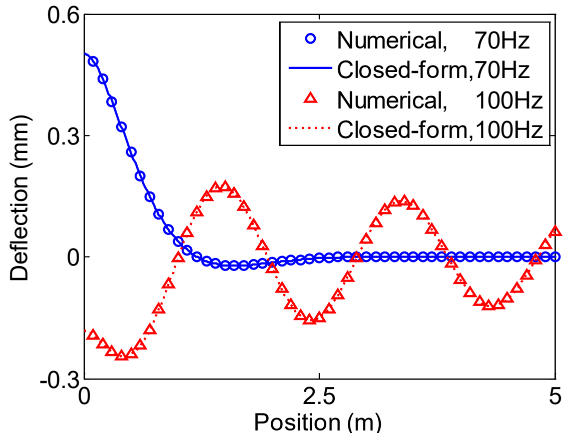


Figure 3: Comparison of the present closed-form and numerical solutions for the deflection of a viscoelastically supported shear beam.

5.3. Comparison with the numerical examples of a Timoshenko beam

Another example is a Timoshenko beam (S = 5 MN and R = 0.4 m) on a viscoelastic foundation. This beam was also studied numerically in reference [10]. The other parameters are similar to those described in Section 5.2. When the load frequencies are 70 and 100 Hz, h = 3.07 and 40.57 and h_1 = –0.47 and –0.84, respectively. Both cases belong to Case 2. Figure 4 indicates that the beam deflection determined with the present closed-form solutions agrees well with the numerical results at both frequencies. In the lower frequency case, deflection is still maximized at the loading center, but its distribution curve differs significantly from that of the static case. Similar variations are also observed in the higher frequency case. These results are consistent with previous numerical investigations [10].

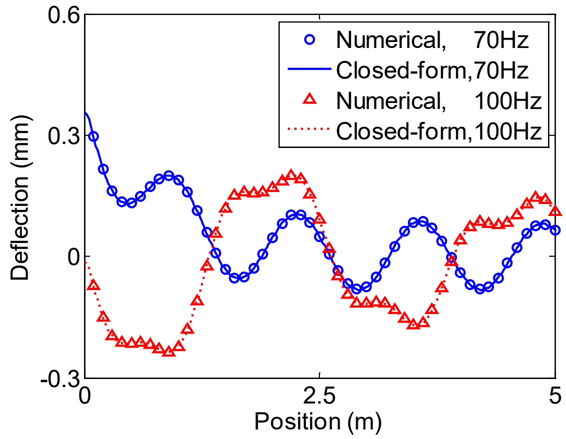


Figure 4: Comparison of the present closed-form and numerical solutions for the deflection of a viscoelastically Timoshenko beam.

Figure 5 displays a parametric study on the influence of the radius of gyration and shear rigidity on beam deflection. The closed-form solutions of these cases also agree well with the numerical results, whereas the numerical results are not shown here such that the closed-form results for different parameters are compared clearly. When the loading frequency is lower than the critical frequency, the variation of the curve along with positive x becomes more significant as the radius of gyration increases. However, an increase of the radius of gyration lowers maximal deflection when the loading frequency is 100 Hz. The influence of shear rigidity is weaker than that of the radius of gyration, especially at f = 70 Hz. In this case, the difference between different shear rigidity is little. These trends are consistent with the parametric study in [10].

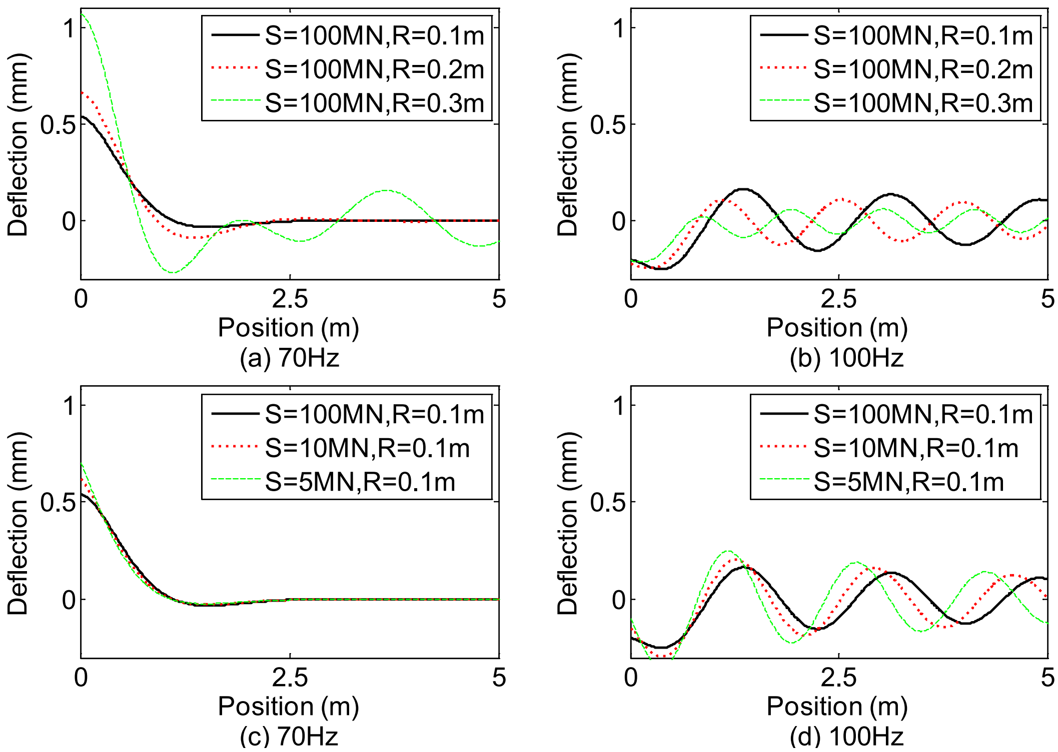


Figure 5: Influence of the radius of gyration and shear rigidity on beam deflection.

6. Conclusions

This paper presents a closed-form solution to a viscoelastically supported Timoshenko beam that is subjected a harmonic line load. The solution is first given in the integral representation through the inverse Fourier transform and further evaluated in five forms through Cauchy's residue theorem. Fourteen cases of roots are discussed given different combinations of viscous damping, frequency, and other sub-conditions. Moreover, the explicit forms of these roots are provided. An example is presented of the existing closed-form solutions for an EB beam on a Winkler foundation; these solutions correspond to the specific case of the present solutions. The other two examples verify the present solutions in comparison with the numerical solutions for a viscoelastically supported shear beam or Timoshenko beam.

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Appendix A

Eq. 10 has at least one complex root [17]. We prove that these two roots are complex in the following paragraph.

Suppose that Eq. (10) has one real root z_1 and one complex root $z_2 = r + is$, where r is a real number and s is a non-zero real number. Substituting z_1 into Eq. (10) yields:

$$\left(z_1^{\ 2} + Az_1 + C\right) + \left(Bz_1 + D\right)i = 0\,. \tag{A.1}$$

Hence, $z_1 = -D/B$ and then $D^2 - ABD + B^2C = 0$, that is, $(S - mR^2\omega^2)\omega^2c^2/EI^2S = 0$. This condition contradicts the assumption of $S > mR^2\omega^2$. Therefore, the assumption of one real and one complex root is incorrect. In other words, two roots of Eq. (10) are complex.

We also prove that these two complex roots are not conjugates. Suppose that Eq. (10) has two complex conjugate roots $z_{1,2} = r \pm is$. Applying Vieta's formulae yields $r = -(A + iB)/2$ and $r^2 + s^2 = C + iD$, thereby implying that $B = 0$ and $D = 0$. These findings contradict the conditions of $B \neq 0$ and $D \neq 0$. Therefore, the assumption of two complex conjugate roots is incorrect.

When the aforementioned results are combined, Eq. (10) has two non-conjugate complex roots.

Appendix B

R_1 to R_{10} in Table 3 are given by as follows:

$$R_1 = 2\left[r_1^4 - 4r_1r_2s_1s_2 + s_1^2\left(r_2^2 + s_1^2 - s_2^2\right) + r_1^2\left(-r_2^2 - 6s_1^2 + s_2^2\right)\right]$$

$$R_2 = 4\left[2r_1^3s_1 + r_1^2r_2s_2 - r_2s_1^2s_2 + r_1s_1\left(-r_2^2 - 2s_1^2 + s_2^2\right)\right]$$

$$R_3 = 2\left[r_2^4 - 4r_1r_2s_1s_2 - s_1^2s_2^2 + s_2^4 + r_2^2\left(s_1^2 - 6s_2^2\right) + r_1^2\left(s_2^2 - r_2^2\right)\right]$$

$$R_4 = 4\left[r_1^2r_2s_2 - r_2s_2\left(2r_2^2 + s_1^2 - 2s_2^2\right) + r_1s_1\left(s_2^2 - r_2^2\right)\right]$$

$$R_5 = 2\left[r_1^4 + 4r_1r_2s_1s_2 + s_1^2\left(r_2^2 + s_1^2 - s_2^2\right) + r_1^2\left(-r_2^2 - 6s_1^2 + s_2^2\right)\right]$$

$$R_6 = -4\left[2r_1^3s_1 - r_1^2r_2s_2 + r_2s_1^2s_2 + r_1s_1\left(-r_2^2 - 2s_1^2 + s_2^2\right)\right]$$

$$R_7 = 2\left[r_2^4 + 4r_1r_2s_1s_2 - s_1^2s_2^2 + s_2^4 + r_2^2\left(s_1^2 - 6s_2^2\right) + r_1^2\left(s_2^2 - r_2^2\right)\right]$$

$$R_8 = 4\left[r_1^2r_2s_2 - r_2s_2\left(2r_2^2 + s_1^2 - 2s_2^2\right) + r_1s_1\left(r_2^2 - s_2^2\right)\right]$$

$$R_9 = a\varepsilon_1\left(EI\varepsilon_1^2 + S - mR^2w^2\right)\left(\varepsilon_1 - \varepsilon_2\right)\cos\left(a\varepsilon_1\right) - \left[S\left(3\varepsilon_1 - \varepsilon_2\right) + mR^2w^2\left(\varepsilon_2 - 3\varepsilon_1\right) + EI\varepsilon_1^2\left(\varepsilon_1 + \varepsilon_2\right)\right]\sin\left(a\varepsilon_1\right)$$

$$R_{10} = \varepsilon_1\left(EI\varepsilon_1^2 + S - mR^2w^2\right)\left(\varepsilon_1 - \varepsilon_2\right)x\sin\left(a\varepsilon_1\right).$$

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