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Free vibration analysis of a structural system with a pair of irrational nonlinearities

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Abstract

An alternative method of deriving accurate and simple analytical approximate solutions to a structural dynamical system governed by a pair of strong irrational restoring forces is presented. This system can be used to represent mathematical models in various engineering problems. Prior to solving the problem, a rational approximation of the nonlinear restoring force function is applied to achieve a convergent truncation. Analytical solutions are then obtained using the combination of the harmonic balance method and Newton's method. This approach shows that lower-order analytical procedures can yield highly accurate and exact solutions that are difficult to obtain with an analytical expression.

Keywords Analytical approximate solutions · Irrational nonlinearities · Rational approximation · Newton's method · Harmonic balance method

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1. Introduction

Many engineering applications, including folding multi-layered truss structures [1], pre-loaded two-bar linkage systems [2], buckled beams [3], and pre-tensioned discrete elastic strings [4], can be modelled as a dynamical system with irrational restoring forces. In considering such a model with geometric nonlinearities, it is of interest to investigate the frequency-amplitude dependent relationship, as this allows the general qualitative behavior of steady-state responses to be established.

Recently, a novel dynamical system with a pair of strong irrational nonlinearities was investigated [5, 6]. This oscillating system comprises a lump mass linked with a couple of inclined elastic springs that are pinned to rigid supports. Although the resistance of both springs is linear in nature, the elastic restoring force of the dynamical system is strongly nonlinear due to the geometric configuration. Further, the dynamical properties of this system rely upon the values of the smoothness parameters that concern the nonlinearities. A comprehensive qualitative analysis was conducted to study the global bifurcation and multiple snap-through buckling phenomena. To analyze the equilibrium stability of the system, the irrational restoring force function was expressed in terms of polynomials up to the fifth order (i.e., the restoring force $f(x)=\sum_{i=0}^{2} a_{2i+1}x^{2i+1}$) via the Taylor series expansion [6]. The snap-through buckling of a structure is a kind of bifurcation behavior in which the structure can suddenly jump from one state to another stable equilibrium configuration [7]. The mechanism of snap-through buckling is a useful technique that can be adopted for energy harvesting [8-10]. Both quantitative and qualitative analyses can concretize the factors involved to understand the nature of this system in practical use.

In this paper, a nonlinear problem that involves a pair of strong irrational restoring forces is solved. Due to the complexity of the restoring force function, it is difficult to derive exact solutions to this problem with analytical expressions. The present work provides an alternative approach that obtains simple and accurate analytical approximate solutions to this problem. Generally, analytical approximate solutions to nonlinear problems are obtained using the perturbative approaches [11-16] and harmonic balance method [17, 18]. The present approach implements the coupling of Newton's method with the harmonic balancing technique [19], which reduces the deficiency of the classical harmonic balance method in dealing with nonlinear problems governed by both small and large parameters. In contrast with the conventional perturbative and harmonic balance approaches, the Newton harmonic balance approach is applicable and flexible for solving various nonlinear problems with odd nonlinearities [19, 20], rational restoring forces [21], strong damping effects [22], and integral-differential forms [23].

To extend the investigation of a dynamical problem with irrational nonlinearities, we first apply a rational approximation [24] for the irrational restoring force function, and then solve the resulting equation by means of the Newton harmonic balance method. The solutions can be found from a set of linear algebraic equations in lieu of nonlinear equations. Illustrative examples are selected and compared with "exact solutions" obtained using the Runge-Kutta numerical method to verify the accuracy of the lower-order analytical approximate solutions.

2. Mathematical Model

A structural system describing a lumped mass linked to a pair of inclined elastic springs that are pinned to the rigid supports is expressed as [5, 6] (see Figure 1):

$$m\frac{d^2X}{dt^2}+k(X+a)\left[1-\frac{L}{\sqrt{(X+a)^2+h^2}}\right]+k(X-a)\left[1-\frac{L}{\sqrt{(X-a)^2+h^2}}\right]=0 \tag{1}$$

in which m is the moving mass, k is the spring stiffness constant, X is the displacement of the mass, a is the half distance between the rigid supports, h is the vertical height between the moving mass and the rigid supports, and L is the free length of the springs.

In terms of a non-dimensional form, the system can be re-written as follows,

$$\ddot{x}+(x+\alpha)\left[1-\frac{1}{\sqrt{(x+\alpha)^2+\beta^2}}\right]+(x-\alpha)\left[1-\frac{1}{\sqrt{(x-\alpha)^2+\beta^2}}\right]=0 \tag{2}$$

with the initial conditions

$$x(0)=X/L=A\,(\leq\alpha), \qquad \dot{x}(0)=0 \tag{3}$$

where a dot denotes the derivative with respect to the transformed temporal coordinate T, $x=X/L$, $\alpha=a/L$, and $\beta=h/L$. Upon integrating Eq. (2), we arrive at

$$\frac{1}{2}\dot{x}^2+x^2-\sqrt{(x+\alpha)^2+\beta^2}-\sqrt{(x-\alpha)^2+\beta^2}=E \tag{4}$$

where E is an integration constant corresponding to the total energy of the system from a physical perspective. From Eq. (4), the exact oscillatory mode is difficult to derive in terms of analytical expressions.

The relationship between the parameters α and β is in the form of $\alpha^2 + \beta^2 = 1$. For $\beta = 0$, Eq. (2) will degenerate from a continuous system into a discontinuous system, that is [5]:

$$\ddot{x} + 2x - sign(x+\alpha) - sign(x-\alpha) = 0 \qquad (5)$$

From Eq. (5), there is an equilibrium point at $x = 0$ when $\alpha = 1$. In this paper, we will focus on solving Eqs. (2) and (3) with the parameters $0 < \alpha < 1$, $0 < \beta < 1$ and $0 < A \le \alpha$.

This paper offers two novel findings:

- The exact solution to Eq. (2) is difficult to derive due to its complexity. It is rather difficult to derive the exact solution in terms of analytical expressions from Eq. (4). The new approach provides an alternative and simple way of deriving accurate analytical approximate solutions to Eq. (2).
- The irrational restoring force function in Eq. (2) directly expressed in terms of polynomials by Taylor's series expansion converges slowly. To accelerate the convergence of the sequences, it is proposed to apply a rational representation for the irrational restoring force function.

3. Solution Methodology

In an attempt to derive analytical approximate solutions, we adopt a rational approximation [24] of the irrational restoring force function in Eq. (2), as follows,

$$\ddot{x} + \frac{\sum_{i=0}^{4} a_{2i+1} x^{2i+1}}{\sum_{i=0}^{4} b_{2i} x^{2i}} = 0 \qquad (6)$$

with the initial conditions

$$x(0) = A, \qquad \dot{x}(0) = 0 \qquad (7)$$

where the coefficients a_{2i+1} and b_{2i} $(i = 0-4)$ can be obtained by Taylor's series expansion and are presented in Appendix A. The coefficients a_{2i+1} and b_{2i} $(i \ge 5)$ are ignored. The irrational restoring force function, expressed in terms of the rational approximation, can accelerate the convergence, in such a way that the range of validity of the analytical approximate solutions for Eq. (2) can also be enhanced.

By introducing an independent variable, $\tau = \sqrt{\Omega}T$, Eqs. (6) and (7) can be rewritten as

$$\Omega x''\left(\sum_{i=0}^{4} b_{2i}x^{2i}\right)+\sum_{i=0}^{4} a_{2i+1}x^{2i+1} = 0 \tag{8}$$

and

$$x(0)= A, \qquad x'(0)= 0 \tag{9}$$

where a prime denotes the derivative with respect to τ . The independent variable is chosen such that the solution to Eq. (8) is a periodic solution to τ in period 2π . In Eq. (6), the restoring force function of the nonlinear equation is an odd function (i.e., $-f(x)= f(-x)$), so that the system oscillates around the equilibrium position between the symmetric limits $[-A, A]$, and the periodic solution is written as $x(\tau)=\sum_{j=0}^{\infty} k_{2j+1}\cos[(2j+1)\tau]$. By means of Newton's method, the squared angular frequency parameter Ω and periodic solution x are set as

$$\Omega = \Omega_1 + \Delta\Omega_1 \tag{10}$$

$$x = x_1 + \Delta x_1 \tag{11}$$

Substituting both Eqs. (10) and (11) into Eq. (8) and linearizing the governing equation gives

$$\begin{aligned}
&\Omega_1 x_1''\left(b_0 + b_2x_1^2+b_4x_1^4+b_6x_1^6+b_8x_1^8\right)+ 2\Omega_1 x_1'' x_1\ \Delta x_1\left(b_2 + 2b_4x_1^2+3b_6x_1^4+4b_8x_1^6\right)+\\
&\Omega_1\Delta x_1''\left(b_0 + b_2x_1^2+b_4x_1^4+b_6x_1^6+b_8x_1^8\right)+ \Delta\Omega_1 x_1''\left(b_0 + b_2x_1^2+b_4x_1^4+b_6x_1^6+b_8x_1^8\right)+\\
&a_1(x_1 + \Delta x_1)+ a_3\left(x_1^3 + 3x_1^2\Delta x_1\right)+ a_5\left(x_1^5 + 5x_1^4\Delta x_1\right)+ a_7\left(x_1^7 + 7x_1^6\Delta x_1\right)+ a_9\left(x_1^9 + 9x_1^8\Delta x_1\right)= 0
\end{aligned} \tag{12}$$

The initial conditions in Eq. (9) should be satisfied by setting:

$$\Delta x_1(0)= 0, \quad \Delta x_1'(0)= 0 \tag{13}$$

where $\Delta x_1(\tau)$ is a periodic function of τ with a period of 2π to be determined later.

For the lowest-order (first-order) analytical approximation, we initially set

$$x_1(\tau)= A\cos\tau \tag{14}$$

$$\Delta\Omega_1 = 0 \text{ and } \Delta x_1 = 0 \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (12) and setting the coefficient of $\cos\tau$ to zero yields,

$$\Omega_1 = \frac{128a_1 + 96A^2 a_3 + 80A^4 a_5 + 70A^6 a_7 + 63A^8 a_9}{128b_0 + 96A^2 b_2 + 80A^4 b_4 + 70A^6 b_6 + 63A^8 b_8} \tag{16}$$

Therefore, the first-order analytical approximation of the periodic solution to Eq. (2) is

$$x_1(T) = A\cos\left(\sqrt{\Omega_1}\, T\right) \tag{17}$$

and the corresponding period (T_1) is equal to

$$T_1 = \frac{2\pi}{\sqrt{\Omega_1}} \tag{18}$$

For the second-order analytical approximation, the following equation is substituted into Eq. (12) to satisfy the initial condition in Eq. (13) at the outset.

$$\Delta x_1(\tau) = c_1(\cos\tau - \cos 3\tau) \tag{19}$$

Making use of Eqs. (14) and (19) and expanding the resulting expression of Eq. (12) into a trigonometric series and setting the coefficients of cos τ and cos 3τ to zero, we obtain two linear equations for the unknowns c_1 and $\Delta\Omega_1$. Solving these two linear equations yields

$$c_1(A) = \frac{M_1(A)}{M_2(A)} \tag{20}$$

$$\Delta\Omega_1(A) = \frac{-A[L_1(A)L_2(A) + L_3(A)L_4(A)]}{K_1(A)K_2(A) - K_3(A)K_4(A)} \tag{21}$$

where the variables $M_i\ (i = 1,2)$, $L_i\ (i = 1-4)$ and $K_i\ (i = 1-4)$ in Eqs. (20) and (21) are given in Appendix B.

Hence, the second-order analytical approximation of the periodic solution for Eq. (2) is

$$x_2(T) = (A + c_1)\cos\left(\sqrt{\Omega_2}\, T\right) - c_1\cos\left(3\sqrt{\Omega_2}\, T\right) \tag{22}$$

$$\Omega_2 = \Omega_1 + \Delta\Omega_1 \tag{23}$$

and the corresponding period (T_2) is equal to

$$T_2 = \frac{2\pi}{\sqrt{\Omega_2}} \tag{24}$$

The derivation of higher-order analytical approximations can be constructed similarly and is omitted for simplicity.

4. Results and Discussion

In this section, illustrative examples are selected to demonstrate the accuracy of the present analytical approximate solutions with respect to the numerical solutions ("exact solutions") obtained from the Runge-Kutta method. Although nonlinear problems can be solved by numerical methods, there are two major merits of analytical approaches over numerical methods.

- The distinct feature of analytical solutions allows the evolution and the dynamic behavior of physical quantities to be traced.
- Analytical approaches do not require a known initial condition at the outset, which is a condition for numerical methods.

To obtain the numerical solutions, the Runge-Kutta method is implemented by solving Eq. (2) directly. The angular frequency and periodic solution obtained by the Runge-Kutta method are denoted as "ω_{num}" and "$x_{num}(T)$", respectively.

In Table 1, a comparison of the coefficients of the irrational restoring force function in terms of a rational representation and a Taylor series form is presented. There are two cases in Table 1, it is observed that the coefficients a_{2i+1} and b_{2i} $(i = 0-10)$ (i.e., the first eleven terms) in the rational representation decrease monotonically and rapidly. However, the sequences $\hat{a}_{2i+1}$ $(i = 0-10)$ (i.e., the first eleven terms) in the direct use of Taylor series expansion for the irrational restoring force are oscillatory. A faster convergence of the coefficients in the rational form leads to the reduction of the computational efforts in the present study. Although only the first fifth terms of the coefficients a_{2i+1} and b_{2i} $(i = 0-4)$ in the rational representation are taken into consideration in Eq. (6), accurate analytical approximate solutions can be achieved and presented in Table 2. Conversely, the coefficients $\hat{a}_{2i+1}$ $(i > 4)$ oscillate, it is difficult to make truncations for further analysis.

In addition, a comparison of the rational form, Taylor series expansion and exact solutions for the irrational restoring force function is presented in Figures 2 and 3. Making use of the first fifth terms of the coefficients a_{2i+1} and b_{2i} $(i=0-4)$ in the rational representation, it can provide a very good approximation to the exact solutions of the irrational restoring force function in the region of $x\in[-\alpha,\alpha]$ and under the parameters $\alpha\to 1$ and $\beta\to 0$. However, the direct use of Taylor series expansion for the irrational restoring force function cannot provide good agreement when $\alpha\to 1$, $\beta\to 0$ and $x\in[-\alpha,\alpha]$ as shown in Figures 2 and 3, even if the first ninth terms (i.e., $\hat{a}_{2i+1}$ $(i=0-8)$) are used for computation.

In Table 2, a comparison of the analytical approximate and numerical solutions is presented. We observe that the second-order analytical solutions (i.e., $\sqrt{\Omega_2}$) are closer than the first-order analytical solutions (i.e., $\sqrt{\Omega_1}$) to the numerical frequency solutions (i.e., ω_{num}), it implies that the second-order analytical solutions enhance the accuracy significantly. When $\alpha\to 1$ and $\beta\to 0$, the nonlinear system tends to degenerate from a continuous system to a discontinuous system. Undergoing the large amplitudes of oscillation (i.e., $A\to 1$) and approaching the bifurcation state (i.e., $\alpha\to 1$ and $\beta\to 0$), the relative errors of the first-order and second-order analytical approximate solutions are comparatively higher as shown in Table 2. To enhance the accuracy of the present approach in this regime (i.e., $A\to 1$, $\alpha\to 1$ and $\beta\to 0$), higher-order analytical approximations can be constructed by following the procedures as aforementioned in Section 3 by setting the k^{th}-order analytical approximations as:

$$\Delta x_{k-1}(\tau)=\sum_{j=1}^{k-1}c_j\left\{\cos[(2j-1)\tau]-\cos[(2j+1)\tau]\right\} \tag{25}$$

where c_j are the unknowns to be determined by the corresponding set of linear algebraic equations. In addition, it is observed that the increment pattern of relative errors is slightly oscillatory when $A\to 1$, $\alpha\to 1$ and $\beta\to 0$, it is mainly affected by the number of coefficients used in Eq. (6). Furthermore, corrections are applied to both the angular frequency and periodic solution in the present method to enhance its accuracy with respect to the numerical solutions obtained from the Runge-Kutta method, but it may not be able to ensure that the increment pattern of the relative errors in angular frequency is monotonic in general. Nevertheless, the present solutions provide very good agreement with respect to the numerical solutions.

In Figures 4(a), 5(a) and 6(a), three illustrative examples are given to compare the numerical periodic solution $x_{num}(T)$ obtained by the Runge-Kutta method and the approximate analytical periodic solutions $x_1(T)$ and $x_2(T)$ for different parameters α, β, and A. Figures 4(b), 5(b), and 6(b) respectively depict the absolute errors (i.e., $|x_i(T)-x_{num}(T)|$) between $x_1(T)$, $x_2(T)$ and $x_{num}(T)$ in each illustrative example. These figures clearly indicate that the second-order approximate solutions provide closer solutions to the numerical solutions for the entire range of permitted amplitudes of oscillation.

5. Conclusions

An effective approach, called the Newton harmonic balance method, is adopted to derive the analytical approximate solutions for the structural dynamical system with a pair of strong irrational nonlinearities. To accelerate the convergence, a rational representation of the irrational restoring force function is applied so that the range of validity of the analytical approximate solutions is augmented. The results of the second-order analytical approximation show excellent convergence compared with the numerical solutions derived from the Runge-Kutta method for a full range of permitted amplitudes of motion. The method presented here demonstrates its simplicity and potential for solving complicated strongly nonlinear problems.

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Appendix A

The coefficients a_{2i+1} and b_{2i} $(i = 0 - 4)$ in Eq. (6) are presented as follows.

$$a_1 = 2\alpha^2 \tag{A1}$$

$$a_3 = -2\alpha^4 - 2\alpha^2\beta^2 + \beta^4 \tag{A2}$$

$$a_5 = \frac{\beta^2\left[16\alpha^6 + 16\alpha^2\beta^4 + \beta^4 - 8\alpha^4\left(1 - 4\beta^2\right)\right]}{4} \tag{A3}$$

$$a_7 = \frac{\beta^2\left[32\alpha^8 - \beta^6 - 16\alpha^6\left(1 - 2\beta^2\right) + 8\alpha^4\left(3\beta^2 - 4\beta^4\right) + \alpha^2\left(6\beta^4 - 32\beta^6\right)\right]}{8} \tag{A4}$$

$$\begin{aligned} a_9 = \beta^2\Big[&256\alpha^{10} + 5\beta^8 + 512\alpha^6\beta^2\left(1 - 2\beta^2\right) - 128\alpha^8\left(1 + 2\beta^2\right) - 32\alpha^4\left(5\beta^4 + 8\beta^6\right) \\ &+ \alpha^2\left(-80\beta^6 + 256\beta^8\right)\Big]/64 \end{aligned} \tag{A5}$$

$$b_0 = 1 \tag{A6}$$

$$b_2 = -\alpha^2 + \beta^2 \tag{A7}$$

$$b_4 = 2\alpha^2\beta^2 \tag{A8}$$

$$b_6 = 2\alpha^2\beta^2\left(\alpha^2 - \beta^2\right) \tag{A9}$$

$$b_8 = 2\alpha^2\beta^2\left(\alpha^4 - 3\alpha^2\beta^2 + \beta^4\right) \tag{A10}$$

Appendix B

The variables $M_i\left(i=1,2\right)$, $L_i\left(i=1-4\right)$ and $K_i\left(i=1-4\right)$ in Eqs. (20) and (21) are presented below.

$$\begin{aligned} M_1\left(A\right)=-2A^3\Big[&-2688A^4a_7b_0-2688A^6a_9b_0+2048a_1b_2-896A^6a_7b_2-1008A^8a_9b_2\\ &+2560A^2a_1b_4-280A^8a_7b_4-420A^{10}a_9b_4+2688A^4a_1b_6-147A^{12}a_9b_6\\ &+2688A^6a_1b_8+147A^{12}a_7b_8-16a_3\left(128b_0-40A^4b_4-56A^6b_6-63A^8b_8\right)\\ &+20A^2a_5\left(-128b_0-32A^2b_2+14A^6b_6+21A^8b_8\right)\Big] \end{aligned} \tag{B1}$$

$$\begin{aligned} M_2\left(A\right)=\,&5120A^4a_5b_0-3456A^8a_9b_0+10240A^6a_5b_2+6272A^8a_7b_2+3456A^{10}a_9b_2\\ &+11200A^8a_5b_4+7840A^{10}a_7b_4+5400A^{12}a_9b_4+11200A^{10}a_5b_6+8232A^{12}a_7b_6\\ &+6048A^{14}a_9b_6+10920A^{12}a_5b_8+8232A^{14}a_7b_8+6237A^{16}a_9b_8\\ &+128a_1\left(128b_0+128A^2b_2+120A^4b_4+112A^6b_6+105A^8b_8\right)\\ &+96A^2a_3\left(128b_0+160A^2b_2+160A^4b_4+154A^6b_6+147A^8b_8\right)\\ &-\Omega_1\big(147456b_0^2+192512A^2b_0b_2+56320A^4b_2^2+151552A^4b_0b_4+82944A^6b_2b_4\\ &+29120A^8b_4^2+129024A^6b_0b_6+67648A^8b_2b_6+45920A^{10}b_4b_6+17640A^{12}b_6^2\\ &+114432A^8b_0b_8+58272A^{10}b_2b_8+38592A^{12}b_4b_8+29064A^{14}b_6b_8+11781A^{16}b_8^2\big) \end{aligned} \tag{B2}$$

$$L_1\left(A\right)=-2A^2\left(16a_3+20A^2a_5+21A^4a_7+21A^6a_9-16b_2\Omega_1-20A^2b_4\Omega_1-21A^4b_6\Omega_1-21A^6b_8\Omega_1\right) \tag{B3}$$

$$\begin{aligned} L_2\left(A\right)=\,&128a_1+192A^2a_3+200A^4a_5+196A^6a_7+189A^8a_9-128b_0\Omega_1+64A^2b_2\Omega_1+120A^4b_4\Omega_1\\ &+140A^6b_6\Omega_1+147A^8b_8\Omega_1 \end{aligned} \tag{B4}$$

$$\begin{aligned} L_3\left(A\right)=\,&128a_1+96A^2a_3+80A^4a_5+70A^6a_7+63A^8a_9-128b_0\Omega_1-96A^2b_2\Omega_1-80A^4b_4\Omega_1\\ &-70A^6b_6\Omega_1-63A^8b_8\Omega_1 \end{aligned} \tag{B5}$$

$$\begin{aligned} L_4\left(A\right)=\,&-128a_1-96A^2a_3-40A^4a_5+27A^8a_9+1152b_0\Omega_1+608A^2b_2\Omega_1+424A^4b_4\Omega_1\\ &+336A^6b_6\Omega_1+285A^8b_8\Omega_1 \end{aligned} \tag{B6}$$

$$K_1\left(A\right)=2\left[16A^3b_2+20A^5b_4+21A^7\left(b_6+A^2b_8\right)\right] \tag{B7}$$

$$\begin{aligned} K_2\left(A\right)=\,&128a_1+192A^2a_3+200A^4a_5+196A^6a_7+189A^8a_9-128b_0\Omega_1+64A^2b_2\Omega_1+120A^4b_4\Omega_1\\ &+140A^6b_6\Omega_1+147A^8b_8\Omega_1 \end{aligned} \tag{B8}$$

$$K_3\left(A\right)=128Ab_0+96A^3b_2+80A^5b_4+70A^7b_6+63A^9b_8 \tag{B9}$$

$$\begin{aligned} K_4\left(A\right)=\,&-128a_1-96A^2a_3-40A^4a_5+27A^8a_9+1152b_0\Omega_1+608A^2b_2\Omega_1+424A^4b_4\Omega_1\\ &+336A^6b_6\Omega_1+285A^8b_8\Omega_1 \end{aligned} \tag{B10}$$

where Ω_1 is derived in Eq. (16).

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Table 1 Comparison of the coefficients of rational representation and Taylor series for the irrational restoring force function.

Table 2 Comparison of the analytical approximate and numerical solutions for $\beta = \sqrt{1-\alpha^2}$ and $A = \alpha$.

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Fig. 1. Structural system with a lump mass and a pair of springs pinned to rigid bases.

Fig. 2. Comparison of various forms of the irrational restoring force for $\alpha = 0.995$, $\beta = 0.0999$ and $x \in [-0.995, 0.995]$ (Rational form: $f(x) = \sum_{i=0}^{4} a_{2i+1} x^{2i+1} / \sum_{i=0}^{4} b_{2i} x^{2i}$; Taylor series: $f(x) = \sum_{i=0}^{8} \hat{a}_{2i+1} x^{2i+1}$).

Fig. 3. Comparison of various forms of the irrational restoring force for $\alpha = 0.999$, $\beta = 0.045$ and $x \in [-0.999, 0.999]$ (Rational form: $f(x) = \sum_{i=0}^{4} a_{2i+1} x^{2i+1} / \sum_{i=0}^{4} b_{2i} x^{2i}$; Taylor series: $f(x) = \sum_{i=0}^{8} \hat{a}_{2i+1} x^{2i+1}$).

Fig. 4. (a) Comparison of the time history response of Eq. (2) for $\alpha = 0.3$, $\beta = 0.9539$, and $A = 0.3$; (b) Comparison of the absolute errors between the approximate and numerical periodic solutions in this case.

Fig. 5. (a) Comparison of the time history response of Eq. (2) for $\alpha = 0.6$, $\beta = 0.8$, and $A = 0.6$; (b) Comparison of the absolute errors between the approximate and numerical periodic solutions in this case.

Fig. 6. (a) Comparison of the time history response of Eq. (2) for $\alpha = 0.99$, , $\beta = 0.1411$ and $A = 0.99$; (b) Comparison of the absolute errors between the approximate and numerical periodic solutions in this case.

Table 1 Comparison of the coefficients of rational representation and Taylor series for the irrational restoring force function.

Rational approximation for the irrational restoring force function *f*(x) in Eq. (2) (Note 1)				Direct use of Taylor series expansion for the irrational restoring force function *f*(x) in Eq. (2) (Note 2)	
$\alpha = 0.1$ and $\beta = 0.9950$					
a_1	0.02000	b_0	1.00000	$\hat{a}_1$	0.02000
a_3	0.96010	b_2	0.98000	$\hat{a}_3$	0.94050
a_5	0.28198	b_4	0.01980	$\hat{a}_5$	-0.64011
a_7	-0.15131	b_6	-0.01940	$\hat{a}_7$	0.45776
a_9	0.09969	b_8	0.01882	$\hat{a}_9$	-0.31836
a_{11}	-0.07202	b_{10}	-0.01806	$\hat{a}_{11}$	0.20115
a_{13}	0.05446	b_{12}	0.01714	$\hat{a}_{13}$	-0.09879
a_{15}	-0.04207	b_{14}	-0.01607	$\hat{a}_{15}$	0.00862
a_{17}	0.03268	b_{16}	0.01488	$\hat{a}_{17}$	0.07013
a_{19}	-0.02522	b_{18}	-0.01360	$\hat{a}_{19}$	-0.13743
a_{21}	0.01911	b_{20}	0.01223	$\hat{a}_{21}$	0.19295
$\alpha = 0.99$ and $\beta = 0.1411$					
a_1	1.96020	b_0	1.00000	$\hat{a}_1$	1.96020
a_3	-1.95980	b_2	-0.96020	$\hat{a}_3$	-0.07762
a_5	0.03979	b_4	0.03901	$\hat{a}_5$	-0.11121
a_7	0.03859	b_6	0.03746	$\hat{a}_7$	-0.13859
a_9	0.03665	b_8	0.03520	$\hat{a}_9$	-0.15819
a_{11}	0.03404	b_{10}	0.03234	$\hat{a}_{11}$	-0.16894
a_{13}	0.03087	b_{12}	0.02898	$\hat{a}_{13}$	-0.17037
a_{15}	0.02726	b_{14}	0.02525	$\hat{a}_{15}$	-0.16257
a_{17}	0.02335	b_{16}	0.02128	$\hat{a}_{17}$	-0.14626
a_{19}	0.01927	b_{18}	0.01722	$\hat{a}_{19}$	-0.12266
a_{21}	0.01518	b_{20}	0.01322	$\hat{a}_{21}$	-0.09347

Note 1: $f(x) = \sum_{i=0}^{10} a_{2i+1} x^{2i+1} / \sum_{i=0}^{10} b_{2i} x^{2i}$

Note 2: $f(x) = \sum_{i=0}^{10} \hat{a}_{2i+1} x^{2i+1}$

Table 2 Comparison of the analytical approximate and numerical solutions for $\beta = \sqrt{1-\alpha^2}$ and $A = \alpha$.

α	β	$\sqrt{\Omega_1}$, from Eq. (16)	$\sqrt{\Omega_2}$, from Eq. (23)	ω_{num}	$\left(\frac{\sqrt{\Omega_1}-\omega_{num}}{\omega_{num}}\right)\times 100\%$	$\left(\frac{\sqrt{\Omega_2}-\omega_{num}}{\omega_{num}}\right)\times 100\%$
0.1	0.9950	0.16438	0.16413	0.16413	0.15	0.00
0.2	0.9798	0.32058	0.32015	0.32015	0.13	0.00
0.3	0.9539	0.46277	0.46231	0.46230	0.10	0.00
0.4	0.9165	0.58846	0.58816	0.58815	0.05	0.00
0.5	0.8660	0.69799	0.69801	0.69805	-0.01	-0.01
0.6	0.8	0.79367	0.79401	0.79413	-0.06	-0.02
0.7	0.7141	0.87959	0.87985	0.87975	-0.02	0.01
0.8	0.6	0.96241	0.96130	0.95985	0.27	0.15
0.9	0.4359	1.05225	1.04415	1.04493	0.70	-0.07
0.99	0.1411	1.28204	1.21382	1.19271	7.49	1.77
0.995	0.0999	1.33577	1.27523	1.22289	9.23	4.28

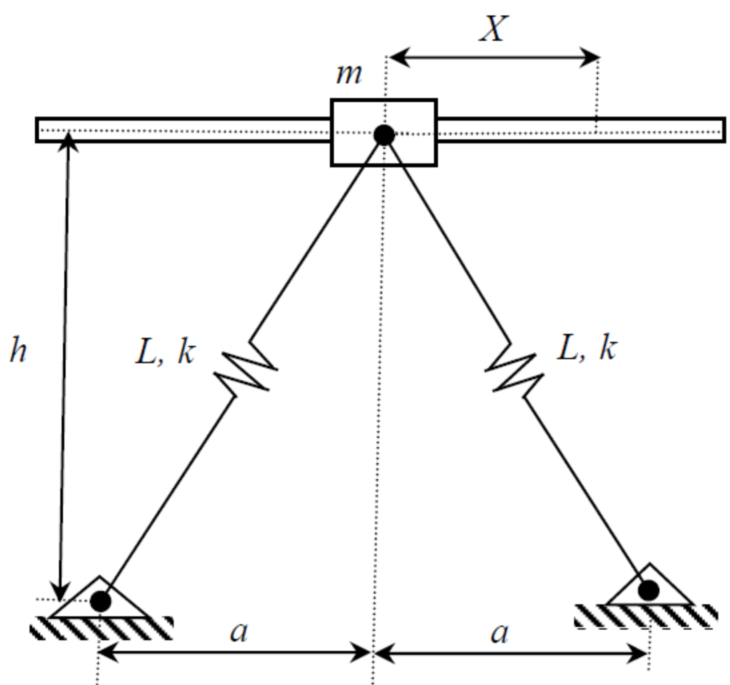


Fig. 1. Structural system with a lump mass and a pair of springs pinned to rigid bases.

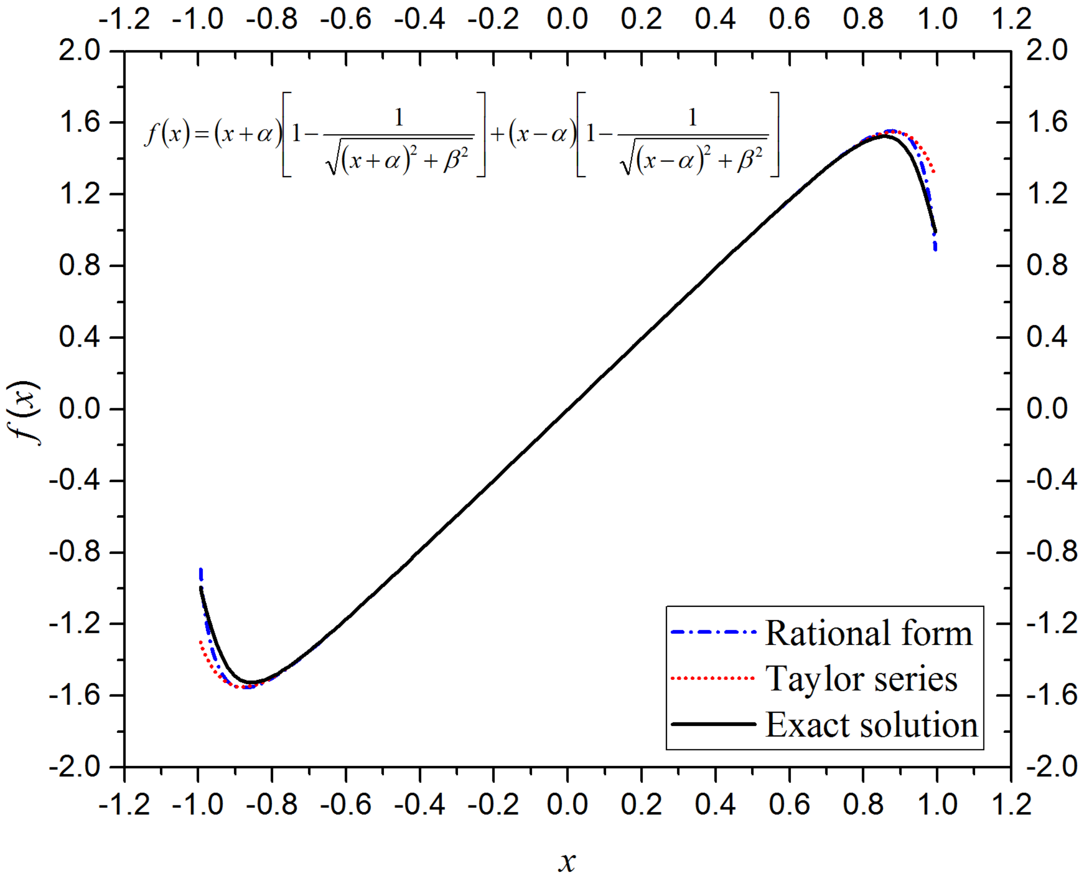


Fig. 2. Comparison of various forms of the irrational restoring force for $\alpha = 0.995$, $\beta = 0.0999$ and $x \in [-0.995, 0.995]$ (Rational form: $f(x) = \sum_{i=0}^{4} a_{2i+1}x^{2i+1} / \sum_{i=0}^{4} b_{2i}x^{2i}$; Taylor series: $f(x) = \sum_{i=0}^{8} \hat{a}_{2i+1}x^{2i+1}$).

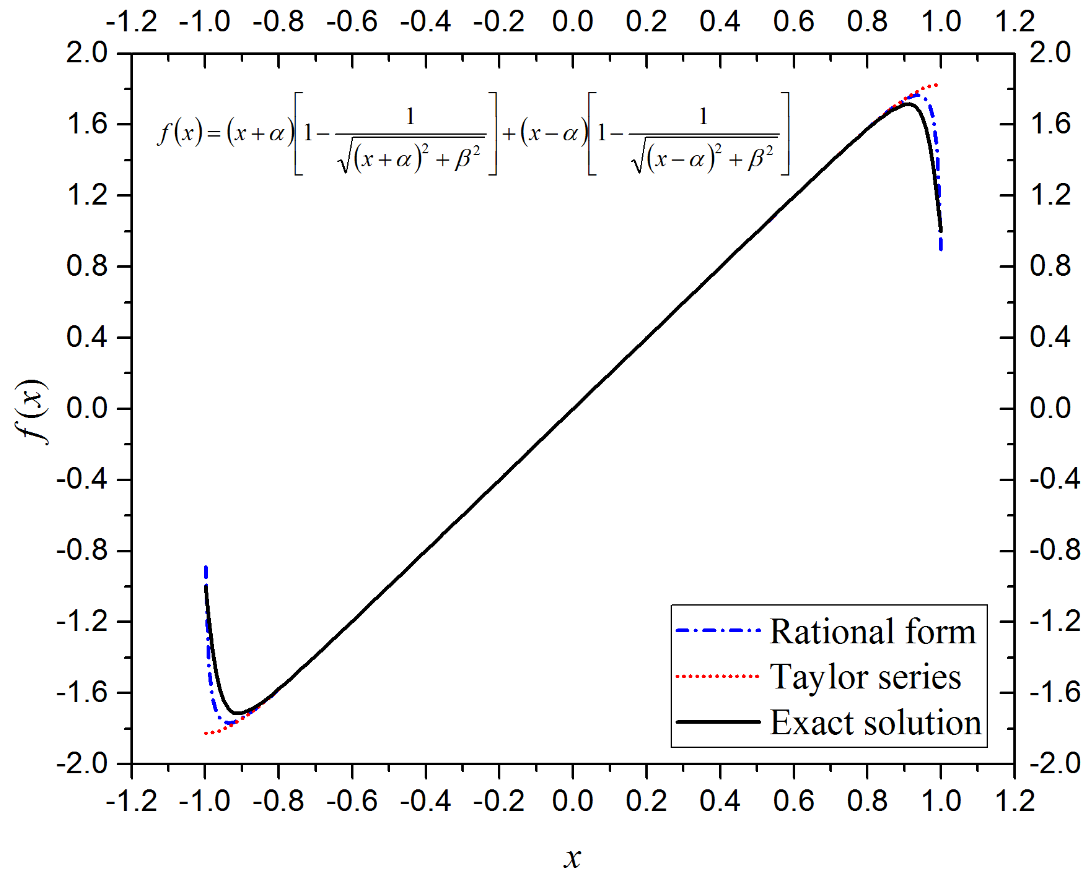
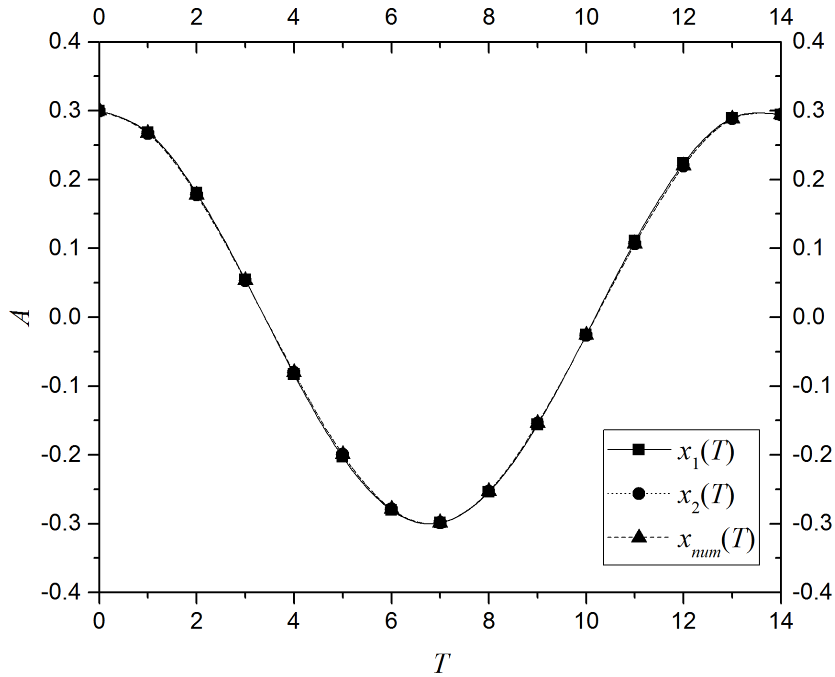
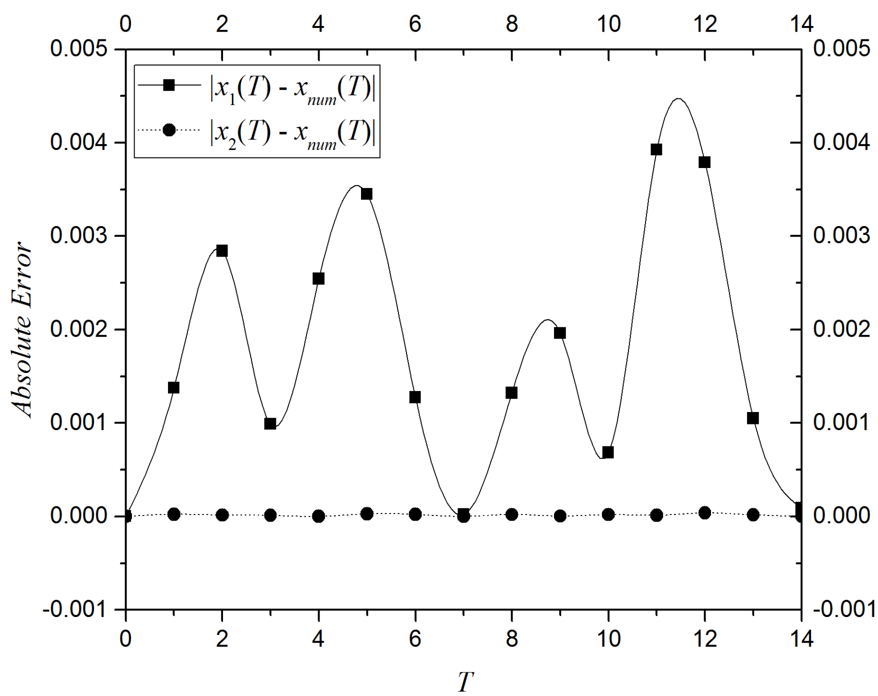


Fig. 3. Comparison of various forms of the irrational restoring force for $\alpha = 0.999$, $\beta = 0.045$ and $x \in [-0.999, 0.999]$ (Rational form: $f(x) = \sum_{i=0}^{4} a_{2i+1} x^{2i+1} / \sum_{i=0}^{4} b_{2i} x^{2i}$; Taylor series: $f(x) = \sum_{i=0}^{8} \hat{a}_{2i+1} x^{2i+1}$).

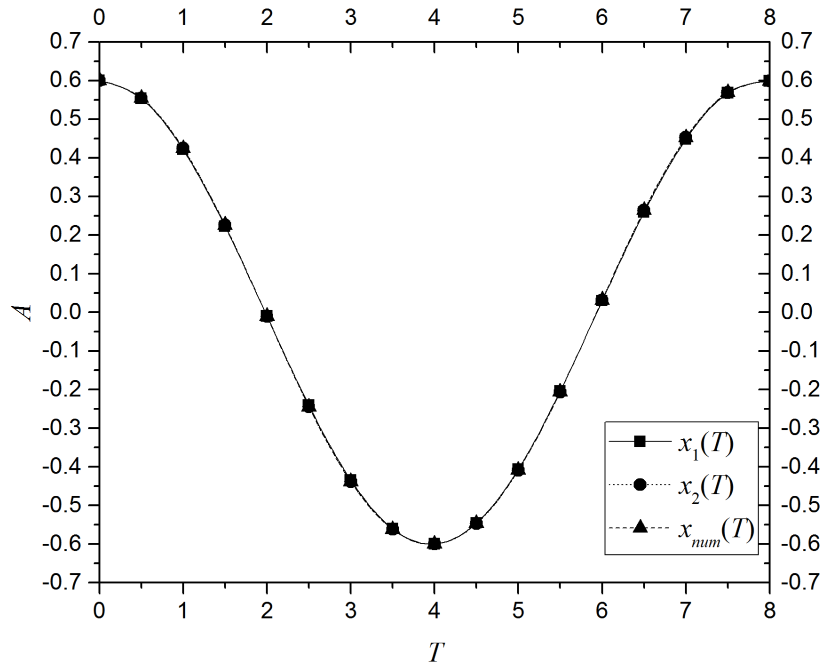


(a)

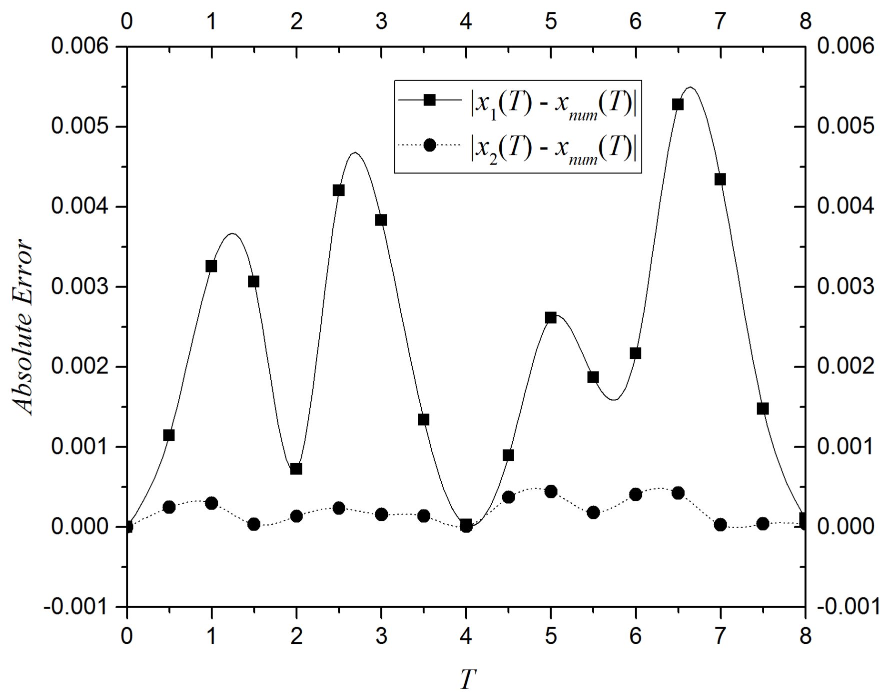


(b)

Fig. 4. (a) Comparison of the time history response of Eq. (2) for $\alpha = 0.3$, $\beta = 0.9539$, and $A = 0.3$; (b) Comparison of the absolute errors between the approximate and numerical periodic solutions in this case.

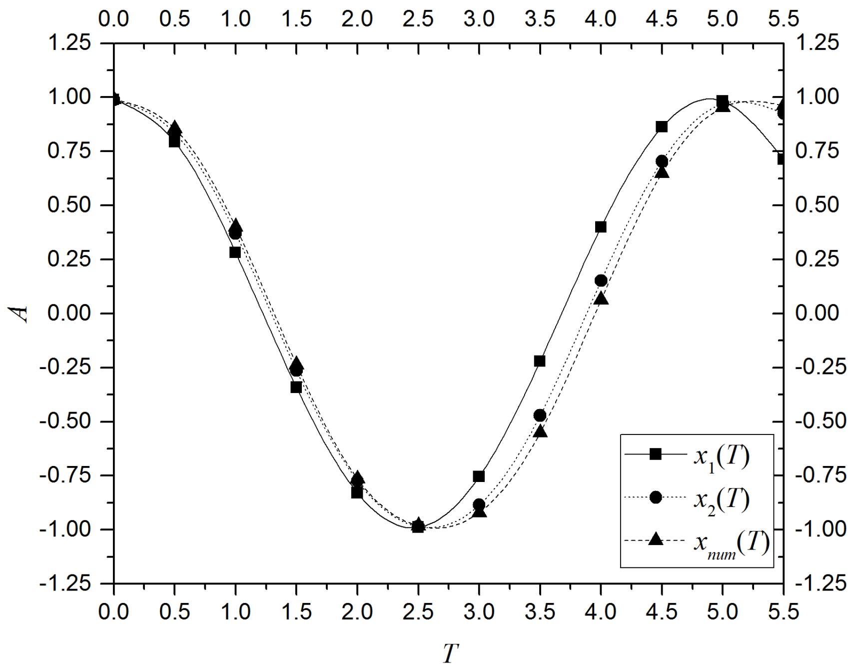


(a)

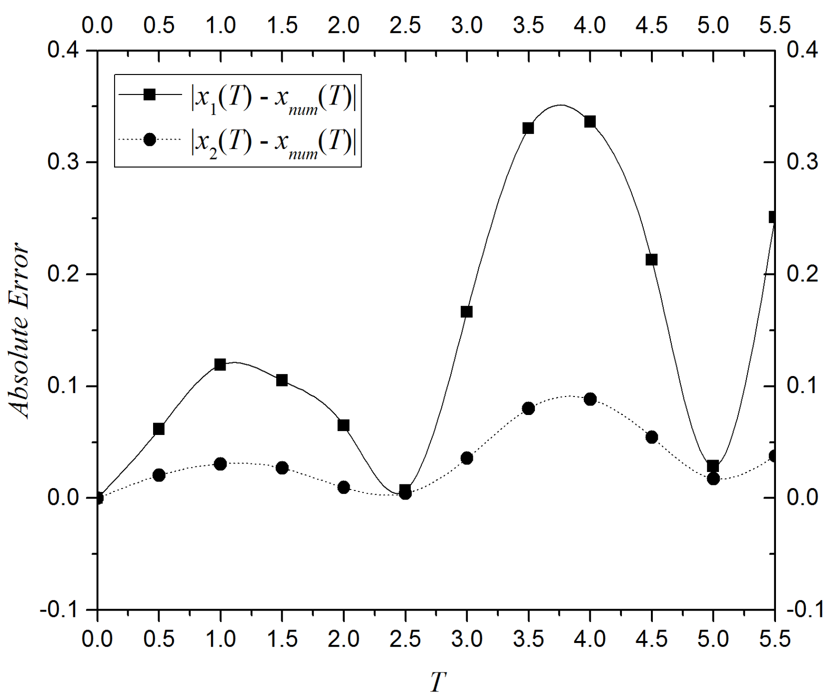


(b)

Fig. 5.(a) Comparison of the time history response of Eq. (2) for $\alpha = 0.6$, $\beta = 0.8$, and $A = 0.6$; (b) Comparison of the absolute errors between the approximate and numerical periodic solutions in this case.



(a)



(b)

Fig. 6. (a) Comparison of the time history response of Eq. (2) for $\alpha = 0.99$, , $\beta = 0.1411$ and $A = 0.99$; (b) Comparison of the absolute errors between the approximate and numerical periodic solutions in this case.