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A LINK-BASED MEAN-EXCESS TRAFFIC EQUILIBRIUM MODEL UNDER UNCERTAINTY

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ABSTRACT

Traffic equilibrium models under uncertainty characterize travelers' route choice behaviors under travel time variability. In this paper, we develop a *link*-based mean-excess traffic equilibrium (L-METE) model by integrating the sub-additivity property and complete travel time variability characterization of mean-excess travel time (METT), and the computationally tractable additive route cost structure of the conventional user equilibrium (UE) problem. Compared to the majority of relevant models formulated in the *route* domain, the *link*-based modeling has two desirable features on modeling flexibility and algorithmic development. First, it avoids the normal route travel time distribution assumption (uniformly imposed for all routes) that inherits from the Central Limit Theorem in most *route*-based models, permitting the use of any suitable link travel time distributions from empirical studies. Second, the additive route cost structure makes the L-METE model solvable by readily adapting existing UE algorithms without the need of storing/enumerating routes while avoiding the computationally demanding nonadditive shortest path problem and route flow allocations in route-based models, which is a significant benefit for large-scale network applications under uncertainty.

Keywords: uncertainty, traffic equilibrium, mean-excess travel time, nonadditive, sub-additivity

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1 INTRODUCTION

Uncertainty is present in many aspects of transportation systems. Recent empirical studies of various cities (see recent reviews by Li *et al*. (2010) and Carrion and Levinson (2012)) have pointed out that travel time variability plays an important role in travelers' route choice decisions. For example, Abdel-Aty *et al*. (1995) found that travel time reliability was either the most or second most important factor for most commuters. Travelers treat travel time variability as a risk in their travel choices, because it introduces uncertainty for an on-time arrival at the destination. On the other hand, observed travel time data exhibit a strong positive skew, very long/fat upper tail, and bimodality (see, e.g., van Lint *et al*., 2008; Fosgereau and Karlstrom, 2010; Susilawati *et al*., 2013). These empirical characteristics impose challenges to characterizing travel time distributions and modeling route choice behavior under travel time variability.

Due to the theoretical and practical importance, modeling route choice behavior and traffic equilibrium problem (TEP) under uncertainty have received great attention in the literature. Different modeling philosophies have been developed to characterize travelers' route choice behavior under travel time variability, e.g., the expected utility (Mirchandani and Soroush, 1987; Tatineni *et al*., 1997; Chen *et al.*, 2002b; Yin *et al.*, 2004; Watling, 2006), reliability (Lo *et al*., 2006; Shao *et al*., 2006; Lam *et al*., 2008; Nie and Wu, 2009; Chen and Zhou, 2010), prospect theory (Connors and Sumalee, 2009; Xu *et al*., 2011), game theory (Bell and Cassir, 2002; Szeto *et al*., 2006), ambiguity-aware CARA (constant absolute risk aversion) travel time model (Qi *et al*., 2016), stochastic dominance (Wu and Nie, 2011), and multi-objective optimization (Tan *et al*., 2014; Wang *et al*., 2014), etc.

The estimation of risk-taking route choice criteria typically requires the knowledge of route travel time probability distribution. In the literature, most TEPs under uncertainty (see some exceptions to be discussed in Section 2.1) use the Central Limit Theorem (CLT) to propagate link travel time distributions (using the mean and variance only). The CLT implicitly assumes *route* travel times to be *normally distributed* and *link* travel times to be *independent*[1], regardless of *link* travel time distributions. This assumption enables a simple approximation of route travel time distributions. However, it does not necessarily concur with empirical characteristics (i.e., strong positive skew, very long/fat upper tail, and bimodality). Also,

[1] The CLT has a number of variants. The basic version states that given certain conditions, the mean of a sufficiently large number of independent random variables will be approximately normally distributed, regardless of the underlying distribution. Further generalizations have weakened the independence assumption and allow the random variables to be weakly dependent or not "too" dependent (Rice, 2007).

besides normal distribution, diverse distribution types have been used to fit travel time distribution. Examples include Lognormal (Emam and Al-Deek, 2006; Kaparias *et al*., 2008; Rakha *et al*., 2010; Arezoumandi, 2011; Chen *et al*., 2014), Shifted Lognormal (Srinivasan *et al*., 2014), Weibull and Exponential (Al-Deek and Emam, 2006), Gamma (Polus, 1979), compound Gamma (Kim and Mahmassani, 2015), Generalized Beta (Castillo *et al*., 2012), Stable (Fosgerau and Fukuda, 2012), and Burr (Susilawati *et al*., 2013)). Hence, the route-based CLT modeling approach may sacrifice too much modeling flexibility (i.e., realistic route travel time distribution and link travel time interdependence) for mathematical tractability.

Other than the above limitation on modeling flexibility, the *route*-based modeling approach also adds complexity to algorithmic development for applications of large-scale networks. The main reason is that the risk-taking route choice criteria, e.g., the widely used travel time budget (TTB), percentile travel time (PTT), and mean-excess travel time (METT), are typically nonadditive. In other words, route cost cannot be easily decomposed into the sum of link costs. Solving the nonadditive TEPs needs either a nonadditive shortest path algorithm (Lo and Chen, 2000; Chen et al., 2001, 2012) to serve as a column generation scheme or a priori working route set created by a route choice set generation scheme (Prato, 2009). The computation of either scheme is quite expensive for large-scale networks, which also hinders the applications of bi-level optimization problems when using the TEP models as the lower-level subprogram.

This paper considers a recent addition to the family of traffic equilibrium models under uncertainty, known as the mean-excess traffic equilibrium (METE) model (Chen and Zhou, 2010). The METE model adopts the *route*-based METT to capture both reliability (on-time arrival) and unreliability (late arrival) aspects of route travel time variability. Thus, METT can be regarded as a more complete and accurate risk-averse measure to describe travelers' route choice decisions under uncertainty. Also, it has received other applications such as in modeling travelers' stochastic perception error (Chen *et al*., 2011; Xu *et al*., 2013), demand elasticity under multiple user classes (Xu *et al*., 2014b), routing hazardous materials on time-dependent networks (Taumazis and Kwon, 2013), travel time robust reliability (Sun and Gao, 2012), strategies cost in schedule-based transit networks with capacity constraints (Rochau *et al*., 2012), risk-based transit schedule design (Zhao *et al*., 2013), network performance assessment (Xu *et al*., 2014a), and parking pricing and modal split (Zhu *et al*., 2014). However, the route-based METT still has the above two shortcomings on modeling flexibility and algorithmic development. In this paper, we address the possibility of formulating the TEP under uncertainty in the *link* domain and propose a *link*-based METE model to resolve the two shortcomings simultaneously.

Specifically, we develop a *link*-based mean-excess traffic equilibrium (L-METE) model by making use of the sub-additivity property of mean-excess travel time (METT). The sum of *link*-based additive METTs on a route provides an upper bound of the *route*-based nonadditive METT. Conceptually, the L-METE model integrates the sub-additivity property of METT, the complete travel time variability characterization of METT, and the computationally tractable additive route cost structure of the conventional user equilibrium (UE) problem. Similar to the classical Beckmann transformation, two equivalent mathematical programming (MP) formulations (i.e., link-route and node-link) are provided for the *link*-based METE model. Compared to the majority of relevant models formulated in the *route* domain, the modeling philosophy in the *link* domain has two desirable features.

(1) *Modeling flexibility*: the *link*-based METE model has no specific assumptions on link and route travel time distributions. Any suitable link travel time distribution from empirical studies can be adopted. Hence, it avoids the assumptions of independent (or weakly dependent) link travel times and normally distributed route travel times that inherited from the Central Limit Theorem in most *route*-based models.
(2) *Algorithmic development*: the existing algorithms for the conventional UE problem in the planning software packages can be readily adapted to solve the L-METE model while avoiding solving the nonadditive shortest path problem in route-based models. The additive route cost structure makes the L-METE model solvable without storing routes, which is a significant benefit for planners working with large-scale networks. For demonstration purposes, we customize the widely used Frank-Wolfe algorithm of the UE problem to solve the L-METE model and then apply it to a large-scale realistic network.

In summary, the main contribution of this paper is the development of a theoretically sound and computationally tractable *link*-based mean-excess traffic equilibrium model to simultaneously alleviate the drawbacks on both modeling flexibility and algorithmic development of the existing *route*-based traffic equilibrium models under uncertainty.

In a broader research picture, the link-based modeling has also been studied in the traffic assignment literature. To list a few, Ran *et al*. (1996) developed a link-based variational inequality formulation for the dynamic user-optimal departure time and route choice problem, so that route enumeration could be avoided in both the formulation and solution procedure. Maher (1998) developed an efficient and practical algorithm for the logit-based stochastic user equilibrium (SUE) which obviates path storage/enumeration. He *et al*. (2010) developed a link-based day-to-day traffic assignment model to relax two essential shortcomings of route-based models (e.g., dependence on an initial route flow pattern and route overlapping problem).

The remainder of this paper is organized as follows. Section 2 details two observations on route-based METE model. Sections 3 and 4 present the link-based METE model and solution algorithm. Section 5 provides a set of numerical examples to demonstrate the features and applicability of the proposed model. Finally, some concluding remarks and future research directions are provided in Section 6.

2 Two Observations on the Route-based METT

In this section, we discuss two observations associated with the *route*-based METE model.

2.1 Observation 1: Route Travel Time Distribution Characterization

Most *route*-based TEP models under uncertainty adopt the Central Limit Theorem (CLT) to propagate link travel time distributions by using the mean and variance only. The CLT implicitly assumes route travel times are normally distributed and link travel times are independent (or weakly dependent), regardless of link travel time distributions. This assumption enables a simple and tractable approximation of route travel time distributions; whereas the concern is whether the CLT using the normal distribution is adequate in approximating the route travel time distributions. Generally, the number of random variables (i.e., link travel times in our context) of using the CLT should be larger than 30 for most distributions and 15 for fairly symmetric distributions (Berenson *et al*., 2011). However, the normal distribution assumption may not be always applicable for all used routes in realistic networks, since they may not include a sufficient number of links to satisfy the approximation requirement, especially for the short- and medium-distance origin-destination (O-D) pairs. In addition, the above assumption does not necessarily coincide with empirical characteristics of travel time distributions (i.e., strongly positive skewness, very long/fat upper tail, bimodality, and diverse distribution types).

Below we assume all *link* travel time distributions follow the same lognormal distribution with the same mean of 10. Consider two coefficients of variation (CV): 0.2 and 0.3, which correspond to the variance of 4 and 9. Our research question is: *does the route travel time distribution approach the normal distribution with the increasing number of links on the route*? For each link travel time distribution, we randomly generate 10,000 samples according to the corresponding specification. For simplicity, all link travel times are independent. When we have n links on the route, the route travel time equals the sum of n link travel times for each realization. The skewness of the route travel time distribution is shown in Figure 1. Note that a high value of skewness indicates that the distribution is asymmetric, and the normal distribution has a skewness of zero. One can see the route travel time distribution could be highly positive skew, particularly when n is less than 30. The route travel time variability cannot be generally and uniformly characterized as a normal distribution using the CLT.

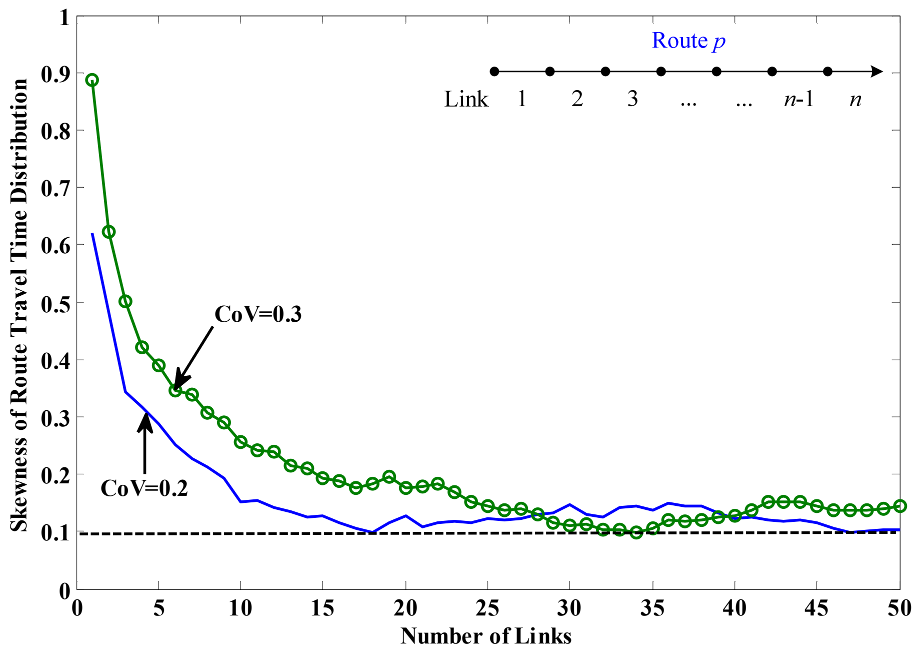


Figure 1 Skewness of route travel time distribution with different numbers of links

Note that the issues of using CLT have also been studied in the literature for different purposes, e.g., Ng and Waller (2010) of using the Fourier transform technique to numerically approximate the probability density function of the system-wide travel time; Nie (2011) of computing the percentile route travel times by convolutions; Castillo *et al*. (2013) of assuming a location-scale family for route travel times while avoiding the use of CLT and convolutions in their proposed percentile system optimal model rather than the user equilibrium model; and Castillo *et al*. (2014) of examining the probabilistic and physical consistency of traffic random variables and models (including models using normal distribution). Convolution is a potential flexible way of circumventing the use of CLT for obtaining route travel time distributions. Theoretically, route travel time distribution can be obtained by recursively convoluting its member links' travel time distributions under the assumption of independent link travel time distributions (Nie, 2011). However, it does not have a closed-form in general cases and has to be numerically evaluated, which is a computationally demanding process for large-scale networks. This paper does not attempt to directly resolve the issues of using CLT or convolution to approximate route travel time. Instead, we avoid the approximation of route travel time distribution by using the link-based modeling approach.

2.2 Observation 2: Sub-additivity of Mean-Excess Travel Time (METT)

The sub-additivity property of a risk measure is defined as follows:

$$\rho\left(X+Y\right)\leq\rho\left(X\right)+\rho\left(Y\right), \tag{1}$$

where X and Y are two random variables; $\rho(.)$ is a risk measure. The METT at the confidence level of α is defined as (Chen and Zhou, 2010)

$$\eta_a\left(\alpha\right)=E\left[T_a\middle|T_a\geq\xi_a\left(\alpha\right)\right], \tag{2}$$

where T_a is the random travel time of link a; $\xi_a(\alpha)$ is the α-percentile of T_a, i.e., the minimum threshold $\bar{\xi}$ that satisfies $\Pr\left(T_a\leq\bar{\xi}\right)\geq\alpha$. In general, Eq. (1) does not hold with equality since risk measures with reliability concerns (e.g., TTB, PTT and METT) are non-additive.

The definition of METT is a customization of the conditional value-at-risk (CVaR) measure (Rockafellar and Uryasev, 2002) in finance to the context of travel time variability. CVaR has been proved as a sub-additive risk measure whereas the widely used value-at-risk (VaR) measure, which is equivalent to PTT and TTB as a risk averse route choice criterion, does not necessarily satisfy the sub-additivity (Artzner *et al*., 1999; Acerbi and Tasche, 2002). An immediate consequence is that VaR might discourage risk diversification/reduction. Tan *et al*. (2014) mentioned that the risk-averse diversifiers accept more expected travel time reduction in exchange for the same amount of risk; while risk-averse plungers accept more risk in exchange for the same amount of expected travel time reduction. Therefore, CVaR has been recognized as a desirable alternative risk measure, and thus is more suitable for modeling flexible travel time distributions (including asymmetric and heavily tailed distributions).

Without loss of generality, below we consider a route p connected by link a and link b (T_p=T_a+T_b) to prove the sub-additivity of METT. Note that the proof follows Acerbi and Tasche (2002) for CVaR but with detailed intermediate process for the context of travel time distribution and METT.

Proposition 1. *The route-based METT is not larger than the sum of link-based METTs on the route under the same confidence level α.*

Proof. First of all, we define the following indicator function

$$\mathbf{1}_S = \begin{cases} 1, & s \in S \\ 0, & s \notin S \end{cases}. \tag{3}$$

For any confidence level α, consider the following relationship

$$\begin{aligned} & \eta_a(\alpha) + \eta_b(\alpha) - \eta_p(\alpha) \\ & = E\left[T_a \mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}}\right] + E\left[T_b \mathbf{1}_{\{T_b \ge \xi_b(\alpha)\}}\right] - E\left[(T_a + T_b)\mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right] \\ & = E\left[T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right)\right] + E\left[T_b\left(\mathbf{1}_{\{T_b \ge \xi_b(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right)\right]. \end{aligned} \tag{4}$$

Since the two expectation terms have the same structure, without loss of generality, we look at the first term.

$$\begin{aligned} & E\left[T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right)\right] \\ & = \int T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF \\ & = \int_{T_a \ge \xi_a(\alpha)} T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF + \int_{T_a < \xi_a(\alpha)} T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF . \end{aligned} \tag{5}$$

where F is the probability distribution of link travel time. Using Eq. (3), if $T_a \ge \xi_a(\alpha)$, then $\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} = 1$ and $\mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}$ is always between 0 and 1; if $T_a < \xi_a(\alpha)$, then $\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} = 0$ and $\mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}$ is always between 0 and 1. Then we have

$$\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}} \begin{cases} \ge 0, & \text{if } T_a \ge \xi_a(\alpha) \\ \le 0, & \text{if } T_a < \xi_a(\alpha) \end{cases}. \tag{6}$$

Multiplying with T_a, we have

$$T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) \begin{cases} \ge \xi_a(\alpha)\cdot\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right), & \text{if } T_a \ge \xi_a(\alpha) \\ \ge \xi_a(\alpha)\cdot\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right), & \text{if } T_a < \xi_a(\alpha) \end{cases}. \tag{7}$$

Then, Eq. (5) becomes

$$\begin{aligned} & \int_{T_a \ge \xi_a(\alpha)} T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF + \int_{T_a < \xi_a(\alpha)} T_a\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF \\ & \ge \int_{T_a \ge \xi_a(\alpha)} \xi_a(\alpha)\cdot\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF + \int_{T_a < \xi_a(\alpha)} \xi_a(\alpha)\cdot\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF \\ & = \int \xi_a(\alpha)\cdot\left(\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right) dF \\ & = \xi_a(\alpha)\cdot E\left[\mathbf{1}_{\{T_a \ge \xi_a(\alpha)\}} - \mathbf{1}_{\{T_p \ge \xi_p(\alpha)\}}\right] = 0 . \end{aligned} \tag{8}$$

In other words, both terms in the last equality of Eq. (4) are greater than or equal to zero. Finally, we have

$$\eta_a(\alpha)+\eta_b(\alpha)-\eta_p(\alpha)\geq 0\,. \tag{9}$$

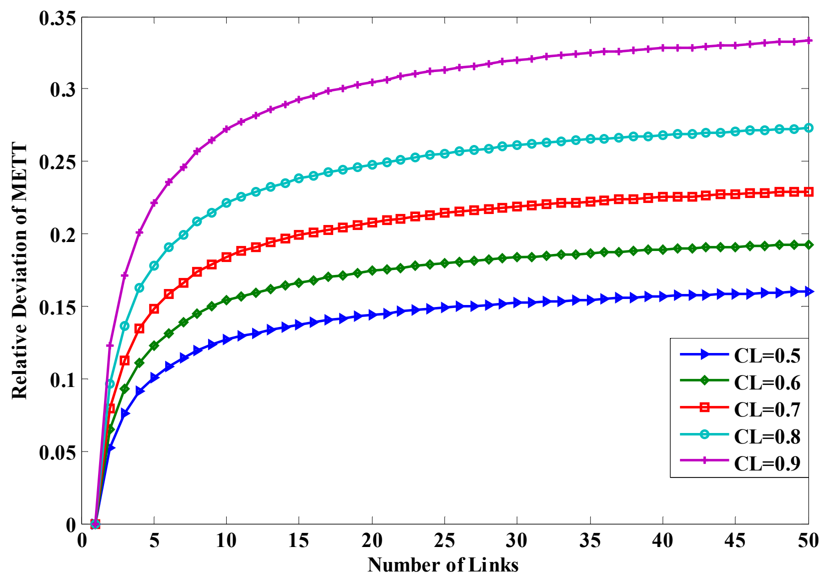
This completes the proof. □

The sub-additivity property of METT means that the *global risk* (i.e., route-based METT) will not be greater than the sum of its *partial risks* (i.e., link-based METTs). This property will be used to develop a link-based METT model in Section 3.

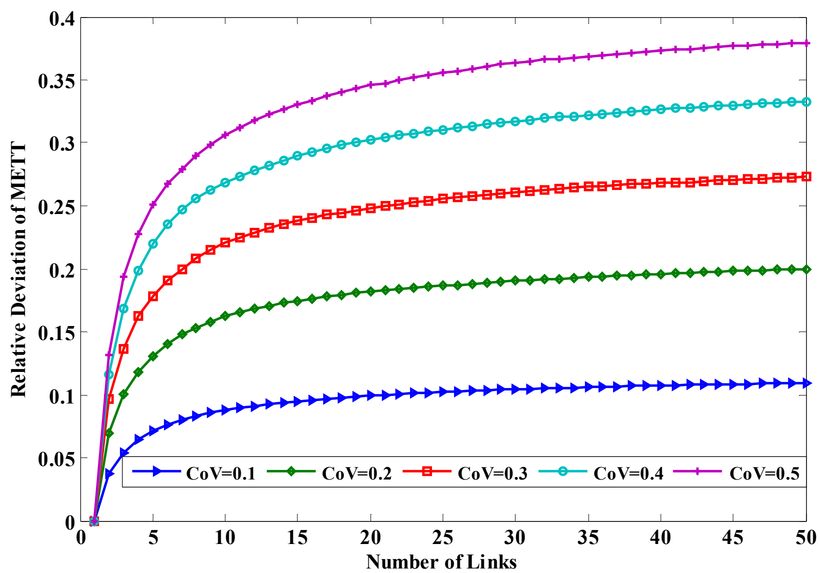
The sub-additivity of METT means the sum of *link*-based additive METTs on a route provides an upper bound for the *route*-based nonadditive METT. However, even with a known probability distribution of link travel times, the route travel time distribution is still unknown. Hence, it is difficult to analytically derive or bound their theoretical deviation due to the lack of a "true" route-based METT. Below we conduct simulation experiments to analyze the deviation between the route-based METT and the sum of link-based METTs, as well as the effect of the travelers' confidence level (CL) and the coefficient of variation (CV) of link travel times. Similar to Figure 1, all link travel times are assumed to be independent and follow the same lognormal distribution with the mean of 10 and the CV of 0.3. For each link, we randomly generate 10,000 samples according to the specification. The CL is set at α=0.8. We define the relative difference between the route-based METT and the sum of link-based METTs as follows. Figure 2 shows the relative difference with respect to the number of links on the route under various CLs and CVs.

$$\text{relative difference}=\left(\left(\sum\nolimits_{a\in A}\eta_a\right)-\eta_p\right)\Big/\sum\nolimits_{a\in A}\eta_a\,. \tag{10}$$

One can see that the relative difference is always positive, further verifying the sub-additivity of METT. For any CL and CV, the relative difference increases with the number of links on the route. However, the relative difference curve seems to have an upper bound after stabilization. A larger CL indicates that the travelers are more conservative when encountering travel time variability; a large CV corresponds to a more uncertain network. In these situations, using the sum of link-based METTs as the route choice risk measure will yield a more reliable (or risk averse) evaluation than the nonadditive route-based METT itself.



(a) Effect of confidence level (CL)



(b) Effect of coefficient of variation (CV)

Figure 2 Relative deviation of METT when using the sub-additivity property

3 A Link-Based Mean-Excess Traffic Equilibrium (L-METE) Model

In this section, we describe the definition of *link*-based METT (LMETT). Then, the *link*-based mean-excess traffic equilibrium (L-METE) is provided, followed by its link-route and node-link equivalent mathematical programming (MP) formulations. Consider a network G= [N, A], where N and A denote the sets of nodes and links, respectively. Let W denote the set of O-D pairs. From each O-D pair, there exists a travel demand q^w that can be accomplished by a set of routes (denoted by P^w) connecting O-D pair w.

3.1 Link-Based Mean-Excess Travel Time (LMETT)

Definition 1: *The* ***Link-based Mean-Excess Travel Time*** **(LMETT)** *$\eta_a(\alpha)$ on link a at the confidence level α is defined as the conditional expectation of link travel time T_a exceeding the corresponding link-based travel time budget (LTTB) $\xi_a(\alpha)$*, i.e.,

$$\eta_a(\alpha) = E\left[T_a \middle| T_a \geq \xi_a(\alpha)\right] = \frac{1}{1-\alpha}\int_{\xi_a(\alpha)}^{+\infty}\left(x \cdot f_{T_a}(x)\right)dx, \ \forall a \in A, \tag{11}$$

where T_a is the random travel time on link a, $f_{Ta}(x)$ is the probability density function (PDF) of T_a, and x is the realization of T_a; $\xi_a(\alpha)$ is the minimal threshold that satisfies the α-reliability requirement, i.e.,

$$\xi_a(\alpha) = \min\left\{\xi \middle| \Pr\left(T_a \leq \xi\right) \geq \alpha\right\} = E\left[T_a\right] + \gamma_a(\alpha) \ , \ \forall a \in A, \tag{12}$$

where $\gamma_a(\alpha)$ is a "buffer time" added to the mean link travel time $E[T_a]$ to ensure the travel time reliability requirement for on-time arrivals at the confidence level α.

For interpretation purposes, LMETT in Eq. (11) can be decomposed as (Chen and Zhou, 2010):

$$\underbrace{\eta_a(\alpha)}_{METT} = \underbrace{\xi_a(\alpha)}_{TTB} + \underbrace{E\left[\left(T_a - \xi_a(\alpha)\right)\middle| T_a \geq \xi_a(\alpha)\right]}_{EED}, \ \forall a \in A, \tag{13}$$

where the first and second terms represent the reliability (in terms of TTB) and unreliability (in terms of expected excess delay (EED)) aspects of link travel time variability, respectively. These two terms explicitly capture the left region of link travel time distribution with α reliability requirement and the right region with (1-α) unreliability in the link travel time distribution tail, respectively. In this sense, METT can capture the whole link travel time distribution and while TTB only considers the left region to the α-percentile (Chen and Zhou, 2010). In addition, from the risk assessment perspective, METT can characterize not only the risk consequence of late arrival (via the expectation of distribution tail) but also the occurrence probability of losses (e.g., on-time arrival with α and late arrival with 1-α).

3.2 Link-Based Mean-Excess Traffic Equilibrium (L-METE)

Using the sub-additivity of METT discussed in Section 2.2, we sum the *link*-based METTs (LMETTs) on the route to substitute the *originally nonadditive route*-based METT (RMETT). With this, we have the following additive route cost structure:

$$\tilde{\eta}_p^w = \sum_{a \in A} \eta_a \delta_{ap}^w, \ \forall p \in P^w, w \in W, \tag{14}$$

where δ_{pa}^w is the link-route incidence indicator: it equals 1 if route p connecting O-D pair w

uses link a, and 0 otherwise. We use $\tilde{\eta}_p^w$ to denote the route-based METT with the above additive cost structure, while η_p^w denotes the nonadditive route-based METT (i.e., the conditional expectation of route travel time).

Here we model the travelers' perception of risk/disutility at the link level (i.e., LMETT, a conditional expectation of *link* travel time distribution), rather than perceiving risk/disutility at the route level (i.e., imposing a perception 'operator' on the *route* travel time distribution directly), which has the features of adaptive models. The modeling approach of link-level risk/disutility has been used in the adaptive routing problem, despite that our study uses it in the static UE problem. In the L-METE model, travelers perceive link travel disutility (i.e., via the LMETT that explicitly considers both travel time uncertainty and users' confidence level) link by link from the origin to the destination. In a broader research picture, non-adaptive routing or traffic assignment models can also be solved by link-based algorithms. For example, Dial's STOCH loading method (Dial, 1971), which is also based on a Markov process, can be used as a multi-path assignment scheme to circumvent route enumeration/storing for calculating the logit-based route choice probability with a definition of reasonable path. Recently, Fosgerau *et al*. (2013) developed a recursive logit (RL) model, which models route choice as a sequence of link choices using a dynamic discrete choice framework. The RL model was shown to be equivalent to a static MNL model over the set of all feasible routes. Our L-METE model can be viewed as a multi-path traffic assignment scheme that considers congestion, network uncertainty, and risk-averse behavior along with the additive route cost structure.

Additive risk measures of travel time uncertainty have been proposed in various travel choice contexts. In the context of *route choice*, Mirchandani and Soroush (1987) and Tatineni *et al*. (1997) used exponential disutility functions to model risk-taking behavior. Due to the tractable feature of exponential disutility functions, the disutility associated with a route can be calculated by summing the link disutilities on this route. This is beneficial for algorithmic development, since it allows the use of classical shortest path algorithm in finding the minimum expected perceived disutility route. Our paper uses the additive LMETE (i.e., the sum of LMETTs) as travelers' risk-averse route choice measure under travel time uncertainty. In the context of *departure time choice*, Engelson and Fosgerau (2011) proposed an additive measure of expected travel time cost of travel time variability for travelers equipped with scheduling preferences given in terms of time-varying utility rates and optimally chosen departure time. Despite from different modeling perspectives (e.g., reliability analysis and utility theory) and choice contexts, a common motivation of developing additive measures is for the convenience of network modeling, e.g., shortest path algorithms are essentially based on the assumption of additivity (Engelson and Fosgerau, 2011). In addition, the LMETT (i.e., conditional expectation of link travel time distribution) can also be considered as a link-based disutility or a scalar transformation of link travel time distribution.

In this paper, assuming additivity is mainly due to the sub-additivity property of METT, which allows us to develop a link-based METE model by bounding the route cost from above. In other words, the sum of link METTs is not less than the route METT under the same confidence level. However, this modeling may not be necessarily applicable to all route-based traffic equilibrium models under uncertainty. Particularly, the widely used TTB model does not have this sub-additivity property as mentioned in the second paragraph of Section 2.2; hence, it may not be meaningful to assume additivity in the TTB model. The link-based model or additive assumption can be applied to certain models that possess good properties (e.g., sub-additivity of the METT model) to overcome some limitations of the route-based or nonadditive models (e.g., normal route travel time distribution inherited from the CLT).

Traffic equilibrium model under uncertainty explicitly characterizes travelers' individual route choice behavior under travel time variability, as well as the aggregate equilibrium pattern under congestion effect. With the above risk-averse route choice criterion in Eq. (14), we can define the following if-then traffic equilibrium condition:

$$\tilde{\eta}_p^w\left(\mathbf{f}^*\right)\begin{cases}=\tilde{u}^w, \text{if } f_p^{w*}>0\\ \geq \tilde{u}^w, \text{if } f_p^{w*}=0\end{cases}, \quad \forall p\in P^w, w\in W, \tag{15}$$

where $\tilde{u}^w$ is the minimum RMETT between O-D pair w, i.e., $\tilde{u}^w=\min\left\{\tilde{\eta}_p^w, p\in P^w\right\}$; f_p^w is the flow on route *p* between O-D pair *w*. The link-based L-METE state is reached by allocating O-D demands to the network such that for each O-D pair, all used routes have equal and minimum RMETT. The final outcome is that no traveler can improve his/her RMETT by unilaterally changing his/her route choice. Note that the L-METE model has an additive route cost structure (i.e., Eq. (14)), while the R-METE model developed by Chen and Zhou (2010) does not hold this property since the conditional expectation of route travel time is nonadditive. To construct equivalent MP formulations, we further assume separable link travel time functions, e.g., the commonly used Bureau of Public Roads (BPR) function. This assumption is the same as that in the well-known Beckmann transformation (Beckmann *et al.*, 1956). We use $\eta_a(\alpha)$ and $\eta_b(\alpha)$ to denote the LMETT of link *a* and link *b*, respectively. METT holds the *monotonicity* property. In other words, if $T_a \le T_b$ (i.e., link *a* always has a shorter travel time than link *b* under almost all scenarios), then $\eta_a(\alpha) \le \eta_b(\alpha)$ under the same confidence level α. LMETT is increasing with respect to link travel time. For a strictly increasing link travel time function with respect to link flow (e.g., the BPR function), LMETT (i.e., a composite function) is also an increasing function with respect to its link flow. In addition, LMETT is monotonically increasing with respect to confidence level α, since its first-order derivative is positive as shown below.

$$\begin{aligned}\frac{d\eta_b(\alpha)}{d\alpha} &= \frac{d}{d\alpha}\left(\frac{1}{1-\alpha}\int_\alpha^1 \xi_b(\tau)\,d\tau\right)\\ &= \frac{1}{(1-\alpha)^2}\int_\alpha^1 \xi_b(\tau)\,d\tau - \frac{1}{1-\alpha}\xi_b(\alpha)\\ &= \frac{1}{1-\alpha}\left(\frac{1}{1-\alpha}\int_\alpha^1 \xi_b(\tau)\,d\tau - \xi_b(\alpha)\right) = \frac{1}{1-\alpha}\left(\eta_b(\alpha) - \xi_b(\alpha)\right) > 0,\end{aligned} \tag{16}$$

where the first equality makes use of the relationship between TTB and METT, and the inequality makes use of the fact that METT is larger than TTB (see Eq. (13)).

3.3 Two Mathematical Programming Formulations

The 'if-then' condition in Eq. (15) can be formally written as the following link-route or node-link type MP formulations for different modeling purposes.

(1) Link-Route Formulation

The L-METE can be equivalently formulated as follows:

$$\min Z(\mathbf{v}) = \sum_{a\in A}\int_0^{v_a} \eta_a(x,\alpha)\,dx, \tag{17}$$

subject to:

$$\sum_{p\in P^w} f_p^w = q^w,\ \forall w \in W, \tag{18}$$

$$v_a = \sum_{w\in W}\sum_{p\in P^w} f_p^w \delta_{pa}^w,\ \forall a \in A, \tag{19}$$

$$f_p^w \ge 0,\ \forall p \in P^w, w \in W, \tag{20}$$

where $\eta_a(x,\alpha)$ is the integrand, denoting the LMETT of link *a* under the flow pattern *x* at the confidence level α; v_a is the flow on link *a* and $\mathbf{v}$ is its vector form; q^w is the travel demand of O-D pair *w*. Eq. (18) is the travel demand conservation constraint that sums up all route flows between an O-D pair; Eq. (19) is a definitional constraint that sums up all route flows that pass through a given link; and Eq. (20) is a non-negativity constraint on the route flows. The feasible set is a convex set. Also, it is easy to verify that the solution to the above MP formulation is equivalent to the *link*-based METE conditions in Eq. (15) by deriving the Karush-Kuhn-Tucker (KKT) conditions.

(2) Node-Link Formulation

Alternatively, we can also formulate the L-METE problem as a node-link MP. The objective function in the node-link formulation is the same as that in the link-route formulation. However, the decision variables are different. The node-link formulation uses origin-specific link flows as the decision variables. Accordingly, the conservation constraints are origin-specific.

$$\min Z(\mathbf{v}) = \sum_{a \in A} \int_0^{v_a} \eta_a(x, \alpha)\, dx, \tag{21}$$

subject to:

$$\sum_{a \in O(n)} v_a^i - \sum_{a \in I(n)} v_a^i = D_{in}, \quad \forall\, n \in N,\ i \in I, \tag{22}$$

$$v_a = \sum_{i \in I} v_a^i, \ \forall\, a \in A, \tag{23}$$

$$v_a^i \geq 0, \ \forall\, a \in A,\ i \in I, \tag{24}$$

where I and J are the sets of origins and destinations, respectively; $O(n)$ is the set of links emanating from node n; $I(n)$ is the set of links going into node n; v_a^i is the flow generating from origin i and assigning to link a; D_{in} is the demand from origin i to node n: if n=i, D_{in}=$\sum_{j \in J} q_{ij}$; if n=j, D_{in}=-q_{ij}; otherwise, D_{in}=0. Note that the solution of the node-link formulation is v_a^i, and its dimension is $|A|\times|I|$, which is generally much smaller than that in the link-route formulation. Hence, the node-link formulation may be beneficial to be used in the bi-level programming problem as the lower-level subprogram (e.g., Ban *et al*., 2006).

Remark 1: The L-METE model structure is similar to the Beckmann transformation of the UE problem (Beckmann *et al*., 1956). When we replace the LMETT function in Eq. (17) by a deterministic link travel time function, the L-METE model becomes the classical UE model. Thus, we are able to adapt existing UE solution algorithms (e.g., the well-known Frank-Wolfe algorithm) for solving the L-METE model. By substituting Eq. (11) into Eq. (17), the objective function has two layers of integral, making it more complicated than the UE objective function (see Figure 3 for an illustration). Similar to the Beckmann transformation, the objective function herein has no intuitive economic interpretation, which is used only for constructing the equivalent MP formulation. Note that the integrand in the UE objective function is a deterministic function with respect to deterministic link flow. The integrand in the L-METE objective function is not a random function; instead it is a deterministic mapping/cost with the consideration of travel time uncertainty (via the travel time distribution of T_a) and travelers' risk-aversion attitude (via confidence level α). Similar to the classical UE problem, the L-METE link flow pattern is unique under some common assumptions (e.g., separable and strictly increasing link travel time functions). However, the route flow pattern is not unique in general. How to select a reasonable route flow pattern from the set of equilibrium solutions will be a valuable research direction in the future.

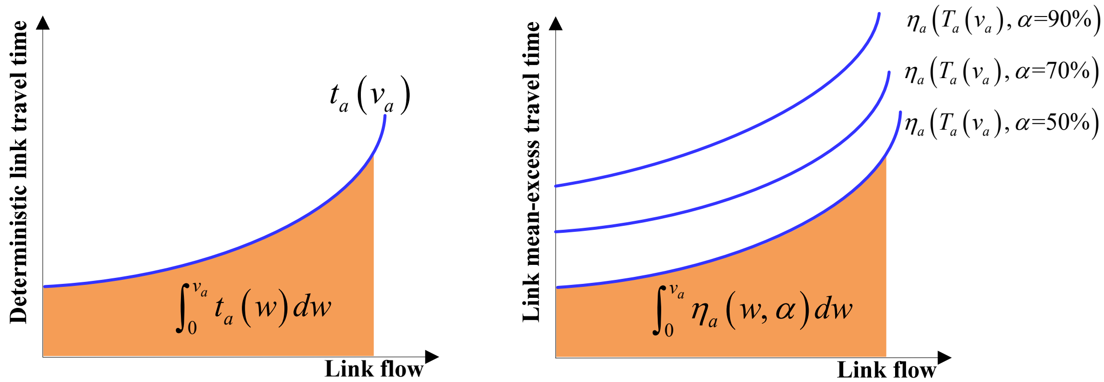


Figure 3 Illustration of UE and L-METE objective functions

In the following, we summarize the main differences between the *link*-based and *route*-based METE models as well as the benchmark of the classical UE model in Table 1.

Table 1 Comparison between the *link*-based and *route*-based METE models

Model	**UE**	**L-METE**	**R-METE**
Uncertainty consideration	No	Reliability & unreliability	Reliability & unreliability
Route choice criterion	Route expected travel time	Sum of link-based METTs	Route-based METT (conditional expectation of route travel time)
Route cost structure	Additive	Additive	Nonadditive
Mathematical formulation	Mathematical programming	Mathematical programming	Variational inequality
Solution algorithm	Can solve without storing/enumerating routes	Can solve without storing/enumerating routes	Cannot solve without storing/enumerating routes; need a non-additive shortest path algorithm

Remark 2 In the literature, few TEPs under uncertainty were also constructed in the link domain. For example, Mirchandani and Soroush (1987) used exponential disutility functions

to model travelers' risk-taking behavior. Due to the tractable feature of exponential disutility functions, the disutility associated with a route can be represented by summing up the link disutilities on this route. This is beneficial for algorithmic development. Also, Szeto *et al*. (2009) developed a *link*-based risk-averse game theoretic traffic assignment model with multiple network-specific demons. This model avoids route enumeration/generation of route-based models and allows one to consider the effect of greater network uncertainty for capacity degradation. In general, these models rely on a certain type of utility function (e.g., exponential function) to realize the decomposition of route utility to link utility. The selection of utility functional forms and the calibration of utility parameters are the main challenges in these applications. In this paper, the proposed L-METE model makes use of the sub-additivity of METT to realize the decomposition, and the risk measure of METT has a complete uncertainty characterization (i.e., both on-time arrival and late arrival), a risk preference control mechanism, and a practical implementation strategy (i.e., via a reliability requirement between 0 and 1).

4 Solution Algorithm

The existing algorithms for the conventional UE problem in the planning software packages can be readily adapted to solve the L-METE model while avoiding the need to solve the nonadditive shortest path problem in route-based models. The additive route cost structure makes the L-METE model solvable without storing/enumerating routes, which is a significant benefit for large-scale network applications. The additive route cost structure also enables the L-METE model to be potentially solved by different efficient traffic assignment algorithms. For demonstration purposes, we customize the widely used Frank-Wolfe (F-W) algorithm (LeBlanc *et al*., 1975) of the UE problem for solving the L-METE model. The F-W algorithm is iteratively performed between the search direction step (or all-or-nothing loading) and the line search step until some convergence criterion is reached. In the section, we only highlight the main adaptation for solving the L-METE model compared to the UE model. For the detailed implementation of F-W algorithm, interested readers may refer to the above reference as well as Chen (2001), Chen *et al*. (2002a; 2013), and Lee *et al*. (2002) for its extensions.

In the deterministic traffic assignment problems (e.g., UE), the deterministic link travel times are directly calculated using the link performance function (e.g., the BPR function) with the deterministic link flows. However, the travel time variability consideration makes the LMETT much more complex. The main difference is the link cost (i.e., LMETT) updating process. Note that the LMETT updating process explicitly captures the uncertainty propagation from the source of uncertainty (e.g., demand or supply uncertainty) to the link travel time variability as well as travelers' risk-aversion strategy toward travel time variability.

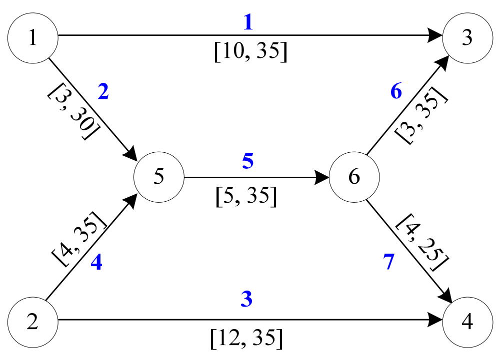
The *link*-based METE model presented in Section 3 is general in the sense that the modeling starts from the link travel time variability without specifying the uncertainty sources. It is well known that day-to-day demand fluctuation and link capacity degradation are two main uncertainty sources in transportation systems. Also, most of the recent studies considered either demand uncertainty or capacity uncertainty or both in the traffic equilibrium models under uncertainty. In the numerical examples of Section 5, we consider the travel time variability induced by day-to-day demand fluctuation as a representative uncertainty source. The lognormal distribution is adopted to characterize travel demand and link travel time uncertainties. It is a nonnegative, asymmetric distribution, and it has been adopted as a more realistic approximation of the fluctuated demand to investigate the uncertainty propagation in the four-step demand forecasting procedure (Zhao and Kockelman 2002). Furthermore, recent empirical studies (Emam and Al-Deek, 2006; Kaparias *et al*., 2008; Rakha *et al*., 2010; Arezoumandi, 2011; Chen *et al*., 2014) have verified its usefulness in characterizing travel time variability. With the modeling of uncertainty propagation (travel demand-route flow-link flow) and the lognormal distribution characterization of link travel times, we are able to analytically derive the LMETT as shown in the Appendix. We should point out that other distributions with empirical support could also be used in the analytical derivation. As to the capacity degradation case, the link flow is deterministic. We can derive the generic moment of link travel time under the link capacity distribution specification (e.g., uniform distribution used in Lo *et al*., 2006). With the link travel time moments, we can either fit the link travel time distributions to analytically derive the LMETT, or directly approximate the LMETT without knowing the complete PDF of link travel time (see Xu *et al*., 2014a for the details).

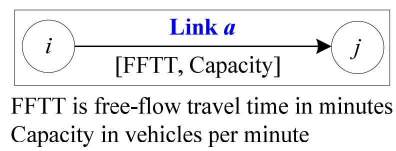
5 NUMERICAL EXAMPLES

In this section, three numerical examples are provided to demonstrate the features of the proposed *link*-based METE model. Specifically, (1) Network 1 verifies the solution correctness and compares the UE, L-METE, and R-METE models; (2) Network 2 uses the well-known Sioux Falls network to compare the above three models in both aggregate and disaggregate manners; and (3) Network 3 uses the ADVANCE network to examine the applicability of the L-METE model in realistic networks.

5.1 Illustrative Example

We use a small network, shown in Figure 4, to illustrate the features of the *link*-based METE model. This network consists of six nodes, seven links, and two O-D pairs. We adopt the BPR function with the parameters of 0.15 and 4. The random O-D demands are assumed to follow the lognormal distribution with the mean of 60 for O-D pair (1, 3) and 50 for O-D pair (2, 4), and the variance-to-mean (VMR) ratio of 2.0 for both O-D pairs, corresponding to the CV of 0.18 and 0.20. We further assume the confidence level of all travelers is 80%.





O-D pair	Route	Sequence of links
O-D (1-3)	1	1
	2	2-5-6
O-D (2-4)	3	4-5-7
	4	3

Figure 4 Illustrative example network

(1) Equilibrium Results

To verify the equilibrium assignment results in Figure 5, we check the METT-based equilibrium conditions and the conservation constraints. One can see, for each O-D pair, all used routes have equal cost (i.e., the corresponding summation of link-based METTs). In addition, the equilibrium route flows satisfy the conservation constraints.

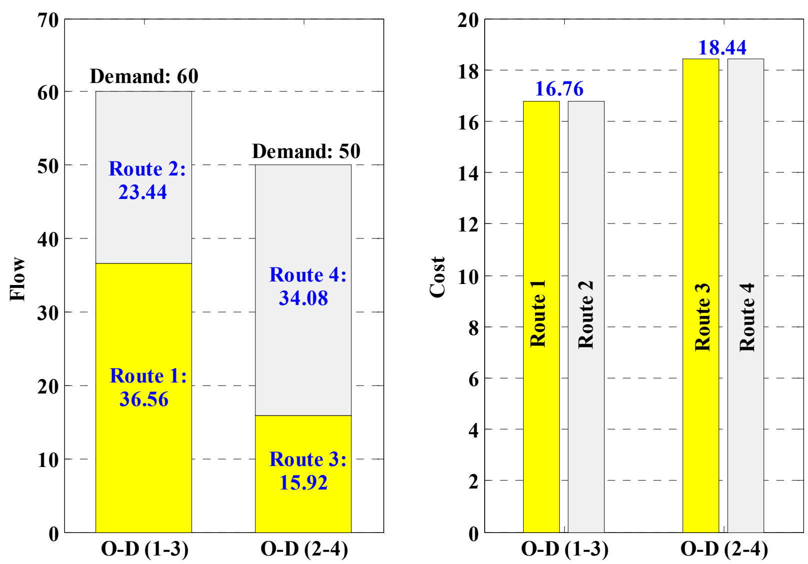


Figure 5 Assignment results of the L-METE model

Recall that the route choice criterion of the L-METE model is additive, since it is the summation of LMETTs. From the left panel of Figure 6, the cost of route 2 is equal to the summation of LMETTs on links 2, 5, and 6, which also equals the cost of route 1 (i.e., LMETT on link 1). Similar relationship also occurs for routes 3 and 4 between O-D pair (2-4). The right panel of Figure 6 shows the nonadditive route-based METT calculated using the CLT and normal route travel time distribution under the L-METE flow pattern. In other words, both panels of Figure 6 are associated with the same L-METE flow pattern. One can see that the sub-additivity of METT makes the summation of LMETTs larger than the route-based METT (i.e., conditional expectation of normally distributed route travel time).

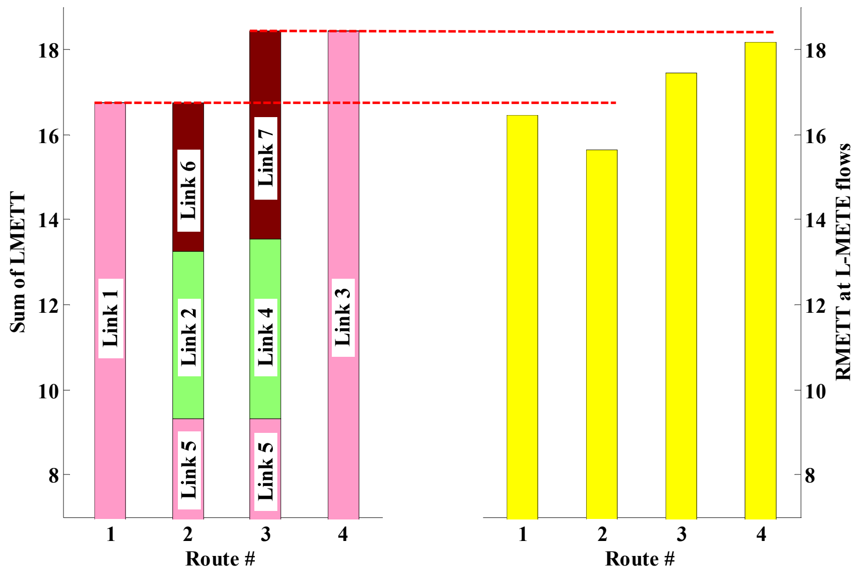


Figure 6 Comparison between RMETT and sum of LMETTs

(2) Comparative Analysis among L-METE, R-METE, and UE Models

Below we compare the equilibrium assignment results from two viewpoints: (a) the link-based METE model versus route-based METE model as shown in Table 2, and (b) the traditional UE model without uncertainty consideration versus and the link-based METE model as shown in Table 3. One can see that the difference between the L-METE and R-METE model is less than the difference between the L-METE and UE model. The large route cost difference between the L-METE and UE models is due to the fact that the L-METE model explicitly considers the effect of travel time variability on travelers' route choice decisions, while the UE model ignores this effect by using the deterministic travel time under the mean flow pattern as the route choice criterion. Different travel time variability considerations may lead to significantly different route choice decisions. On the other hand, both the L-METE and R-METE models capture the travelers' on-time arrival reliability requirement and the late arrival unreliability consequence. Their main difference lies in the route cost structure: additive versus nonadditive.

Table 2 Comparison between the link-based and route-based METE models

Route	Route Flow			Route Cost		
	L-METE	R-METE	Difference	L-METE	R-METE	Difference
1	36.56	35.91	0.65	16.76	16.08	0.68
2	23.44	24.09	-0.65	16.76	16.08	0.68
3	15.92	16.42	-0.50	18.44	17.89	0.55
4	34.08	33.58	0.50	18.44	17.89	0.55

Table 3 Comparison between the UE and link-based METE models

Route	Route Flow			Route Cost		
	L-METE	UE	Difference	L-METE	UE	Difference
1	36.56	37.95	-1.39	16.76	12.07	**4.69**
2	23.44	22.05	1.39	16.76	12.07	**4.69**
3	15.92	14.28	1.64	18.44	13.95	**4.49**
4	34.08	35.72	-1.64	18.44	13.95	**4.49**

In this example, all four routes are used in all three models (i.e., the same used route set). Hence, we are able to provide a more intuitive interpretation of the resulting route flow patterns. Routes 1 and 4 are respectively the shortest length route of O-D pairs (1, 3) and (2, 4) as shown in Figure 4. The UE model assigns larger flows on these two routes. Due to the uncertainty consideration (including reliability and unreliability aspects of travel time uncertainty) in both L-METE and R-METE models, their corresponding route costs become larger, which are further away from the minimum UE routes. Hence, compared to the UE model, both L-METE and R-METE models assign relatively lower flows to routes 1 and 4. Between the L-METE and R-METE models, the sub-additivity used in the L-METE model results in a greater enlarging effect to route costs. Routes 1 and 4 both consists of a single link, while routes 2 and 3 both consists of three links. Due to the sub-additivity, the relative difference between the sum of link METTs and route METT increases with the number of links on the route (see Figure 2). Accordingly, routes 2 and 3 become less attractive with lower assigned flows in the L-METE model.

5.2 Sioux Falls Network

In this section, we use the well-known Sioux Falls network to compare the above three models in both aggregate and disaggregate manners. The Sioux Falls network consists of 24 nodes, 76 links and 550 O-D pairs. The O-D demands are assumed to follow the lognormal distribution with the variance-to-mean ratio (VMR) of 0.30. All travelers have the same reliability requirement of 80%. A main purpose of using this network is that a behaviorally generated working route set (Bekhor *et al*., 2008) is available for providing a fair comparison of different models. In this route set, the total number of routes is 3,441, the maximum number of routes for any O-D pair is 13, and the average number of routes is 6.3 per O-D pair. This route set has also been used in the route-based METE model (Chen and Zhou, 2010). We should point out when examining a more general network, it is much more complicated to compare the three models with different route choice criteria (i.e., additive travel time of UE, additive sum of LMETTs, and non-additive RMETT). In general, not all feasible routes are used at equilibrium. Due to the different route choice criteria, the three models are unlikely to produce the same route flow pattern. Also, the different route cost structures (additive vs. non-additive) further require different algorithms to solve the models.

(1) Composition of Used Routes

At the UE flow pattern, the number of used routes is 703, i.e., the average number of used routes is 1.28 per O-D pair. From Figure 7, about 93% of the 703 used routes are composed of less than 5 links, and the largest number of links on a used route is 7. By analyzing the skewness of route travel time distribution with respect to number of links on a route (see Figure 1), Figure 7 further shows the inapplicability of using the CLT to uniformly approximate route travel time distribution as a normal distribution.

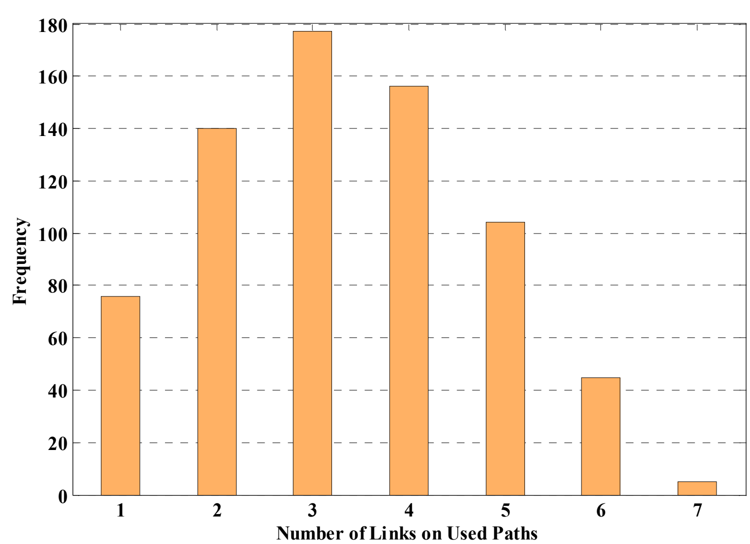


Figure 7 Distribution of number of links on used paths

What if we use CLT to uniformly approximate route travel time distribution? First of all, we examine the difference of route costs. Figure 8 shows the relative deviation between the route-based METT (RMETT) and the sum of link-based METTs. Note that both RMETT and LMETT are evaluated at the same L-METE flow pattern. Particularly RMETT is calculated using the conditional expectation of normal route travel time distribution. From this figure, the relative deviation is always positive and the sum of LMETTs is always larger than the RMETT due to its sub-additivity. For most routes, the relative deviation is between 15% and 25%. Hence, the biased effect of uniformly using CLT should not be overlooked. In the following subsection, we further examine the difference of L-METE and R-METE models in terms of link flows, as well as the effects of uncertainty level and risk-aversion attitude.

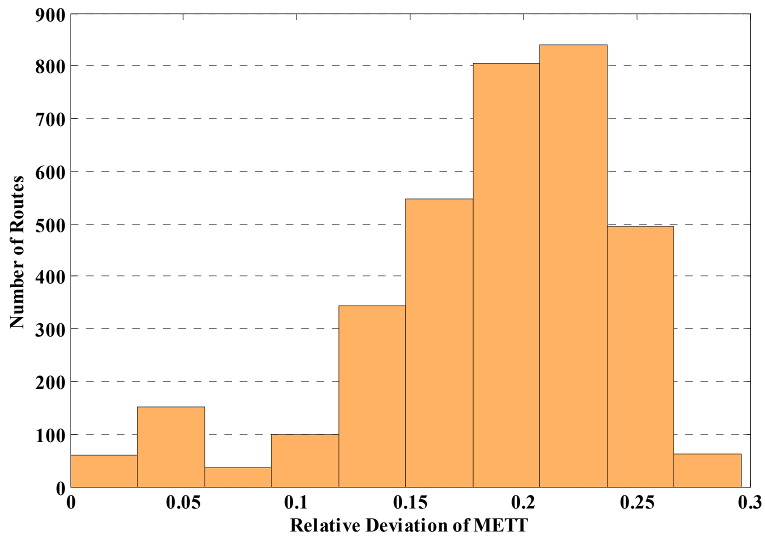


Figure 8 Comparison between route-based METT and sum of link-based METT (Sioux Falls)

(2) Comparison of Link Flows among Three Models

Figure 9 shows the scatter plot of link flows from the UE, R-METE and L-METE models. If two models have a close/similar link flow pattern, their scatter distributions should be close to the 45-degree line. From the figure, one can see that the UE and L-METE models have the largest difference of link flows, followed by the difference between the UE and R-METE models, while the L-METE and R-METE have the smallest difference.

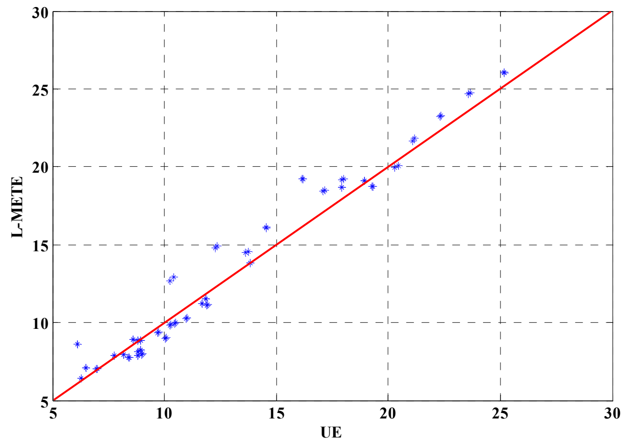
From Figure 9, it appears that the link flow differences of the three models are not significant. This visual ‘indifference’ is probably caused by the characteristics of this particular network. The Sioux Falls network with the above data setting is highly congested with the average volume-to-capacity (V/C) ratio being larger than 1.5, partially lessening the model difference. Similar phenomena also occur on the diminishing difference between UE and SUE with the increase of demand level. Another possible reason is that the route set is established by generating significantly different routes with large cost differences, and the average number of used routes is only 1.28 per O-D pair at the UE flow pattern. This may result in similar route flow allocations for each O-D pair among different models.

For a further quantification, the relative deviation of link flows is defined as follows:

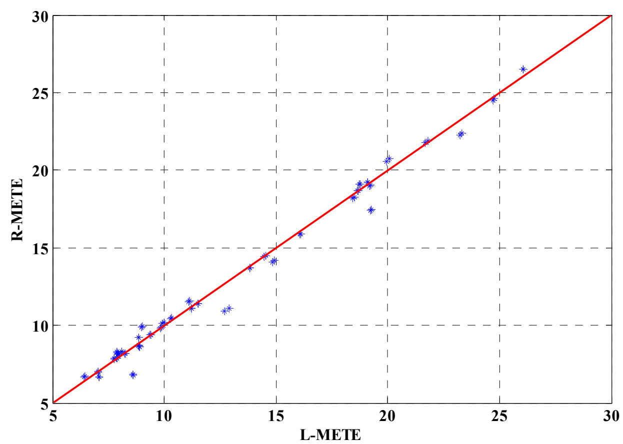
$$\left|v_a^{\text{L-METE}} - v_a^{\text{UE}}\right| \Big/ v_a^{\text{L-METE}}, \quad \left|v_a^{\text{L-METE}} - v_a^{\text{R-METE}}\right| \Big/ v_a^{\text{L-METE}}. \tag{25}$$

Figure 10 shows that the relative deviation between the L-METE and R-METE models is much smaller than that between the UE and L-METE models. For example, about 85% of the links have less than 5% relative deviation between the L-METE and R-METE models; whereas only 50% of the links have less than 5% relative deviation between the UE and L-METE models. Also, the Kolmogorov–Smirnov test further verifies that the link flow patterns from the three models follow different distributions with the significance of 0.05.

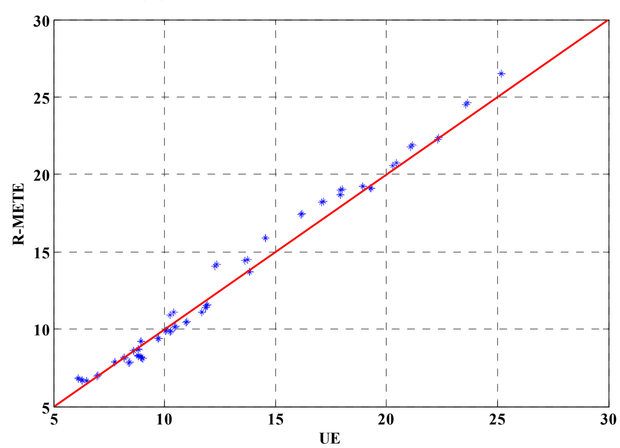
The above observations could be explained as follows. The UE model assumes travelers are risk-neutral and ignores travel time variability by using the deterministic travel time under the mean flow pattern as the route choice criterion. The L-METE and R-METE models explicitly consider travel time variability and traveler’s risk-aversion attitude. It is the travel time variability consideration that makes the L-METE and R-METE models (e.g., their minimum cost routes) more different from the UE model, while the similar travel time variability consideration via the risk measure of METT makes the difference between L-METE and R-METE models much smaller. Their main difference is the route cost structure (i.e., the additive sum of LMETTs in the L-METE model and the route-based nonadditive METT in the R-METT model). Due to the sub-additivity of METT, the sum of LMETTs provides an upper bound of the RMETT. From this perspective, the L-METE model could be considered as a more reliable (or risk averse) traffic equilibrium model than the R-METE model. Relative to the R-METE model, the enlarging effect of sub-additivity of the L-METE model makes the minimum cost routes further deviated from the minimum UE routes. Consequently, the link flow difference between UE and L-METE models is larger than that between UE and R-METE models.



(a) UE versus **L**-METE



(b) **L**-METE versus **R**-METE



(c) UE versus **R**-METE

Figure 9 Scatter plot of link flows from different equilibrium models

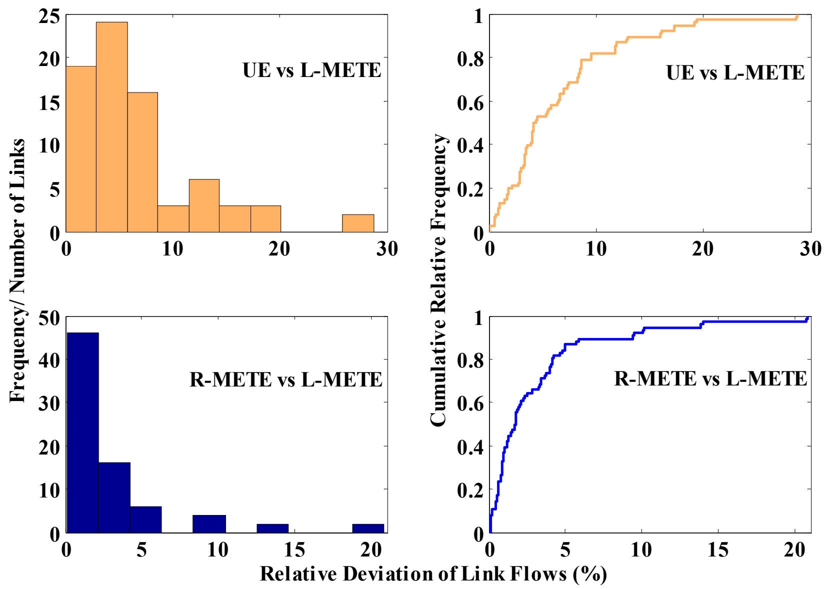


Figure 10 Relative deviation of link flows from different equilibrium models

(3) Effects of Uncertainty Degree and Risk-Aversion Attitude

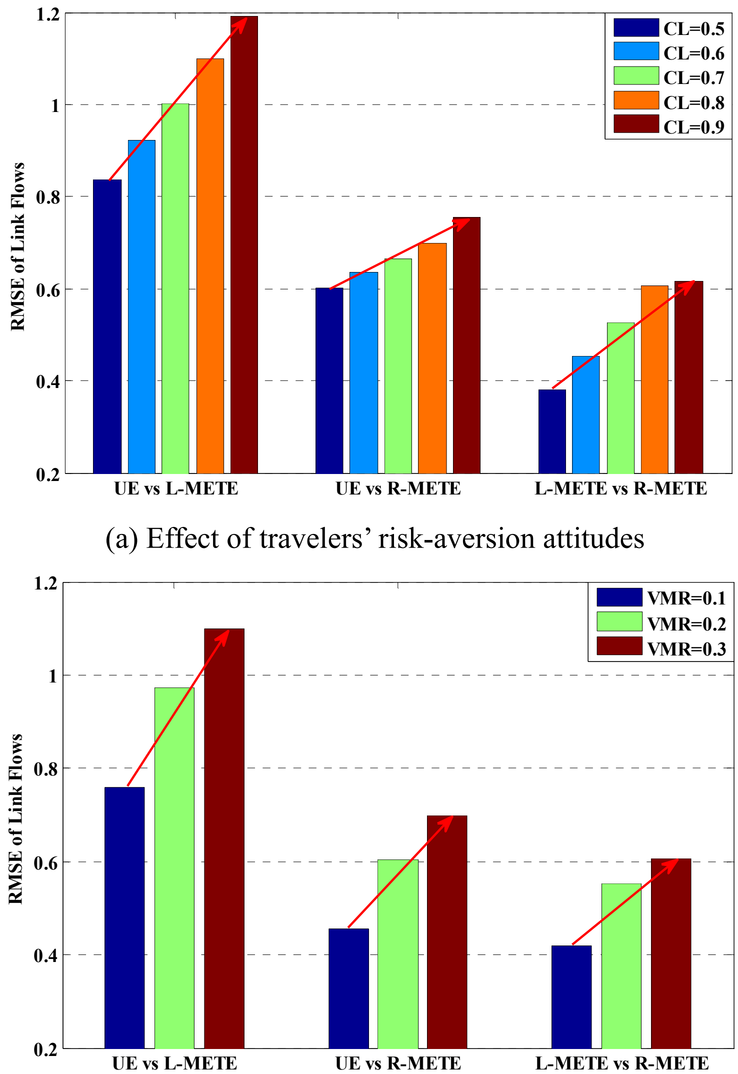
Travel demand fluctuation and travelers' risk-averse route choice behavior are critical characteristics of transportation systems. The uncertainty degree of travel demand can be characterized by the variance-to-mean ratio (VMR), and the travelers' risk-aversion attitude can be characterized by the confidence level (CL) or on-time arrival reliability requirement. Below we analyze the effects of VMR and CL on the model differences in terms of root mean square error (RMSE) as defined below.

$$\sqrt{\frac{1}{|A|}\left(v_a^{\text{L-METE}}-v_a^{\text{UE}}\right)^2},\ \sqrt{\frac{1}{|A|}\left(v_a^{\text{R-METE}}-v_a^{\text{UE}}\right)^2},\ \sqrt{\frac{1}{|A|}\left(v_a^{\text{L-METE}}-v_a^{\text{R-METE}}\right)^2}, \qquad (26)$$

where $|A|$ is the total number of links in the network. Note that the UE could be treated as a benchmark since it ignores the travel time variability and travelers' risk-aversion attitude in route choice decisions.

Figure 11 shows that for each VMR and CL, the difference between UE and L-METE models is larger than that between UE and R-METE models, and even larger than that between L-METE and R-METE models. This relationship is consistent with Figure 9 and Figure 10. In addition, the link flow difference of each two models gets larger as the CL and VMR increase. Especially, the difference between UE and L-METE models becomes more significant. Travelers with a larges CL are generally more conservative facing travel time variability and pursue a higher reliability of on-time arrivals. A larger uncertainty degree corresponds to a more fluctuated travel demand pattern, and the network tends to be more uncertain.

Accordingly, with the increase of CL and VMR, travelers may budget a larger buffer time and tardy time besides the mean travel time, for guaranteeing a higher reliability requirement and reducing the late arrival delay. Therefore, the difference between UE and L-METE/R-METE models are getting larger, and the effect of ignoring travel time variability becomes more significant. Similarly, the difference between L-METE and R-METE models also becomes larger with the increase of CL and VMR, which is consistent with the difference between RMETT and sum of LMETTs on a single route as shown in Figure 2.



(a) Effect of travelers' risk-aversion attitudes

(b) Effect of demand uncertainty levels

Figure 11 Effects of travelers' risk-aversion attitudes and demand uncertainty levels

5.3 Real Case Study: ADVANCE Network

In this section, we use the ADVANCE network in Chicago to examine the applicability of the proposed L-METE model in realistic networks. This network was extracted from the

ADVANCE (advanced driver and vehicle advisory navigation concept) program. It is located in the northwestern suburbs of Chicago and covers about 330 miles^2 (800 km^2). As shown in Figure 12, the topology is a regular grid with a few diagonal major arterials directed toward the Chicago central business district. The network includes 2,552 nodes, 7,850 links, 447 traffic analysis zones, and 137,417 O-D pairs. Note that a few studies in the literature also used this network to investigate traffic assignment algorithms for solving the UE problem (e.g., Chen, 2001; Chen *et al*., 2002c, and Lee *et al*., 2003). We continue to use the lognormal distribution to characterize travel demand uncertainty. For simplicity, all O-D pairs are assumed to have the VMR of 0.50, and the travelers' reliability requirement is 80%. In this section, we only compare the computational performance of the F-W algorithm for solving the two additive models (i.e., UE and L-METE models) and focus on the additional computational efforts required for solving the L-METE model; while the route-based METE model has a nonadditive route cost structure, rendering its burdensome computation.

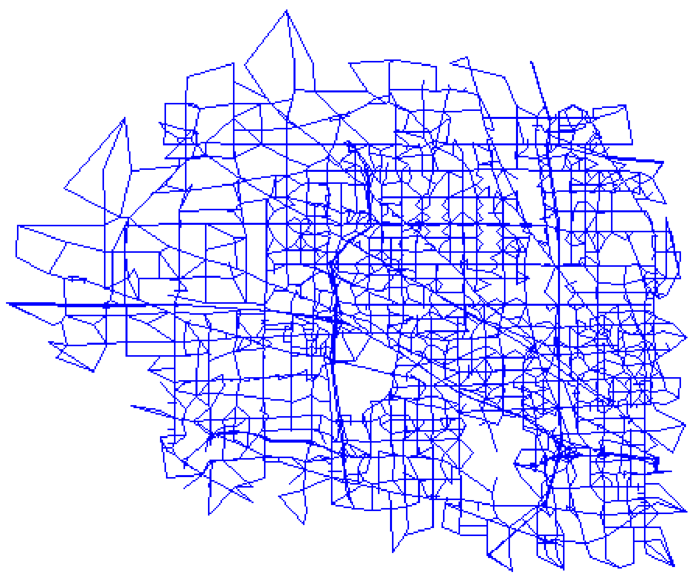


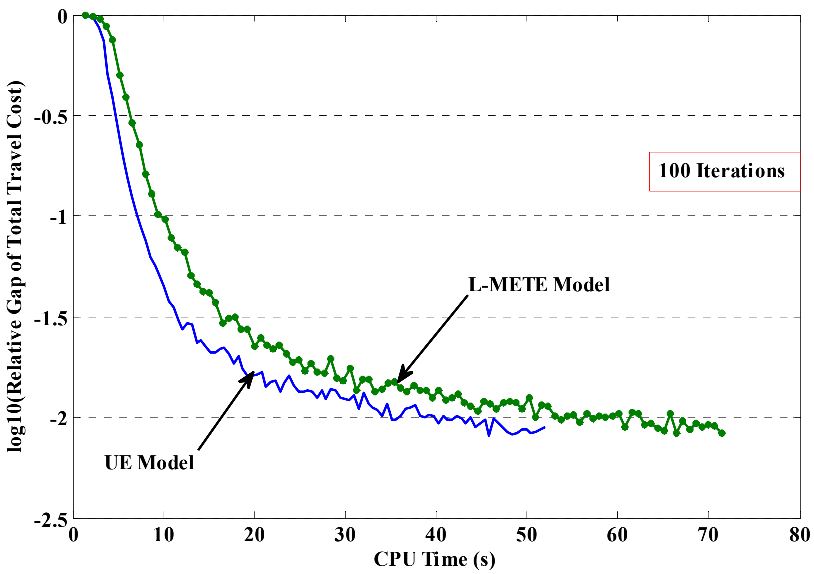
Figure 12 ADVANCE network

To facilitate comparisons, the F-W algorithm is executed 100 iterations for both UE and L-METE models. We define the following relative gap to measure the algorithmic convergence performance for solving the two models as follows:

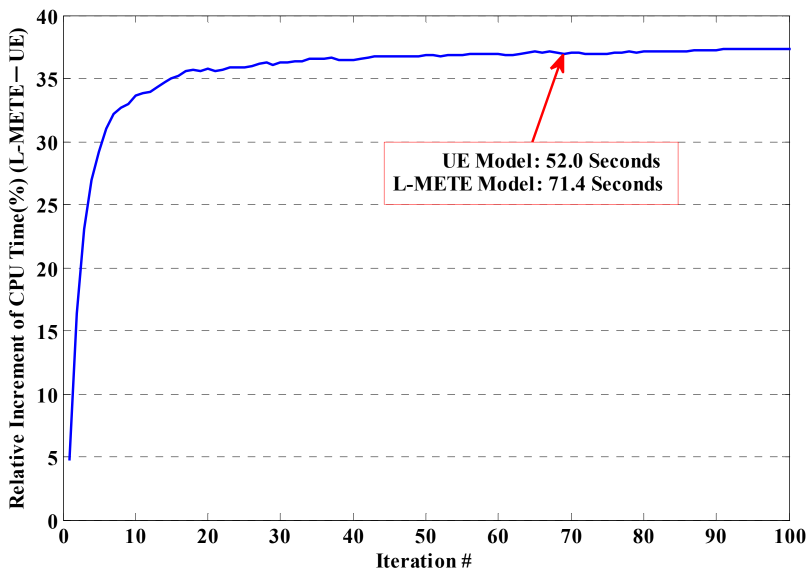
$$\text{Relative Gap}(\text{UE}) = \left(\sum_{a\in A} t_a\left(v_a^n\right)\cdot v_a^n - \sum_{a\in A} t_a\left(v_a^n\right)\cdot \tilde{v}_a^{n,\text{AON}} \right) \Big/ \sum_{a\in A} t_a\left(v_a^n\right)\cdot v_a^n, \tag{27}$$

$$\text{Relative Gap}(\text{L-METE}) = \left(\sum_{a\in A} \eta_a\left(v_a^n\right)\cdot v_a^n - \sum_{a\in A} \eta_a\left(v_a^n\right)\cdot \tilde{\tilde{v}}_a^{n,\text{AON}} \right) \Big/ \sum_{a\in A} \eta_a\left(v_a^n\right)\cdot v_a^n, \tag{28}$$

where n is the iteration number; $\tilde{v}_a^{n,\mathrm{AON}}$ is the auxiliary link flow obtained from the all-or-nothing (AON) loading based on travel time $t_a\left(v_a^n\right)$; $\tilde{\tilde{v}}_a^{n,\mathrm{AON}}$ is the auxiliary link flow obtained from the AON loading based on the LMETT $\eta_a\left(v_a^n\right)$.



(a) Convergence of the two models in terms of relative gap



(b) CPU time deviation of solving the two models

Figure 13 Convergence performances of solving UE and L-METE models

From Figure 13(a), one can observe that the convergence curve of the L-METE model is on the right-hand side of the UE model. It indicates that the F-W algorithm converges faster for the UE model than the L-METE model. The reason is that the cost update process in solving the UE model only needs to update the link travel time based on the BPR function; whereas the cost update process in solving the L-METE model needs to calculate the complex LMETT as discussed in Section 4. Figure 13(b) further shows the computational requirements for solving the two models. With the increase of iteration number, solving the L-METE model requires more computational efforts. However, after 30 iterations, the CPU deviation seems to stabilize at about 37%. In summary, the link-based METE model eases its algorithmic development and large-scale network applications of the traffic equilibrium under uncertainty. With a simple adaption of the classical F-W algorithm, we are able to solve the L-METE model with uncertainty consideration in realistic networks.

Again, this example aims to demonstrate the applicability and solvability of the L-METE model in realistic networks, rather than to demonstrate the computational efficiency of F-W algorithm for solving the L-METE model. In fact, it is quite valuable to enhance the classical F-W algorithm with different extensions in this context. For example, as mentioned above, the cost update process is much more computationally demanding; hence, it is valuable to implement an efficient flow update strategy such as one origin at a time and one O-D at a time schemes (see Chen, 2001; Chen *et al.*, 2002a). Also, evaluating the objective function is nontrivial relative to the conventional UE problem; hence implementing an efficient stepsize determination strategy will be helpful (see Chen *et al.*, 2013).

6 Concluding Remarks

This paper advanced the state-of-the-art on traffic equilibrium models under uncertainty by providing a theoretically sound and computationally tractable link-based METE model. Two observations were provided with respect to the necessity (i.e., the incapability of uniformly using the Central Limit Theorem in route travel time distribution characterization) and the possibility (i.e., the sub-additivity of the risk measure METT) of developing a *link*-based METT model. Conceptually, the L-METE model integrated the sub-additivity and complete travel time variability characterization of METT, and the computationally tractable additive route cost structure of the conventional UE problem. Mathematically, link-route and node-link Beckmann-type equivalent MP formulations were provided for the L-METE model.

The L-METE model has no specific assumptions on link and route travel time distributions, permitting any suitable link travel time distributions from empirical studies. Hence, it avoids the normal route travel time assumption (uniformly imposed for all routes) that inherited from the CLT in most *route*-based models. In addition, the L-METE model can be solved by readily adapting the existing UE algorithms, while avoiding the computationally demanding nonadditive shortest path problem and route flow allocations in route-based models. The additive route cost structure further makes it solvable without storing/enumerating routes, a significant benefit for large-scale network applications under uncertainty.

We provided a set of numerical examples using different networks to demonstrate the features of the L-METE model. We found significant discrepancies of equilibrium link flow pattern arising from different behavioral foundations, e.g., with or without travel time variability consideration, and route travel time distribution characterization. The link flow difference between UE and L-METE models was larger than that between UE and R-METE models, followed by that between L-METE and R-METE models. A larger reliability requirement and a more uncertain travel demand pattern would further increase the model result deviations. Accordingly, the biased effect of ignoring or inappropriately modeling travel time variability and travelers' risk-aversion attitude became more significant. To demonstrate the applicability, we customized the widely used F-W algorithm to solve the L-METE model in the realistic ADVANCE network in Chicago.

For future research, we plan to investigate different types of UE traffic assignment algorithms for solving the L-METE model, including link-based, bush-based, and path-based algorithms (see a latest comparison by Perederieieva *et al*., 2015). Also, the L-METE model has a heuristic nature for resolving the issues in the route-based METE model, in the sense that it cannot ensure to preserve the ranking of paths with respect to the route-based model. This issue drives us to explore different ways of reducing the deviation between the RMETT and the sum of LMETTs, e.g., the possibility of rescaling travelers' reliability requirement. On the other hand, with the widespread implementation of continuous vehicle tracking technologies, it is valuable to systemically evaluate the existing traffic equilibrium models under uncertainty (as mentioned in Section 1, including the L-METE model) regarding their replication capability relative to empirical data.

APPENDIX: ANALYTICAL DERIVATION OF LMETT

(1) Estimation of *Link* Travel Time Distribution

For the case of log-normally distributed demand uncertainty, the mean and variance of link flow V_a (i.e., v_a and $\varepsilon_{a,v}$) can be derived by following Zhou and Chen (2008). For the lognormal link flow distribution $V_a \sim LN(\mu_{a,v}, \sigma_{a,v})$, the two parameters can be inversely calculated according to v_a and $\varepsilon_{a,v}$:

$$\mu_{a,v} = \ln\left(v_a\right) - \frac{1}{2}\ln\left(1 + \varepsilon_{a,v} \big/ \left(v_a\right)^2\right), \tag{29}$$

$$\left(\sigma_{a,v}\right)^2=\ln\left(1+\varepsilon_{a,v}/\left(v_a\right)^2\right). \tag{30}$$

The generic n-th moment of $V_a \sim LN(\mu_{a,v}, \sigma_{a,v})$ is as follows:

$$E\left[\left(V_a\right)^n\right]=\exp\left[n\cdot\left(\mu_{a,v}\right)+\frac{n^2}{2}\left(\sigma_{a,v}\right)^2\right]. \tag{31}$$

We adopt the commonly used BPR -type link performance function:

$$T_a=t_a^0\left[1+\theta\left(V_a/C_a\right)^\beta\right], \tag{32}$$

where t_a^0 and C_a are the deterministic free-flow travel time and capacity of link a; V_a is the random link flow; θ and β are two parameters. From Eqs. (31)-(32), we have the generic n-th moment of T_a:

$$E\left[\left(T_a\right)^n\right]=\left(t_a^0\right)^n\sum_{i=0}^{n}\frac{n!}{i!\cdot(n-i)!}\frac{\theta^i}{\left(C_a\right)^{\beta\cdot i}}\cdot E\left[\left(V_a\right)^{\beta\cdot i}\right]. \tag{33}$$

Particularly, the mean t_a and variance $\varepsilon_{a,t}$ of T_a are as follows:

$$t_a=E\left[T_a\right]=t_a^0\sum_{i=0}^{1}\frac{1}{i!\cdot(1-i)!}\frac{\theta^i}{\left(C_a\right)^{\beta\cdot i}}\cdot E\left[\left(V_a\right)^{\beta\cdot i}\right]=t_a^0\left[1+\frac{\theta}{\left(C_a\right)^\beta}\exp\left[\beta\mu_{a,v}+\frac{\beta^2\left(\sigma_{a,v}\right)^2}{2}\right]\right], \tag{34}$$

$$\begin{aligned}E\left[\left(T_a\right)^2\right]&=\left(t_a^0\right)^2\sum_{i=0}^{2}\frac{2!}{i!\cdot(2-i)!}\frac{\theta^i}{\left(C_a\right)^{\beta\cdot i}}\cdot E\left[\left(V_a\right)^{\beta\cdot i}\right]\\&=\left(t_a^0\right)^2\left\{1+\frac{2\theta}{\left(C_a\right)^\beta}\exp\left[\beta\mu_{a,v}+\frac{\beta^2}{2}\left(\sigma_{a,v}\right)^2\right]+\frac{\theta^2}{\left(C_a\right)^{2\beta}}\exp\left[2\beta\mu_{a,v}+\frac{(2\beta)^2}{2}\left(\sigma_{a,v}\right)^2\right]\right\},\end{aligned} \tag{35}$$

$$\varepsilon_{a,t}=E\left[\left(T_a\right)^2\right]-\left(E\left[T_a\right]\right)^2=\left(\frac{\theta t_a^0}{\left(C_a\right)^\beta}\right)^2\exp\left[2\beta\mu_{a,v}+\beta^2\left(\sigma_{a,v}\right)^2\right]\cdot\left(\exp\left[\beta^2\left(\sigma_{a,v}\right)^2\right]-1\right). \tag{36}$$

Note that when the link travel time function is nonlinear, i.e., $\beta>1$, the expected link travel time ($E[T_a]$) is not equal to the deterministic travel time under the expected link flow pattern ($t_a(E[V_a])$). We continue to use lognormal distribution to characterize link travel time variability, i.e., $T_a \sim LN(\mu_{a,t}, \sigma_{a,t})$. The two parameters can be inversely calculated as follows:

$$\mu_{a,t}=\ln\left(t_a\right)-\frac{1}{2}\ln\left(1+\varepsilon_{a,t}/\left(t_a\right)^2\right), \tag{37}$$

$$\left(\sigma_{a,t}\right)^2=\ln\left(1+\varepsilon_{a,t}/\left(t_a\right)^2\right). \tag{38}$$

(2) Derivation of *Link*-based METT

After determining the two parameters, the PDF of link travel time has been known. From Eq.

(12), the *link*-based TTB with respect to confidence level α can be written as:

$$\Phi\left(\frac{\ln\left(\xi_a(\alpha)\right)-\mu_{a,t}}{\sigma_{a,t}}\right)=\alpha$$
$$\Rightarrow \xi_a(\alpha)=\exp\left(\mu_{a,t}+\Phi^{-1}(\alpha)\sigma_{a,t}\right)=\exp\left(\mu_{a,t}+\sqrt{2}\operatorname{erf}^{-1}(2\alpha-1)\sigma_{a,t}\right). \tag{39}$$

From Eq. (11), the LMETT with respect to confidence level α can be expressed as

$$\eta_a(\alpha)=\frac{1}{1-\alpha}\int_{\xi_a(\alpha)}^{+\infty}T_a\frac{1}{\sqrt{2\pi}T_a\sigma_{a,t}}\exp\left(-\frac{\left(\ln T_a-\mu_{a,t}\right)^2}{2\left(\sigma_{a,t}\right)^2}\right)d\left(T_a\right). \tag{40}$$

By setting $x_{a,t}=\left(\ln T_a-\mu_{a,t}\right)/\sigma_{a,t}$, i.e., $T_a=\exp\left(\mu_{a,t}+\sigma_{a,t}x_{a,t}\right)$, we have

$$\begin{aligned}
\eta_a(\alpha)&=\frac{1}{1-\alpha}\int_{\frac{\ln(\xi_a(\alpha))-\mu_{a,t}}{\sigma_{a,t}}}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma_{a,t}}\exp\left(-\left(x_{a,t}\right)^2/2\right)\exp\left(\mu_{a,t}+\sigma_{a,t}x_{a,t}\right)\sigma_{a,t}dx_{a,t}\\
&=\frac{\exp\left(\mu_{a,t}\right)}{1-\alpha}\int_{\frac{\ln(\xi_a(\alpha))-\mu_{a,t}}{\sigma_{a,t}}}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\left(x_{a,t}\right)^2/2+\sigma_{a,t}x_{a,t}\right)dx_{a,t}\\
&=\frac{\exp\left(\mu_{a,t}\right)}{1-\alpha}\int_{\frac{\ln(\xi_a(\alpha))-\mu_{a,t}}{\sigma_{a,t}}}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(x_{a,t}\right)^2-2\sigma_{a,t}x_{a,t}+\left(\sigma_{a,t}\right)^2}{2}+\frac{\left(\sigma_{a,t}\right)^2}{2}\right)dx_{a,t}\\
&=\frac{\exp\left(\mu_{a,t}+\left(\sigma_{a,t}\right)^2/2\right)}{1-\alpha}\int_{\frac{\ln(\xi_a(\alpha))-\mu_{a,t}}{\sigma_{a,t}}}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(x_{a,t}\right)^2-2\sigma_{a,t}x_{a,t}+\left(\sigma_{a,t}\right)^2}{2}\right)dx_{a,t}\\
&=\frac{\exp\left(\mu_{a,t}+\left(\sigma_{a,t}\right)^2/2\right)}{1-\alpha}\int_{\frac{\ln(\xi_a(\alpha))-\mu_{a,t}}{\sigma_{a,t}}}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\left(x_{a,t}-\sigma_{a,t}\right)^2}{2}\right)dx_{a,t}\,.
\end{aligned}$$

By further setting $y_{a,t}=x_{a,t}-\sigma_{a,t}$, we have

$$\begin{aligned}
\eta_a(\alpha)&=\frac{\exp\left(\mu_{a,t}+\left(\sigma_{a,t}\right)^2/2\right)}{1-\alpha}\int_{\frac{\ln(\xi_a(\alpha))-\mu_{a,t}}{\sigma_{a,t}}-\sigma_{a,t}}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\left(y_{a,t}\right)^2/2\right)dy_{a,t}\\
&=\frac{\exp\left(\mu_{a,t}+\left(\sigma_{a,t}\right)^2/2\right)}{1-\alpha}\left(1-\Phi\left(\frac{\ln\left(\xi_a(\alpha)\right)-\mu_{a,t}}{\sigma_{a,t}}-\sigma_{a,t}\right)\right)\\
&=\frac{\exp\left(\mu_{a,t}+\left(\sigma_{a,t}\right)^2/2\right)}{1-\alpha}\Phi\left(-\frac{\ln\left(\xi_a(\alpha)\right)-\mu_{a,t}}{\sigma_{a,t}}+\sigma_{a,t}\right).
\end{aligned}$$

Finally, the LMETT can be written as

$$\eta_a(\alpha)=\frac{\exp\left(\mu_{a,t}+\left(\sigma_{a,t}\right)^2/2\right)}{1-\alpha}\Phi\left(-\sqrt{2}\operatorname{erf}^{-1}(2\alpha-1)+\sigma_{a,t}\right). \tag{41}$$

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