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Construction of orthogonal projector for the damage identification by measured substructural flexibility

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Construction of orthogonal projector for the damage identification by measured substructural flexibility

Abstract

The substructuring methods have been popularly used in model updating, system identification and damage assessment. In the substructuring methods, the global structure is divided into free-free substructures. The independent substructures move freely, and their stiffness matrix is singular and rank-deficient. The flexibility matrix of the free-free substructure, which is associated with the inverse of the stiffness matrix, is not easy to be determined. This study expands on the previous research of the substructuring methods by taking a deeper look at the analysis of a free-free substructure. An orthogonal projector is formulated to add/remove the rigid body components from the generalized stiffness and flexibility matrices of a free-free substructure, and thus make the substructural flexibility useful to model updating or damage identification. The orthogonal projector is derived both for the full and partial measured flexibility, and it can remove all rigid body components regardless its participation factor. The accuracy of the proposed method in extraction of the free-free flexibility and in damage identification is verified by an experimental beam. The properties addressed in this paper are not limited to be used for the analysis of a free-free substructure in many substructuring methods, and they are promising to be generalized to a range of analysis relevant to a free-free structure.

Keywords: Substructuring method; Damage identification; Rigid body modes; Stiffness matrix; Flexibility matrix; Orthogonal projector.

1 Introduction

In the past several decades, a large number of long-term structural health monitoring (SHM) systems have been designed and implemented worldwide on civil engineering structures such as large-scale bridges and high-rise buildings [1-5]. The accurate and efficient model updating and damage detection is significant for the long-term SHM systems. The dynamically measured flexibilities, residual flexibility and local flexibility are frequently used as the indexes for model updating and damage identification. It is observed that the flexibility is more sensitive to damage than the natural frequency or mode shape [6-8]. In particular, the local flexibility is inherently more sensitive to the local damage than the modal flexibility on the global structure, and the calculation of local flexibility attracts many research attention [9-11].

The substructuring methods have been extensively utilized in extraction of local flexibility or local stiffness of a structure [12-14]. The substructuring methods possess many advantages than the traditional global methods which analyze a structure as a whole. First, as the global structure is replaced by smaller and more manageable substructures, it is much easier and quicker to analyze the small system matrices. Second, the substructuring methods allow for the analysis of local parts. When the substructuring method is applied in model updating or damage identification, only one or a few substructures are involved in an optimization procedure. The size of the model and the number of uncertain parameters are much smaller than those in the global structure. Finally, in practical testing, the experimental instruments can be saved if it is necessary to measure the whole structure only for one or more substructures [9-25].

Alvin and Park [15] proposed a force method to extract the substructural flexibility matrices. Doebling *et al*. [16] and Park *et al*. [17] disassembled the measured stiffness matrix and flexibility matrix into substructural stiffness matrices by projecting the measured matrices onto substructural strain energy distribution. Terrell *et al*. [18] proposed a substructure parameterisation technique, and the eigenvalues and eigenvectors of the super-elements were used for model updating and damage identification. Weng [12-14] proposed an inverse substructuring method to disassemble the modal properties of the global structure to the substructure level by satisfying the constraints at the interfaces. Afterwards, the independent substructures can be singled out to be used for the static analysis, dynamic analysis, nonlinear analysis, fatigue analysis and so forth.

The substructuring methods require dividing the global structure into independent free or fixed substructures. After partition, the substructures are usually analyzed independently under the free-free constraints. The substructural movement is usually contributed by both the rigid body motion and deformational motion. The rigid body motion of a free structure is always undetermined or even infinite. It is necessary to remove the rigid body components and thus reflect a real property of a structure. In addition, since a free-free structure includes the rigid body motion, its stiffness matrix **K** is singular and rank-deficient. The singular value decomposition of **K** is not only expensive, but notoriously sensitive to rank decisions when carried out in floating-point arithmetic [26, 27]. In consequence, the flexibility matrix and residual flexibility matrices, associated with the inverse of the rank-deficient stiffness matrix, are not easy to be determined [27]. Some researchers avoided the rigid body modes by introducing a small shift on the rank deficient stiffness matrix [28] or extract the Moore-Penrose generalized inverses [29] of the stiffness matrix. This inevitably introduces some errors and computationally time consuming.

This paper addresses some frequently encountered difficulties associated with the analysis of the free substructures when the authors studied on the substructuring methods in the previous research [12-14, 21-25]. In our previous work, the zero-frequency modes are solved by an shift eigensolver, and the zero frequency is replace by a small value, not zero exactly. Afterwards, the rigid body modes are treated equivalently with the deformational modes to be analyzed. This inevitably introduces some errors, although it is acceptable for most engineering application. In this paper, an orthogonal projector is proposed to remove the rigid body components of the generalized stiffness and flexibility matrices which are contributed by both the deformational components and the rigid body components. The proposed orthogonal projector is used to extract the substructural modal flexibility matrices that are disassembled from the global flexibility, and thus makes them applicable in the substructure-based model updating and damage identification. The orthogonal projector is derived for both the full measurement and partial measurement of flexibility. The formulae proposed in this paper are not only useful for the analysis of a free-free substructure in many substructuring methods, but also generally applicable in the analysis of a free structure. For example, the measured flexibility is frequently used for damage identification, and the measured flexibility is measured under the free condition [7, 30]. The proposed orthogonal projector can be employed to extract the modal flexibility from the flexibility matrices measured under the free condition.

2 Construction of orthogonal projector

2.1 Eigenanalysis for a general singular matrix

If matrix **A** is a nondefective square matrix of order *N*, which may be unsymmetric and singular, matrix A is decoupled by its eigenvalues and eigenvectors as [26]

$$\mathbf{A} = \sum_i \lambda_i \{\phi_i\}\{\varphi_i\}^T \tag{1}$$

$$\phi_i^T \varphi_j = \delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta, λ_i are the eigenvalues, and ϕ_i and φ_i are the associated left and right bi-orthonormalized eigenvectors, respectively. The inverse or the Moore-Penrose inverse has the form of

$$\mathbf{A}^{-1} = \sum_i \frac{1}{\lambda_i}\{\phi_i\}\{\varphi_i\}^T \quad \text{or} \quad \mathbf{A}^{+} = \sum_i \frac{1}{\lambda_i}\{\phi_i\}\{\varphi_i\}^T \tag{3}$$

In structural engineering, a structure with *N* degrees of freedom (DOFs) has the stiffness matrix **K**, which is a symmetric and nondefective matrix. In physical viewpoint, the columns of the stiffness matrix **K** gives the loads to generate an unit displacement on a DOF. The stiffness matrix **K** of a linearly elastic structure relates node displacements to node forces through the stiffness equation [31]

$$\mathbf{K}\{x\} = f \tag{4}$$

where *f* includes the external force or constraints. According to Eq.(1), the stiffness matrix can be decoupled by the normalized eigenvectors as

$$\mathbf{K}=\sum_{i=1}^{N}\lambda_i\left(\phi_i\right)\left(\phi_i\right)^T=\mathbf{\Phi\Lambda\Phi}^T \tag{5}$$

where $\mathbf{\Lambda}=\mathrm{Diag}\begin{pmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_N\end{pmatrix}$ are the eigenvalues, and $\mathbf{\Phi}=\begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_N\end{bmatrix}$ are the corresponding orthogonal eigenvectors. They satisfy the following orthogonal relation

$$\mathbf{\Phi}^T\mathbf{K\Phi}=\mathbf{\Lambda}\,,\quad \mathbf{\Phi}^T\mathbf{\Phi}=\mathbf{I} \tag{6}$$

A flexibility matrix has a very straightforward physical interpretation: the displacement response caused by an applied unit load [30]. The flexibility matrix can be also written by the eigenmodes as

$$\mathbf{F}=\sum_{i=1}^{N}\frac{1}{\lambda_i}\phi_i\phi_i^T=\mathbf{\Phi\Lambda}^{-1}\mathbf{\Phi}^T \tag{7}$$

For a fixed structure, the stiffness matrix and flexibility matrix are normally formed a dual inverse of each other

$$\mathbf{KF}=\mathbf{I}\,,\quad \mathbf{F}=\mathbf{K}^{-1}\,,\quad \mathbf{K}=\mathbf{F}^{-1} \tag{8}$$

2.2 Orthogoal projector for the singular stiffness and flexibility matrices

For a free structure, the stiffness matrix **K** is singular and rank-deficient, and its flexibility can not be calculated from the inverse of the stiffness directly. Decoupling the singular and rank-deficient stiffness matrix **K** according to Eq.(1), it has two kinds of eigenpairs:

1) N_r zero eigenvalues pertaining to the rigid body motions. The associated rigid body eigenvectors $\mathbf{R}=\begin{bmatrix}r_1 & r_2 & \cdots & r_{N_r}\end{bmatrix}$ span the null space of **K**, which are orthogonal as $\mathbf{R}^T\mathbf{R}=\mathbf{I}$.

2) $N_d = N\text{-}N_r$ nonzero eigenvalues λ_i (i=1, 2, …, N_d). The associated orthogonal deformational eigenvectors $\mathbf{\Phi}_d=\begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_{N_d}\end{bmatrix}$ span the range space of **K**, which satisfy $\mathbf{K}\left\{\phi_i\right\}=\lambda_i\left\{\phi_i\right\}$, $\mathbf{\Phi}_d^T\mathbf{\Phi}_d=\mathbf{I}$ and $\mathbf{R}\left\{\phi_i\right\}=\mathbf{0}$.

In consequence, the displacement $\{x\}$ of a free structure can be written as a superposition of the deformational and rigid body motions, and it is decoupled by the sum of orthogonal vectors as

$$\{x\} = \{x_d\} + \{x_r\} = \mathbf{\Phi}_d \{q\} + \mathbf{R}\{\alpha\} \tag{9}$$

where $\{x_d\}$ is the displacement due to the deformational motion, $\{x_r\}$ is the displacement due to the rigid body motion, $\mathbf{\Phi}_d$ and $\mathbf{R}$ are the deformational modes and rigid body modes, and $\{q\}$ and $\{\alpha\}$ are the participation factors of deformational modes and rigid body modes respectively. In this research, the rigid body modes are proposed to be formulated by the geometric location of nodes, other than be extracted from a shift eigensolver or be determined by the null space of the rank deficient stiffness matrix [26]. For a two-dimensional structure having n nodes, the three independent rigid body modes are the x translation ($\mathbf{R}_x = 1$, $\mathbf{R}_y = 0$), the y translation ($\mathbf{R}_x = 0$, $\mathbf{R}_y = 1$) and the z rotation ($\mathbf{R}_x = -y$, $\mathbf{R}_y = x$), i.e.,

$$\mathbf{R}^T = \begin{bmatrix} 1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 1 & 0 \\ -y_1 & x_1 & 1 & -y_2 & \cdots & x_N & 1 \end{bmatrix} \tag{10}$$

The columns of $\mathbf{R}$ can be orthogonalized and be normalized with respect to mass matrix [6, 8].

For a free structure, the stiffness matrix and modal flexibility matrix composed by the deformational eigenmodes are

$$\mathbf{K} = \sum_{i=1}^{N_d} \lambda_i \phi_i \phi_i^T \,, \quad \mathbf{F} = \sum_{i=1}^{N_d} \frac{1}{\lambda_i} \phi_i \phi_i^T \tag{11}$$

Mathematically, the stiffness matrix and the modal flexibility matrix formed with the deformational eigenmodes are singular and rank deficient.

To consider the rigid body modes, a generalized stiffness matrix and a generalized flexibility matrix, which include the contribution made by both the rigid body modes and deformational modes, are defined as follows,

$$\overline{\mathbf{K}} = \mathbf{K} + \alpha\mathbf{R}\mathbf{R}^T = \sum_{i=1}^{N_d} \lambda_i \phi_i \phi_i^T + \sum_{i=1}^{N_r} \alpha_i r_i r_i^T \tag{12}$$

$$\overline{\mathbf{F}} = \mathbf{F} + \beta\mathbf{R}\mathbf{R}^T = \sum_{i=1}^{N_d} \frac{1}{\lambda_i} \phi_i \phi_i^T + \sum_{i=1}^{N_r} \beta_i r_i r_i^T \tag{13}$$

The generalized stiffness matrix $\overline{\mathbf{K}}$ and flexibility matrix $\overline{\mathbf{F}}$ are full-rank, and can be transformed with each other by inversion of

$$\left(\mathbf{K} + \alpha\mathbf{R}\mathbf{R}^T\right)^{-1} = \left(\sum_{i=1}^{N_d} \lambda_i \phi_i \phi_i^T + \sum_{i=1}^{N_r} \alpha_i \left(\phi_i \phi_i^T\right)\right)^{-1} = \left(\sum_{i=1}^{N_d+N_r} \lambda_i \phi_i \phi_i^T\right)^{-1} = \sum_{i=1}^{N_d} \frac{1}{\lambda_i} \phi_i \phi_i^T + \sum_{i=1}^{N_r} \frac{1}{\alpha_i} \mathbf{R}\mathbf{R}^T = \mathbf{F} + \frac{1}{\alpha} \mathbf{R}\mathbf{R}^T \tag{14}$$

$$\left(\mathbf{F} + \beta\mathbf{R}\mathbf{R}^T\right)^{-1} = \sum_{i=1}^{N_d} \lambda_i \phi_i \phi_i^T + \beta\mathbf{R}\mathbf{R}^T = \mathbf{K} + \frac{1}{\beta} \mathbf{R}\mathbf{R}^T \tag{15}$$

Theoretically, a free structure has random displacement in physical space due to its random rigid body motion. In other words, since the free structure moves freely in the space, the participation factors α and β of rigid body modes is undetermined or even infinite. This is usually the case for the measured flexibility or measured stiffness under the free constraints [7, 30], and for the extraction of substructural flexibility where the independent substructures are free substructures without constraints [12-13].

Due to the freely rigid body movement, the generalized flexibility, which is estimated from the measured flexibility or from the substructural flexibility, is undermined or even infinite, and thus it is not able to be used in model updating or damage identification directly. To make the undetermined generalized stiffness and flexibility useful for model updating or damage identification, an orthogonal projector **P** is formed by

$$\mathbf{P} = \mathbf{I} - \mathbf{R}\left(\mathbf{R}^T\mathbf{R}\right)^{-1}\mathbf{R}^T \quad = \mathbf{\Phi}_d\mathbf{\Phi}_d^T = \sum_{i=1}^{N_d}\phi_i\phi_i^T \tag{16}$$

which has the properties

$$\mathbf{P}^2 = \mathbf{P}\,, \quad \mathbf{PR} = \mathbf{R}^T\mathbf{P} = \mathbf{0} \tag{17}$$

Given the orthogonal projector P, the generalized stiffness can be transformed to the stiffness by

$$\mathbf{P}\left(\mathbf{K} + \alpha\mathbf{RR}^T\right) = \left(\sum_{i=1}^{N_d}\phi_i\phi_i^T\right)\left(\sum_{i=1}^{N_d}\lambda_i\phi_i\phi_i^T + \sum_{i=1}^{N_r}\alpha_i r_i r_i^T\right) = \sum_{i=1}^{N_d}\lambda_i\phi_i\phi_i^T + \sum_{i=1}^{N_d}\phi_i\phi_i^T\sum_{r=1}^{N_r}\alpha_i\phi_r\phi_r^T = \sum_{i=1}^{N_d}\lambda_i\phi_i\phi_i^T = \mathbf{K} \tag{18}$$

The modal flexibility can be calculated from the generalized flexibility by

$$\mathbf{P}\left(\mathbf{F} + \beta\mathbf{RR}^T\right) = \left(\sum_{i=1}^{N_d}\phi_i\phi_i^T\right)\left(\sum_{i=1}^{N_d}\frac{1}{\lambda_i}\phi_i\phi_i^T + \sum_{i=1}^{N_r}\beta_i r_i r_i^T\right) = \sum_{i=1}^{N_d}\frac{1}{\lambda_i}\phi_i\phi_i^T = \mathbf{F} \tag{19}$$

The projector **P** can filter out the rigid body modes and leave only the deformational modes in the stiffness and flexibility matrices. The modal flexibility, contributed by the deformational modes solely, can be used for model updating and damage identification [6-8, 30].

The modal flexibility matrix **F** can also be obtained from the inversion of the full-rank stiffness matrix $\bar{\mathbf{K}}$ by multiplying the orthogonal projector as

$$\mathbf{F} = \mathbf{P}\left(\mathbf{K} + \alpha\mathbf{RR}^T\right)^{-1}\mathbf{P} \tag{20}$$

Projecting the stiffness matrix or flexibility matrix onto the range space with projector **P** can always obtain the clean modal flexibility regardless the mode participation factor of rigid body modes.

In the vibration-based finite element (FE) model updating, the FE model is iteratively modified to ensure its vibration properties reproduce the measured counterparts in an optimal way [8, 14, 32]. The elemental parameters in the analytical model are adjusted iteratively to minimize the discrepancy between the analytical modal flexibility and the experimental modal flexibility. In model updating procedure, the modal flexibility matrix of the analytical model are calculated from the singular stiffness matrix and normalized by the orthogonal projector according to Eq.(20). The experimental modal flexibility is calculated from the measured flexibility according to Eq. (19).

2.3 Orthogonal projector for condensed stiffness and flexibility matrices

Practically, the stiffness or modal flexibility matrices are difficult to be estimated or measured on the full DOFs, it is necessary to construct the condensed stiffness and partial flexibility matrices for the free structure. If the full-DOF model is divided into the measured part and the unmeasured part, the stiffness matrix is divided as

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \tag{21}$$

where the subscript '*a*' represents the measured DOFs, while the subscript '*b*' the unmeasured DOFs. The static condensed stiffness matrix by the Guyan condensation is [33]

$$\mathbf{K}_D = \mathbf{K}_{aa} - \mathbf{K}_{ab}\mathbf{K}_{bb}^{-1}\mathbf{K}_{ba} \tag{22}$$

The modal flexibility and generalized flexibility are written according to the measured and unmeasured DOFs as [30]

$$\mathbf{F}=\mathbf{\Phi}_d\mathbf{\Lambda}^{-1}\mathbf{\Phi}_d^T=\begin{bmatrix}\mathbf{\Phi}_a\mathbf{\Lambda}^{-1}\mathbf{\Phi}_a^T & \mathbf{\Phi}_a\mathbf{\Lambda}^{-1}\mathbf{\Phi}_b^T \\ \mathbf{\Phi}_b\mathbf{\Lambda}^{-1}\mathbf{\Phi}_a^T & \mathbf{\Phi}_b\mathbf{\Lambda}^{-1}\mathbf{\Phi}_b^T\end{bmatrix}=\begin{bmatrix}\mathbf{F}_{aa} & \mathbf{F}_{ab} \\ \mathbf{F}_{ba} & \mathbf{F}_{bb}\end{bmatrix} \tag{23}$$

$$\overline{\mathbf{F}}=\mathbf{\Phi}_d\mathbf{\Lambda}^{-1}\mathbf{\Phi}_d^T+\beta\mathbf{R}\mathbf{R}^T=\begin{bmatrix}\mathbf{\Phi}_a\mathbf{\Lambda}^{-1}\mathbf{\Phi}_a^T+\beta\mathbf{R}_a\mathbf{R}_a^T & \mathbf{\Phi}_a\mathbf{\Lambda}^{-1}\mathbf{\Phi}_b^T+\beta\mathbf{R}_a\mathbf{R}_b^T \\ \mathbf{\Phi}_b\mathbf{\Lambda}^{-1}\mathbf{\Phi}_a^T+\beta\mathbf{R}_b\mathbf{R}_a^T & \mathbf{\Phi}_b\mathbf{\Lambda}^{-1}\mathbf{\Phi}_b^T+\beta\mathbf{R}_b\mathbf{R}_b^T\end{bmatrix} \tag{24}$$

The partial flexibility matrix estimated at the measured DOFs is directly equal to the corresponding lines and columns of the full flexibility

$$\mathbf{F}_{aa}=\mathbf{\Phi}_a\mathbf{\Lambda}^{-1}\mathbf{\Phi}_a^T \quad \text{and} \quad \overline{\mathbf{F}}_{aa}=\mathbf{\Phi}_a\mathbf{\Lambda}^{-1}\mathbf{\Phi}_a^T+\beta\mathbf{R}_a\mathbf{R}_a^T \tag{25}$$

In experiment, the partial flexibility matrices are estimated directly by the modal experiments or static testing on the measured DOFs [30, 31]. In Eq. (25), $\mathbf{\Lambda}$ is the measured frequencies, and $\mathbf{\Phi}_a$ is the mode shapes at the measured DOFs. $\mathbf{R}_a$ is the rigid body modes at the measured DOFs, and it can be obtained according to the nodal location of the measured DOFs by Eq.(10).

Based on the reduced model, an orthogonal projector is formed as

$$\mathbf{P}_R=\mathbf{I}-\mathbf{R}_a\left(\mathbf{R}_a^T\mathbf{R}_a\right)^{-1}\mathbf{R}_a^T \tag{26}$$

which has the properties

$$\mathbf{P}_R^2=\left(\mathbf{I}-\mathbf{R}_a\mathbf{R}_a^T\right)\left(\mathbf{I}-\mathbf{R}_a\mathbf{R}_a^T\right)=\mathbf{I}-2\mathbf{R}_a\mathbf{R}_a^T+\mathbf{R}_a\mathbf{R}_a^T\mathbf{R}_a\mathbf{R}_a^T=\mathbf{I}-\mathbf{R}_a\mathbf{R}_a^T=\mathbf{P}_R$$

$$\mathbf{P}\mathbf{R}_a=\mathbf{R}_a^T\mathbf{P}=\mathbf{0} \tag{27}$$

The orthogonal projector filtering out the rigid body modes by the dual inverse of

$$\mathbf{F}_{aa}=\mathbf{P}_R\left(\mathbf{K}_D+\mathbf{R}_a\left(\mathbf{R}_a^T\mathbf{R}_a\right)^{-1}\mathbf{R}_a^T\right)^{-1}\mathbf{P}_R \tag{28}$$

$$\mathbf{K}_D=\mathbf{P}_R\left(\mathbf{F}_{aa}+\mathbf{R}_a\left(\mathbf{R}_a^T\mathbf{R}_a\right)^{-1}\mathbf{R}_a^T\right)^{-1}\mathbf{P}_R \tag{29}$$

The modal flexibility can also be extracted from the generalized flexibility as

$$\mathbf{F}_{aa}=\mathbf{P}_R\overline{\mathbf{F}}_{aa}\mathbf{P}_R \tag{30}$$

In FE model updating, the experimental modal flexibility is calculated from the measured partial flexibility and the orthogonal projector according to Eq. (30). The elemental parameters in the analytical model are adjusted iteratively and the flexibility matrix of the analytical model is calculated from the singular stiffness matrix repeatedly according to Eq.(28), to minimize the discrepancy between the analytical flexibility and the experimental partial flexibility [8, 32].

3 Indentification of local flexibility by substructuring method

The substructure-based model updating requires extracting the substructural flexibility from the global flexibility [12]. This section will introduce the inverse substructuring method to extract the substructural flexibility matrix.

Without loss of generality, a global structure with N DOFs is divided into two substrctures: a fixed-free substructure of $N^{(1)}$ DOFs (Substructure 1) and a free-free substructure of $N^{(2)}$ DOFs (Substructure 2), connected by N_B interface DOFs (Figure 1). The partitioned substructures have DOFs of $NP = N^{(1)} + N^{(2)} = N + N_B$ in total.

To be independent substructures, the substructural variables are written in primitive forms of

$$\left\{x^p\right\} = \begin{Bmatrix} x^{(1)} \\ x^{(2)} \end{Bmatrix},\ \left\{f^p\right\} = \begin{Bmatrix} f^{(1)} \\ f^{(2)} \end{Bmatrix}$$

$$\mathbf{K}^p = \begin{bmatrix} \mathbf{K}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{(2)} \end{bmatrix},\ \mathbf{F}^p = \begin{bmatrix} \mathbf{F}^{(1)} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{F}}^{(2)} \end{bmatrix},\ \mathbf{R}^p = \begin{bmatrix} \mathbf{R}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{R}^{(2)} \end{bmatrix} \tag{31}$$

where $\mathbf{K}$, $\mathbf{F}$, x, and f represent the stiffness matrix, flexibility matrix, nodal displacements, and external forces, respectively. Superscript p denotes the primitive matrices or vectors, which directly encompass the variables of the independent substructures without imposing any constraints on them. The primitive matrices or vectors take length NP. Superscript (1) and (2) indicate the variables associated with the first and second substructures, respectively. Since the first substructure is a fixed structure after partition, the rigid body mode of the first substructure is null.

The substructural displacements and forces are linked to the global displacement and force vectors by the geometric relation

$$\left\{x^{p}\right\} = \mathbf{L}^{p}\left\{x_{g}\right\} \tag{32}$$

$$\left[\mathbf{L}^{p}\right]^{T}\left\{f^{p}\right\} = \left\{f_{g}\right\} \tag{33}$$

where $\{x_g\}$ denote the nodal displacement vector of the global structure and $\{f_g\}$ the external force vector, $\mathbf{L}^p$ is the geometric operator with size $NP \times N$, and is determined by the geometric relationship between the substructures and the global structure. For example, if the jth DOF of the global structure corresponds to the ith DOF in the separated substructures, then $\mathbf{L}_{ij}^{p} = 1$.

The displacement of an independent substructure can be written as a superposition of its deformational and rigid body motions [12] as follows.

$$\left\{x^{p}\right\} = \mathbf{F}^{p}\left\{f^{p}\right\} + \mathbf{R}^{p}\left\{\alpha^{p}\right\} \tag{34}$$

where $\left\{f^{p}\right\}$ represents the forces imposed on the independent substructures. As an independent structure, a substructure is loaded by the external force and the connecting force from the adjacent substructures

$$\left\{f^{p}\right\}=\left(\left[\mathbf{L}^{p}\right]^{T}\right)^{+}\left\{f_{g}\right\}+\mathbf{C}\{\tau\}=\left\{\tilde{f}_{g}\right\}+\mathbf{C}\{\tau\} \tag{35}$$

where $\left\{\tilde{f}_{g}\right\}=\left(\left[\mathbf{L}^{p}\right]^{T}\right)^{+}\left\{f_{g}\right\}=\tilde{\mathbf{L}}^{p}\left\{f_{g}\right\}$, $\tilde{\mathbf{L}}^{p}=\left(\left[\mathbf{L}^{p}\right]^{T}\right)^{+}$ is the generalized inverse of $\left[\mathbf{L}^{p}\right]^{T}$; $\{\tau\}$ is the Lagrange multiplier representing the connecting forces along the interfaces of the substructures; and matrix $\mathbf{C}$ implicitly defines the general connections between the independent substructures. In matrix $\mathbf{C}$, each row contains two non-zero elements, 1 and –1, for a rigid connection.

From Eqs. (32)-(35), the global displacement is expressed by the substructural variables as

$$\left\{x_{g}\right\}=\left[\mathbf{L}^{p}\right]^{+}\left\{x^{p}\right\}=\left[\tilde{\mathbf{L}}^{p}\right]^{T}\mathbf{F}^{p}\left(\left\{\tilde{f}_{g}\right\}+\mathbf{C}\{\tau\}\right)+\left[\tilde{\mathbf{L}}^{p}\right]^{T}\mathbf{R}^{p}\left\{\alpha^{p}\right\} \tag{36}$$

In Eq.(36), $\{\tau\}$ and $\left\{\alpha^{p}\right\}$ are unknown and needs to be determined. To solve the two variables, the force equilibrium compatibility and displacement compatibility are constructed in the following:

1) The rigid body mode and substructural force satisfy the force equilibrium compatibility

$$\left[\mathbf{R}^{p}\right]^{T}\left\{f^{p}\right\}=\{\mathbf{0}\} \tag{37}$$

2) From the physical perspective, matrix $\mathbf{C}$ bears the displacement compatibility [22]

$$\mathbf{C}^{T}\left\{x^{p}\right\}=\{\mathbf{0}\} \tag{38}$$

Submitting Eqs. (34) and (35) into Eqs. (37) and (38), $\{\tau\}$ and $\left\{\alpha^{p}\right\}$ are solved as

$$\left\{\alpha^{p}\right\}=\mathbf{K}_{R}^{-1}\left(\left[\mathbf{R}^{p}\right]^{T}-\mathbf{R}_{C}^{T}\mathbf{F}_{C}^{-1}\mathbf{C}^{T}\mathbf{F}^{p}\right)\left\{\tilde{f}_{g}\right\} \tag{39}$$

$$\{\tau\}=-\mathbf{F}_{C}^{-1}\mathbf{C}^{T}\mathbf{F}^{p}\left\{\tilde{f}_{g}\right\}+\mathbf{F}_{C}^{-1}\mathbf{R}_{C}\mathbf{K}_{R}^{-1}\left(\mathbf{R}_{C}^{T}\mathbf{F}_{C}^{-1}\mathbf{C}^{T}\mathbf{F}^{p}-\left[\mathbf{R}^{p}\right]^{T}\right)\left\{\tilde{f}_{g}\right\} \tag{40}$$

where $\mathbf{K}_{R}=\mathbf{R}_{C}^{T}\mathbf{F}_{C}^{-1}\mathbf{R}_{C}$.

As long as $\{\tau\}$ and $\{\alpha^p\}$ are solved, Eq.(36) lead to

$$\{x_g\} = \left[\tilde{\mathbf{L}}^p\right]^T \left(\mathbf{F}^p - \mathbf{F}^p\mathbf{K}_C\mathbf{F}^p + \mathbf{F}^p\mathbf{K}_C\mathbf{F}_R\mathbf{K}_C\mathbf{F}^p - \mathbf{F}^p\mathbf{K}_C\mathbf{F}_R - \mathbf{F}_R\mathbf{K}_C\mathbf{F}^p - \mathbf{F}^p\mathbf{H}\mathbf{F}^p + \mathbf{F}_R\right)\{\tilde{f}_g\}$$
$$= \left[\tilde{\mathbf{L}}^p\right]^T \left(\mathbf{F}^p - \mathbf{F}^p\mathbf{H}\mathbf{F}^p - \mathbf{F}^p\mathbf{K}_C\mathbf{F}_R - \mathbf{F}_R^T\mathbf{K}_C^T\mathbf{F}^p + \mathbf{F}_R\right)\tilde{\mathbf{L}}^p\{f_g\} \tag{41}$$

in which

$$\mathbf{F}_R = \mathbf{R}^p\left(\left[\mathbf{R}^p\right]^T \mathbf{K}_C\mathbf{R}^p\right)^{-1}\left[\mathbf{R}^p\right]^T, \quad \mathbf{H} = \mathbf{K}_C - \mathbf{K}_C\mathbf{F}_R\mathbf{K}_C, \quad \mathbf{K}_C = \mathbf{C}\mathbf{F}_C^{-1}\mathbf{C}^T \tag{42}$$

Therefore,

$$\mathbf{F}_g = \left[\tilde{\mathbf{L}}^p\right]^T \left(\mathbf{F}^p - \mathbf{F}^p\mathbf{H}\mathbf{F}^p - \mathbf{F}^p\mathbf{K}_C\mathbf{F}_R - \mathbf{F}_R^T\mathbf{K}_C^T\mathbf{F}^p + \mathbf{F}_R\right)\tilde{\mathbf{L}}^p \tag{43}$$

or the global flexibility matrix can be expressed by the substructural flexibility matrix as

$$\mathbf{L}^p\mathbf{F}_g\left[\mathbf{L}^p\right]^T = \mathbf{F}^p - \mathbf{F}^p\mathbf{K}_C\mathbf{F}_R - \mathbf{F}_R^T\mathbf{K}_C^T\mathbf{F}^p - \mathbf{F}^p\mathbf{H}\mathbf{F}^p + \mathbf{F}_R \tag{44}$$

Given the relation of Eq. (44), $\mathbf{F}^p$ can be solved from the global flexibility matrix $\mathbf{F}_g$ by numerical method [27]. The substructural flexibility matrices are therefore the diagonal sub-blocks of $\mathbf{F}^p$. It is noted that, the substructural flexibility matrices in $\mathbf{F}^p$ are the generalized flexibility matrices, which compass both the deformational and rigid body components. The orthogonal projector in Section 2 is required to extract the modal flexibility matrices, which are thereafter used as references for updating the analytical models of the independent substructures.

4 Substructure-based model updating

This section will illustrate the substructure-based model updating procedure, by treating a substructure as an independent structure. Taking Substructure 2 in Figure 1(b) as an example, after the substructural flexibility matrix $\left(\overline{\mathbf{F}}^{(2)}\right)^E$ is extracted from the global flexibility matrix $\left(\mathbf{F}_g\right)^E$, the sub-model of Substructure 2 is updated independently by the following procedure. For clearance, superscript E represents the modal data from the experimental measurement, and A represents the data of the analytical model.

1) The rigid body modes $\mathbf{R}^{(2)}$ are constructed according to the nodal location of Substructure 2, and then the orthogonal projector $\mathbf{P}^{(2)}$ is calculated by Eq.(16) or Eq. (26).

2) The rigid body modes are removed from the generalized flexibility matrix $\left(\overline{\mathbf{F}}^{(2)}\right)^E$ using the orthogonal projector $\mathbf{P}^{(2)}$: $\left(\mathbf{F}^{(2)}\right)^E = \left[\mathbf{P}^{(2)}\right]^T \left(\overline{\mathbf{F}}^{(2)}\right)^E \mathbf{P}^{(2)}$.

3) The analytical model of Substructure 2 is updated by treating it as an independent structure. In each iteration, the substructural flexibility matrix $\left(\mathbf{F}^{(2)}\right)^A$ is calculated from the stiffness matrix of the analytical model and the orthogonal projector $\mathbf{F}^{(2)} = \mathbf{P}^{(2)}\left(\mathbf{K}^{(2)} + \alpha\mathbf{R}^{(2)}\left(\mathbf{R}^{(2)}\right)^T\right)^{-1}\mathbf{P}^{(2)}$. The elemental parameters in Substructure 2 are adjusted to minimize the difference of $\Delta\mathbf{F} = norm\left(\left(\mathbf{F}^{(2)}\right)^E - \left(\mathbf{F}^{(2)}\right)^A\right)$ through some optimization algorithms, for example the Trust Region Newton method [8, 32].

5 Case studies

5.1 Spring-mass model

The 6-DOF lumped spring-mass model (Figure 2) is used here to illustrate the construction of orthogonal projector and its accuracy directly.

The spring-mass system is a one-dimensional structure, which is fixed at one end and free at the other. The stiffness parameters of the six springs are set to $k_1 = k_2 = k_3 = 10$ N/m, $k_4 = k_5 = k_6 = 20$ N/m. The six masses are set to $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 1$ kg, $m_4 = 2$ kg, $m_5 = 2$ kg, $m_6 = 1$ kg. The stiffness matrix of the model is

$$\mathbf{K}_g = \begin{bmatrix} 20 & -10 & & & & \\ -10 & 20 & -10 & & & \\ & -10 & 30 & -20 & & \\ & & -20 & 40 & -20 & \\ & & & -20 & 40 & -20 \\ & & & & -20 & 20 \end{bmatrix} \tag{45}$$

The frequencies f^E and mode shapes $\mathbf{\Phi}^E$ of the global structure, which are assumed to be the experimental modal data measured on the global structure (Figure (2)), are calculated as

$$\mathbf{\Lambda}^E = \left(2\pi f^E\right)^2 = \text{Diag}\left(\left[0.4198 \quad 4.9812 \quad 13.8865 \quad 23.4349 \quad 33.7875 \quad 43.4901\right]\right)$$

$$\mathbf{\Phi}^E = \begin{bmatrix} 0.1208 & 0.3639 & 0.2840 & 0.8735 & -0.0842 & 0.0458 \\ 0.2366 & 0.5466 & 0.1736 & -0.3001 & 0.1161 & -0.1077 \\ 0.3325 & 0.1847 & -0.4190 & -0.0673 & -0.4682 & 0.6753 \\ 0.3735 & -0.0422 & -0.4244 & 0.1279 & 0.0306 & -0.4017 \\ 0.3988 & -0.2481 & 0.1595 & 0.0234 & 0.4260 & 0.2682 \\ 0.4073 & -0.3304 & 0.5219 & -0.1360 & -0.6180 & -0.2284 \end{bmatrix} \tag{46}$$

As a result, the experimental flexibility matrix of the global structure $\mathbf{F}_g^E$ is determined as

$$\mathbf{F}_g^E = \mathbf{\Phi}^E \left(\mathbf{\Lambda}^E\right)^{-1} \left[\mathbf{\Phi}^E\right]^T = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.35 & 0.35 & 0.35 \\ 0.1 & 0.2 & 0.3 & 0.35 & 0.4 & 0.4 \\ 0.1 & 0.2 & 0.3 & 0.35 & 0.4 & 0.45 \end{bmatrix} \tag{47}$$

The global structure is partitioned into two substructures as demonstrated in Figure 2. The substructural flexibility matrix is calculated from the global flexibility matrix by the inverse substructuring method proposed in Section 3. Concerning the second substructure, the substructural flexibility is obtained as

$$\overline{\mathbf{F}}^{(2)} = \mathbf{F}_g^E\left(3:6 \quad 3:6\right) = \begin{bmatrix} 0.3 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.35 & 0.35 & 0.35 \\ 0.3 & 0.35 & 0.4 & 0.4 \\ 0.3 & 0.35 & 0.4 & 0.45 \end{bmatrix} \tag{48}$$

The extracted substructural flexibility is contributed by both the deformational modes and rigid body modes. An orthogonal projector is required to remove the rigid body components in the substructural flexibility matrix. The mass-normalized rigid body mode and the orthogonal projector are constructed for the second substructure as

$$\mathbf{R}^{(2)} = \begin{bmatrix} 0.4082 \\ 0.4082 \\ 0.4082 \\ 0.4082 \end{bmatrix}, \quad \mathbf{P}^{(2)} = \mathbf{I} - \mathbf{M}^{(2)}\mathbf{R}^{(2)}\left[\mathbf{R}^{(2)}\right]^T = \begin{bmatrix} 0.8333 & -0.1667 & -0.1667 & -0.1667 \\ -0.3333 & 0.6667 & -0.3333 & -0.3333 \\ -0.3333 & -0.3333 & 0.6667 & -0.3333 \\ -0.1667 & -0.1667 & -0.1667 & 0.8333 \end{bmatrix} \tag{49}$$

The modal flexibility matrix of the second substructure is therefore obtained by multiplying the orthogonal projector as

$$\left(\mathbf{F}^{(2)}\right)^E = \left[\mathbf{P}^{(2)}\right]^T \overline{\mathbf{F}}^{(2)}\mathbf{P}^{(2)} = \begin{bmatrix} 0.0486 & 0.0069 & -0.0181 & -0.0264 \\ 0.0069 & 0.0153 & -0.0097 & -0.0181 \\ -0.0181 & -0.0097 & 0.0153 & 0.0069 \\ -0.0264 & -0.0181 & 0.0069 & 0.0486 \end{bmatrix} \tag{50}$$

To verify the accuracy of the extracted substructural flexibility, the real modal flexibility matrix of the second substructure is calculated from the analytical model by treating the second substructure as an independent structure (Figure 3). The modal flexibility of this structure is

$$\left(\mathbf{F}^{(2)}\right)^A = \mathbf{\Phi}_d^{(2)}\left(\mathbf{\Lambda}_d^{(2)}\right)^{-1}\left[\mathbf{\Phi}_d^{(2)}\right]^T = \begin{bmatrix} 0.0486 & 0.0069 & -0.0181 & -0.0264 \\ 0.0069 & 0.0153 & -0.0097 & -0.0181 \\ -0.0181 & -0.0097 & 0.0153 & 0.0069 \\ -0.0264 & -0.0181 & 0.0069 & 0.0486 \end{bmatrix} \tag{51}$$

Using the orthogonal projector, the substructural flexibility matrix $\left(\mathbf{F}^{(2)}\right)^E$ extracted from the global modal data exactly reconstructs that calculated from the independent analytical model of the second substructure $\left(\mathbf{F}^{(2)}\right)^{A}$. The proposed orthogonal projector removes the rigid body component and recovers the substructural modal flexibility successfully. The substructural modal flexibility is therefore effective to be used in model updating.

5.2 Application to experimental beam structure

In this section, the proposed orthogonal projector is applied to an experimental cantilever beam for model updating and damage identification in the substructure level. The substructural flexibility, which is normalized by the orthogonal projector, will be used to update the beam structure in the undamaged state and damaged states.

The beam specimen is 50.1 mm wide, 3.0 mm high and 750 mm long as shown in Figures 4 and 5. The mass density is 8.026×10^{3} kg/m^{3}. To assure that the boundary condition was not altered in each testing, two thick blocks were welded on both sides of the clamped end. The structure was tested in the intact state and four damaged configurations in Table 1. The beam was first tested without introduction of any damage as 'Case 0'. Afterwards, the beam was cut at Location 1 as shown in Figure 6, with the depth of d = 5 mm, 10 mm, and 15 mm gradually, corresponding to 'Case 1', 'Case 2', 'Case 3', respectively. In 'Case 4', the beam was additionally cut at Location 2 with depth of d = 10 mm. The width of the cuts is b = 5 mm in all of the damaged cases.

Ten accelerometers were mounted evenly on the beam as shown in Figure 5 to record the responses at the vertical direction of the ten points, based on which six pairs of the frequencies and mass-normalized mode shapes were extracted in all of the five states with confidence (Figure 6). The measured frequencies and mode shapes are then employed to calculate the global mass-normalized flexibility as [8, 30]

$$\mathbf{F}_g^E = \mathbf{\Phi}^E \left(\mathbf{\Lambda}^E\right)^{-1} \left[\mathbf{\Phi}^E\right]^T \tag{52}$$

The global flexibility matrix takes the size of 10×10.

The beam is modeled using 10 two-dimensional beam elements and 11 nodes, which are divided into two substructures at Node 5 as shown in Figure 7. After partition, the first substructure is fixed, while the second one is free. The sub-models of the two substructures are constructed independently in Figure 8.

In all of the five states, the substructural flexibility matrices of the two substructures are extracted from the measured global flexibility by the inverse substructuring method. The undamaged substructural flexibility matrices are used to update the two sub-models initially. The stiffness of the elements is chosen as the updating parameters. There are four updating parameters in the first substructure and six in the second. The refined sub-models of the two substructures are subsequently used for damage identification.

In each of the undamaged state and four damaged states, the two sub-models are tuned independently. The first substructure is fixed after partition, and the substructural flexibility matrices $\left(\mathbf{F}_{aa}^{(1)}\right)^E$ is used directly for model updating. In model updating, the substructural flexibility is calculated from the analytical model by $\left(\mathbf{F}^{(1)}\right)^A = \left(\mathbf{K}^{(1)}\right)^{-1}$ and is reduced to the measured points $\left(\mathbf{F}_{aa}^{(1)}\right)^A$. The elemental parameters are adjusted in each iteration to make $\left(\mathbf{F}_{aa}^{(1)}\right)^A$ reproduce the extracted flexibility matrix $\left(\mathbf{F}_{aa}^{(1)}\right)^E$ through an optimization process, i.e., to minimize the objective function $\Delta\mathbf{F} = norm\left(\left(\mathbf{F}_{aa}^{(1)}\right)^A - \left(\mathbf{F}_{aa}^{(1)}\right)^E\right)$.

The second substructure is free after partition, and it is updated by the following procedure.

1) The substructural flexibility $\overline{\mathbf{F}}_{aa}^{(2)}$ is extracted from the global flexibility by the inverse substructuring method.
2) Construct the projector $\mathbf{P}_R$ for the second substructure. The rigid body modes $\mathbf{R}_a^{(2)}$ are formed from the nodal location of the second substructure. Accordingly, the projector is formulated as $\mathbf{P}_R^{(2)} = \mathbf{I} - \mathbf{R}_a^{(2)}\left(\left[\mathbf{R}_a^{(2)}\right]^T \mathbf{R}_a^{(2)}\right)^{-1}\left[\mathbf{R}_a^{(2)}\right]^T$.
3) The substructural modal flexibility is obtained by the experimental partial flexibility and orthogonal projector

$$\left(\mathbf{F}_{aa}^{(2)}\right)^E = \left[\mathbf{P}_R^{(2)}\right]^T \overline{\mathbf{F}}_{aa}^{(2)}\mathbf{P}_R^{(2)} \tag{53}$$

4) On the other hand, the reduced substructural flexibility matrix is calculated from the analytical model and the projector $\mathbf{P}_R^{(2)}$ by

$$\left(\mathbf{F}_{aa}^{(2)}\right)^A = \left[\mathbf{P}_R^{(2)}\right]^T \left(\mathbf{K}_C^{(2)} + \mathbf{R}_a^{(2)}\left(\left[\mathbf{R}_a^{(2)}\right]^T \mathbf{R}_a^{(2)}\right)^{-1}\left[\mathbf{R}_a^{(2)}\right]^T\right)^{-1} \mathbf{P}_R^{(2)} \tag{54}$$

The second substructure is similarly updated as an independent structure to minimize the discrepancy between the modal flexibility matrix of the analytical model and those extracted from experimental measurement, i.e., $\Delta\mathbf{F} = norm\left(\left(\mathbf{F}_{aa}^{(2)}\right)^{A} - \left(\mathbf{F}_{aa}^{(2)}\right)^{E}\right)$, through an optimization process [8, 32].

The stiffness reduction factor (SRF) is employed to indicate the change ratio of the updated parameter to the initial value before updating [26].

$$\mathrm{SRF} = \frac{\Delta r}{r} = \frac{r^{U} - r^{O}}{r^{O}} \tag{55}$$

where superscript O represents the original parameters before updating and U represents the updated values after updating. The SRF values of the first substructure in the aforementioned five states are shown in Figure 9. Element 2 is found to have noticeable SRF values in the four damaged states (Cases 1 to 4), whereas the SRF values of the other elements are close to zero. The identified Element 2 coincides with the location of the artificial cut made in the experiment. As the depth of the cut increases progressively with d = 5 mm (Case 1), 10 mm (Case 2), and 15 mm (Case 3 and Case 4), the magnitudes of the SRF values increase as expected. The SRF value quantifies the overall equivalent change in the elemental parameter due to a local cut. As a result, Element 2 is believed to be damaged with the equivalent stiffness reductions of about 10% (Case 1), 14% (Case 2), and 22% (Case 3 and Case 4), respectively.

The six elemental parameters of the second substructure are updated similarly in each of the five states. The SRF values are given in Figure 10. In Cases 1 to 3, very small SRF values are observed in some elements. No artificial cut was introduced within the second substructure in any of these three cases. The non-zero SRF values are due to the inevitable measurement noise. In Case 4, the SRF value of Element 3 (Element 7 of the global structure) is remarkably identified with -16% as shown in Figure 10(e). This agrees with the location of the cut introduced in the experiment.

The updated parameters are then used in the global FE model to calculate the frequencies and mode shapes of the global structure and compare them with the measured ones, as listed in Tables 2 to 6. In all of the five states, the frequencies and mode shapes of the updated structure agree better with the measured ones than do those obtained before the model was updated.

For comparison, the beam structure is also analyzed according to the traditional global method using the same measurement data and updating parameters [8, 321]. The difference between the measured modal flexibility and analytical modal flexibility is minimized by adjusting the 10 elemental stiffness parameters. The initial model is updated in the undamaged state and the refined model is subsequently employed for damage identification. The SRF values identified in the five states are illustrated in Figure 11.

In Cases 1 to 4, the SRF values identified by the traditional global method are consistent with those observed by the substructure-based model updating, and the results from both methods reveal the real locations and severity of the artificial cuts made in the experiment. This again proves that the proposed orthogonal projector is effective to remove the rigid body components from the generalized substructural flexibility extracted from the experiment, and thus the resulting substructural modal flexibility is applicable in model updating and damage identification.

The above examples show the construction of the orthogonal projector and verify its accuracy in the flexibility-based model updating. It can be generalized to be used for all free-free structures and for the free-free substructures in substructuring methods.

6 Conclusions

This paper provides a deep look at the properties of a free structure. It addresses the difficulties associated with the analysis of a free substructure that frequently encountered in the substructuring methods. An orthogonal projector is formulated to remove the rigid body component from the generalized stiffness and flexibility matrices, and thus make the modal flexibility useful for model updating and damage identification. The projector is constructed for both the full model and reduced model. The proposed orthogonal projector merits extensive applications in dealing with a free structure. One of the applications of the projector is to normalize the substructural flexibility extracted from the experimental measurement, and the normalized substructural modal flexibility is applicable to the substructure-based model updating and damage identification. Although the present research intends to assist the analysis of the free substructures in substructuring methods, it can be generalized to extract the modal flexibility from any flexibility matrix measured under free condition.

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