Dynamic condensation approach to the calculation of structural responses and response sensitivities

ABSTRACT

Structural responses and response sensitivities are widely used in the updating of the finite element model, including damage identification and optimization of the design. Calculation of the responses and response sensitivities of a large structure consumes considerable computation storage and is usually a time-consuming process. This paper proposes an improved dynamic condensation approach to calculate the structural responses and response sensitivities. The condensed vibration equation is achieved by a simplified iterative scheme. By selecting the DOFs associated with the concerned element to be master DOFs, the response sensitivity is rapidly calculated from the derivatives of the master stiffness and mass matrices. Since the condensed vibration equation has a much smaller size than the original equation, the proposed method is quite efficient to calculate the structural responses and response sensitivities. Finally, applications of the proposed method to an eight-story frame and a cantilever plate demonstrate that the proposed method is more accurate and efficient in the calculation of structural responses and response sensitivities.

Keywords: dynamic condensation, model updating, structural response, response sensitivity

1. Introduction

The sensitivity method is probably the most successful approach in updating the finite element (FE) model of engineering structures based on vibration testing data [1]. In vibration-based model updating and damage identification, the FE model is iteratively modified to ensure its vibration properties reproduce the measured counterparts in an optimal manner. In the optimization process, the structural responses are used to construct the objective function. The response sensitivities, which give the first derivatives of the structural responses with respect to the designed parameters, indicate the searching direction [2-4]. To accurately describe the practical structures, the analytical model is usually represented by a large model, including a large number of elements, degrees of freedom (DOFs) and structural parameters. Calculation of the structural responses of a large structure and the response sensitivities with respect to numerous designed parameters usually requires considerable computation storage and is a time-consuming process [1-4].

Model condensation, as an efficient technique to reduce the computational loads of large structures, was first applied to large FE models for faster computation of the natural frequencies and mode shapes. Model condensation methods remove some DOFs (slave DOFs) of the original FE model and represent the discarded DOFs with the retained DOFs (master DOFs). Afterwards, the eigen-functions of the condensed model are solved to approximate those of the original structures [5, 6]. The model condensation technique is advantageous for solving a variety of engineering and mechanical problems [5-19]. Since the number of the master DOFs is much less than the total number of DOFs of the full model, the model condensation technique is helpful to reduce the large-size model and thus saves both computational resources

and time [5-9]. In addition, the model condensation technique becomes more promising if it is combined with the substructuring methods. The model condensation is performed on the independent substructures and on the interface coordinates of the substructures to improve computational efficiency [10-19].

Guyan [20] first proposed the static condensation technique to calculate eigensolutions. Friswell et al. [21, 22] developed a dynamic and improved reduced system (IRS) using an iterative method to obtain eigensolutions. Xia and Lin [6] improved the IRS method and achieved much faster convergence of the IRS method. Choi et al. [14] presented an iterated improved reduced procedure and a substructuring scheme for both the undamped and nonclassically damped structures. Weng et al. [23] extended the model condensation technique to calculate the eigensensitivities using the iterated dynamic condensation algorithm. These methods provided an efficient solution for large-scale eigenvalue problems. Soheilifard [24] extended the Guyan condensation method to the damped structures with a hierarchical non-iterative reduction method. Lima et al. [25] addressed a time domain condensation strategy for the viscoelastic linear and nonlinear systems, in which the viscoelastic behaviour is modelled with a four parameter fractional derivative model. As the precision and efficiency of model condensation methods are closely related to the selection of master DOFs, Bouhaddi and Fillod [26] proposed an approach to selection of the master DOFs of the Guyan condensation method. Jeong et al. [27] proposed a rational primary DOFs selection method for a damped system according to the energy distribution of a structure. A variety of condensation methods have been developed to calculate the structural responses, whereas the sensitivity analysis by the condensed model is rarely studied. As calculation of response sensitivity usually

consumes the majority of computation time in model updating, an effective condensed model for efficient calculation of response sensitivities is valuable.

In this paper, the structural responses and response sensitivities are calculated based on an improved dynamic condensation algorithm. A dynamic transformation matrix, obtained using a simplified iterative scheme, is derived to relate the responses of the master DOFs to the slave DOFs. Next, the large-scale global vibration equation is reduced into a condensed equation. By including the DOFs of the concerned element in the master DOFs, the response sensitivities are computed efficiently from the derivatives of the master stiffness and the mass matrices. The proposed method for the calculation of structural responses and response sensitivities is illustrated by an eight-story frame and a cantilever plate.

2. Dynamic condensation to structural responses

The vibration equation of a structure with N DOFs is expressed as

$$\mathbf{M}\ddot{\mathbf{x}}(\tau)+\mathbf{C}\dot{\mathbf{x}}(\tau)+\mathbf{K}\mathbf{x}(\tau)=\mathbf{F}(\tau) \tag{1}$$

where **M**, **C** and **K** represent the mass, damping and stiffness matrices, respectively. $\mathbf{F}(\tau)$ is the excitation of the structure at time step τ. The structure is assumed to exhibit Rayleigh damping as $\mathbf{C}=a_1\mathbf{M}+a_2\mathbf{K}$, where a_1 and a_2 are the Rayleigh damping coefficients. $\ddot{\mathbf{x}}(\tau)$, $\dot{\mathbf{x}}(\tau)$ and $\mathbf{x}(\tau)$ are the acceleration, velocity and displacement of the structure, respectively, at time step τ.

Dividing the total DOFs of a structure into n_m master DOFs and n_s slave DOFs, the vibration equation is divided according to the master and slave DOFs into the following [5, 6]:

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{ms}^{\mathrm{T}} & \mathbf{M}_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_m(\tau) \\ \ddot{\mathbf{x}}_s(\tau) \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_{mm} & \mathbf{C}_{ms} \\ \mathbf{C}_{ms}^{\mathrm{T}} & \mathbf{C}_{ss} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{x}}_m(\tau) \\ \dot{\mathbf{x}}_s(\tau) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_m(\tau) \\ \mathbf{x}_s(\tau) \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_m(\tau) \\ \mathbf{F}_s(\tau) \end{Bmatrix} \quad (2)$$

where the subscript 'm' represents the master DOFs, the subscript 's' represents the slave DOFs, and $N=n_m+n_s$. The superscript 'T' represents the transposed matrix.

A transformation matrix $\mathbf{t}$ is employed to relate the master DOFs and slave DOFs, and it is invariant to time and excitation, as follows [28]:

$$\mathbf{x}(\tau) = \begin{bmatrix} \mathbf{x}_m(\tau) \\ \mathbf{x}_s(\tau) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_m(\tau) \\ \mathbf{t}\mathbf{x}_m(\tau) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{t} \end{bmatrix} \mathbf{x}_m(\tau) = \mathbf{T}\mathbf{x}_m(\tau) \quad (3)$$

where $\mathbf{I}_m$ is an identity matrix with the size of $n_m \times n_m$. $\mathbf{T} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{t} \end{bmatrix}$ has the size of $N \times n_m$, and $\mathbf{t}$ has the size of $n_s \times n_m$.

Substituting Eq. (3) into Eq. (1) and premultiplying $\mathbf{T}^{\mathrm{T}}$ gives

$$\mathbf{M}_R \ddot{\mathbf{x}}_m(\tau) + \mathbf{C}_R \dot{\mathbf{x}}_m(\tau) + \mathbf{K}_R \mathbf{x}_m(\tau) = \mathbf{F}_R(\tau) \quad (4)$$

where

$$\mathbf{M}_R = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T} = \left(\mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}\right) + \mathbf{t}^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right) \quad (5a)$$

$$\mathbf{K}_R = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} = \left(\mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}\right) + \mathbf{t}^{\mathrm{T}}\left(\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}\right) \quad (5b)$$

$$\mathbf{C}_R = \mathbf{T}^{\mathrm{T}}\mathbf{C}\mathbf{T} = \mathbf{T}^{\mathrm{T}}\left(a_1\mathbf{M} + a_2\mathbf{K}\right)\mathbf{T} = a_1\mathbf{M}_R + a_2\mathbf{K}_R \quad (5c)$$

$$\mathbf{F}_R(\tau) = \mathbf{T}^{\mathrm{T}}\mathbf{F}(\tau) = \mathbf{F}_m(\tau) + \mathbf{t}^{\mathrm{T}}\mathbf{F}_s(\tau) \quad (5d)$$

The size of the condensed vibration equation is equal to the number of the master DOFs ($n_m \times n_m$), which is much smaller than that of the original vibration equation ($N \times N$). The pivotal task to form the condensed vibration equation is to calculate the transformation matrix $\mathbf{t}$.

As the matrix $\mathbf{t}$ is a constant matrix to relate the master and slave DOFs and it is irrelevant to the time and external force (including the external excitation and damping force), the free vibration system shares the same matrix $\mathbf{t}$ with the excited system [28]. For the sake of clarity, the same free vibration equation of an undamped system is used for illustration and is written as

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{ms}^{\mathrm{T}} & \mathbf{M}_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_m(\tau) \\ \ddot{\mathbf{x}}_s(\tau) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^{\mathrm{T}} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_m(\tau) \\ \mathbf{x}_s(\tau) \end{Bmatrix} = \mathbf{0} \tag{6}$$

The second line of Eq. (6) gives

$$\mathbf{x}_s(\tau) = -\mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}}\ddot{\mathbf{x}}_m(\tau) + \mathbf{M}_{ss}\ddot{\mathbf{x}}_s(\tau) + \mathbf{K}_{ms}^{\mathrm{T}}\mathbf{x}_m(\tau)\right) \tag{7}$$

Considering Eq. (3) & Eq. (7) can be rewritten as

$$\mathbf{t}\mathbf{x}_m(\tau) = -\mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\ddot{\mathbf{x}}_m(\tau) + \mathbf{t}_G\mathbf{x}_m(\tau) \tag{8}$$

where $\mathbf{t}_G = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}$ is equivalent to the transformation matrix in the Guyan condensation method, and it is only precise for static problems [20, 24].

Since the proportional damping does not affect the dynamic condensation of the system matrices [28], Eq. (4) leads to the free vibration equation of an undamped condensed system, as follows:

$$\mathbf{M}_R\ddot{\mathbf{x}}_m(\tau) + \mathbf{K}_R\mathbf{x}_m(\tau) = \mathbf{0} \tag{9}$$

The accelerations of the master DOFs are written by their displacements as

$$\ddot{\mathbf{x}}_m(\tau) = -\mathbf{M}_R^{-1}\mathbf{K}_R\mathbf{x}_m(\tau) \tag{10}$$

Substituting Eq. (10) into Eq. (8) leads to

$$\mathbf{t} = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\mathbf{M}_R^{-1}\mathbf{K}_R + \mathbf{t}_G = \mathbf{t}_G + \mathbf{t}_d \tag{11}$$

The matrix $\mathbf{t}_d = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\mathbf{M}_R^{-1}\mathbf{K}_R$ is nonlinear and is a function of the undetermined variables $\mathbf{t}$, $\mathbf{M}_R$ and $\mathbf{K}_R$. To improve the computational efficiency of $\mathbf{t}$, in this paper, the dynamic condensation matrices $\mathbf{M}_R$ and $\mathbf{K}_R$ are simplified as follows:

$$\begin{aligned}\mathbf{K}_R &= \mathbf{K}_{mm} + \mathbf{K}_{ms}\left(\mathbf{t}_G + \mathbf{t}_d\right) + \left(\mathbf{t}_G^{\mathrm{T}} + \mathbf{t}_d^{\mathrm{T}}\right)\left[\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\left(\mathbf{t}_G + \mathbf{t}_d\right)\right] \\ &= \mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}_G + \left(\mathbf{K}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{K}_{ss}\right)\mathbf{t}_d + \left(\mathbf{t}_G^{\mathrm{T}} + \mathbf{t}_d^{\mathrm{T}}\right)\left(\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}_G\right) \\ &\quad + \mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d\end{aligned} \tag{12}$$

Noting that $\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}_G = \mathbf{0}$ and, similarly, $\mathbf{K}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{K}_{ss} = \mathbf{0}$, Eq. (12) can be simplified as

$$\mathbf{K}_R = \mathbf{K}_G + \mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d \tag{13}$$

where $\mathbf{K}_G$=$\mathbf{K}_{mm}$+$\mathbf{K}_{ms}\mathbf{t}_G$ is the stiffness matrix of the Guyan condensation method [20].

Similarly, the condensed mass matrix can be written as

$$\begin{aligned}\mathbf{M}_R &= \mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}_G + \mathbf{t}_G^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right) + \mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right) + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d + \mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d \\ &= \mathbf{M}_G + \mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right) + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d \\ &= \mathbf{M}_G + \mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right) + \left(\mathbf{M}_{ms} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d + \mathbf{t}_d^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d\end{aligned} \tag{14}$$

where $\mathbf{M}_G = \mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}_G + \mathbf{t}_G^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}_G\right)$ is the mass matrix of the Guyan condensation method [20].

Substituting Eqs. (13) and (14) into Eq. (9) gives

$$\begin{aligned}&\mathbf{M}_R\ddot{\mathbf{x}}_m(\tau)+\mathbf{K}_R\mathbf{x}_m(\tau)\\&=\left[\mathbf{M}_G+\left(\mathbf{M}_{ms}+\mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d+\mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}}+\mathbf{M}_{ss}\mathbf{t}\right)\right]\ddot{\mathbf{x}}_m(\tau)+\left(\mathbf{K}_G+\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d\right)\mathbf{x}_m(\tau)\\&=\left[\mathbf{M}_G+\left(\mathbf{M}_{ms}+\mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d\right]\ddot{\mathbf{x}}_m(\tau)+\mathbf{K}_G\mathbf{x}_m(\tau)+\mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}}+\mathbf{M}_{ss}\mathbf{t}\right)\ddot{\mathbf{x}}_m(\tau)+\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d\mathbf{x}_m(\tau)\end{aligned} \tag{15}$$

As a consequence, Eq. (9) can be rewritten as

$$\left[\mathbf{M}_G+\left(\mathbf{M}_{ms}+\mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\right)\mathbf{t}_d\right]\ddot{\mathbf{x}}_m(\tau)+\mathbf{K}_G\mathbf{x}_m(\tau)+\mathbf{t}_d^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}}+\mathbf{M}_{ss}\mathbf{t}\right)\ddot{\mathbf{x}}_m(\tau)+\mathbf{t}_d^{\mathrm{T}}\mathbf{K}_{ss}\mathbf{t}_d\mathbf{x}_m(\tau)=\mathbf{0} \tag{16}$$

Substituting Eq. (8) into Eq. (16), the left side of Eq. (16) is written as

$$\begin{aligned}&\mathbf{M}_d\ddot{\mathbf{x}}_m(\tau)+\mathbf{K}_G\mathbf{x}_m(\tau)+\mathbf{t}_d^{\mathrm{T}}\left(-\mathbf{K}_{ss}\left(\mathbf{t}-\mathbf{t}_G\right)+\mathbf{K}_{ss}\mathbf{t}_d\right)\mathbf{x}_m(\tau)\\&=\mathbf{M}_d\ddot{\mathbf{x}}_m(\tau)+\mathbf{K}_G\mathbf{x}_m(\tau)+\mathbf{t}_d^{\mathrm{T}}\left(-\mathbf{K}_{ss}\mathbf{t}_d+\mathbf{K}_{ss}\mathbf{t}_d\right)\mathbf{x}_m(\tau)\\&=\mathbf{M}_d\ddot{\mathbf{x}}_m(\tau)+\mathbf{K}_G\mathbf{x}_m(\tau)\end{aligned} \tag{17a}$$

where

$$\mathbf{M}_d=\mathbf{M}_G+\mathbf{M}_{ms}\mathbf{t}_d+\mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_d=\mathbf{M}_{mm}+\mathbf{M}_{ms}\mathbf{t}+\mathbf{t}_G^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}}+\mathbf{M}_{ss}\mathbf{t}\right) \tag{17b}$$

Consequently, Eq. (16) can be straightforwardly simplified as

$$\mathbf{M}_d\ddot{\mathbf{x}}_m(\tau)+\mathbf{K}_G\mathbf{x}_m(\tau)=\mathbf{0} \tag{18}$$

Eq. (18) leads to

$$\ddot{\mathbf{x}}_m(\tau)=-\mathbf{M}_d^{-1}\mathbf{K}_G\mathbf{x}_m(\tau) \tag{19}$$

Substituting Eq. (19) into Eq. (8) gives

$$\mathbf{t}=\mathbf{t}_G+\mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}}+\mathbf{M}_{ss}\mathbf{t}\right)\mathbf{M}_d^{-1}\mathbf{K}_G \tag{20}$$

In the present condensation technique, the transformation matrix **t** is estimated based on Eq. (20). Note that $\mathbf{K}_G$ is constant, and only $\mathbf{M}_d$ is required to estimate **t,** avoiding

the repeated calculation of $\mathbf{K}_R$ and $\mathbf{M}_R$ in Eq. (11). This simplified procedure is achieved by the following steps.

1) The iterations are started by the Guyan static condensation, as follows:

$$\mathbf{t}^{[0]} = \mathbf{t}_G = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}} \tag{21a}$$

$$\mathbf{M}_d^{[0]} = \mathbf{M}_G = \mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}_G + \left(\mathbf{M}_{ms}\mathbf{t}_G\right)^{\mathrm{T}} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_G \tag{21b}$$

$$\mathbf{K}_G = \mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}_G \tag{21c}$$

2) The transformation matrix $\mathbf{t}$ is updated iteratively. In the kth (k=1, 2, 3…) iteration, $\mathbf{t}$ is obtained by

$$\mathbf{t}^{[k]} = \mathbf{t}_G + \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{[k-1]}\right)\left(\mathbf{M}_d^{[k-1]}\right)^{-1}\mathbf{K}_G \tag{22}$$

$$\mathbf{M}_d^{[k]} = \mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{[k]} + \mathbf{t}_G^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{[k]}\right) \tag{23}$$

3) The iterations continue until the differences of eigenvalues $\lambda^{[k]} = eig\left(\left(\mathbf{M}_d^{[k]}\right)^{-1}\mathbf{K}_G\right)$ from two consecutive iterations are less than the predefined tolerance, as follows:

$$error = \left|\frac{\lambda^{[k]} - \lambda^{[k-1]}}{\lambda^{[k-1]}}\right| < tol \tag{24}$$

The transformation matrix $\mathbf{t}$ can be achieved by $\mathbf{t} = \mathbf{t}^{[k]}$.

4) Next, the condensed matrices are formed by the transformation matrix $\mathbf{t}$ by

$$\mathbf{M}_R = \mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t} + \mathbf{t}^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right) \tag{25a}$$

$$\mathbf{K}_R = \mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t} + \mathbf{t}^{\mathrm{T}}\left(\mathbf{K}_{ms}^{\mathrm{T}} + \mathbf{K}_{ss}\mathbf{t}\right) \tag{25b}$$

$$\mathbf{C}_R = a_1\mathbf{M}_R + a_2\mathbf{K}_R \tag{25c}$$

$$\mathbf{F}_R(\tau) = \mathbf{F}_m(\tau) + \mathbf{t}^{\mathrm{T}}\mathbf{F}_s(\tau) \tag{25d}$$

5) Finally, the displacement, velocity and acceleration of the master DOFs can be solved according to Eq. (4) with the Newmark method [3, 29]. The responses of the whole DOFs are recovered from the master DOFs by the transformation matrix $\mathbf{T}$.

$$\mathbf{T}=\begin{bmatrix}\mathbf{I}_m\\ \mathbf{t}\end{bmatrix},\ddot{\mathbf{x}}(\tau)=\mathbf{T}\ddot{\mathbf{x}}_m(\tau),\dot{\mathbf{x}}(\tau)=\mathbf{T}\dot{\mathbf{x}}_m(\tau),\mathbf{x}(\tau)=\mathbf{T}\mathbf{x}_m(\tau) \tag{26}$$

Because the computational effort of the structural analysis is approximately proportional to the cubic of the size of the system [5], the computational work could be reduced drastically if the size of the vibration equation to calculate the structural responses and response sensitivities is reduced.

3. Dynamic condensation to structural response sensitivities

The structural response sensitivities are the derivatives of the structural responses with respect to the designed elemental parameters. In this section, the dynamic condensation approach is used to calculate the structural response sensitivities.

Eq. (4) is differentiated with respect to an elemental parameter r on both sides [3, 4]:

$$\mathbf{M}_R\frac{\partial\ddot{\mathbf{x}}_m(\tau)}{\partial r}+\mathbf{C}_R\frac{\partial\dot{\mathbf{x}}_m(\tau)}{\partial r}+\mathbf{K}_R\frac{\partial\mathbf{x}_m(\tau)}{\partial r}=-\frac{\partial\mathbf{M}_R}{\partial r}\ddot{\mathbf{x}}_m(\tau)-\frac{\partial\mathbf{K}_R}{\partial r}\mathbf{x}_m(\tau)-a_1\frac{\partial\mathbf{M}_R}{\partial r}\dot{\mathbf{x}}_m(\tau)-a_2\frac{\partial\mathbf{K}_R}{\partial r}\dot{\mathbf{x}}_m(\tau) \tag{27}$$

In the original full model, the stiffness matrix **K** and the mass matrix **M** are assembled from the elemental stiffness matrices and the elemental mass matrices of all elements in the discrete full model. Therefore, **K** and **M** can be expressed as [1]

$$\mathbf{K}=\sum_{j=1}^{n}\mathbf{K}_j=\sum_{j=1}^{n}\alpha_j\mathbf{K}_j^e,\mathbf{M}=\sum_{j=1}^{n}\mathbf{M}_j=\sum_{j=1}^{n}\beta_j\mathbf{M}_j^e \tag{28}$$

where $\mathbf{K}_j$ and $\mathbf{M}_j$ are the jth elemental stiffness matrix and the jth elemental mass matrix, respectively, and α_j and β_j are the "elemental stiffness parameter" and "elemental mass parameter", respectively. The elemental parameters are linearly related to the elemental stiffness and mass matrices. Eq. (28) shows that the change of

an elemental parameter is localized in the stiffness and mass matrices of the original full model (**K** and **M**). However, if the stiffness and mass matrices (**K** and **M**) of the original full model are condensed to $\mathbf{K}_R$ and $\mathbf{M}_R$, then the effects caused by the change of the specific elemental parameter r are not localized any further and can spread over the whole condensed matrices.

In this paper, the DOFs associated with the elemental parameter r are included in the master DOFs, and the perturbation caused by the change of an elemental parameter is constrained within the stiffness and mass matrices of the master DOFs, whereas the derivative matrices related to the slave DOFs are zeros, i.e.,

$$\frac{\partial \mathbf{M}_{ms}}{\partial r}=\mathbf{0},\frac{\partial \mathbf{M}_{ss}}{\partial r}=\mathbf{0},\frac{\partial \mathbf{K}_{ms}}{\partial r}=\mathbf{0},\frac{\partial \mathbf{K}_{ss}}{\partial r}=\mathbf{0} \tag{29}$$

As a result, the derivatives of the condensed mass and stiffness matrices are simplified as

$$\frac{\partial \mathbf{M}_R}{\partial r}=\frac{\partial \mathbf{M}_{mm}}{\partial r}+\mathbf{M}_{ms}\frac{\partial \mathbf{t}}{\partial r}+\frac{\partial \mathbf{t}^{\mathrm{T}}}{\partial r}\left(\mathbf{M}_{ms}^{\mathrm{T}}+\mathbf{M}_{ss}\mathbf{t}\right)+\mathbf{t}^{\mathrm{T}}\mathbf{M}_{ss}\frac{\partial \mathbf{t}}{\partial r} \tag{30}$$

$$\frac{\partial \mathbf{K}_R}{\partial r}=\frac{\partial \mathbf{K}_{mm}}{\partial r}+\mathbf{K}_{ms}\frac{\partial \mathbf{t}}{\partial r}+\frac{\partial \mathbf{t}^{\mathrm{T}}}{\partial r}\left(\mathbf{K}_{ms}^{\mathrm{T}}+\mathbf{K}_{ss}\mathbf{t}\right)+\mathbf{t}^{\mathrm{T}}\mathbf{K}_{ss}\frac{\partial \mathbf{t}}{\partial r} \tag{31}$$

It implies that the derivatives of the condensed stiffness and mass matrices depend on the derivatives of the stiffness and mass matrices of the master DOFs ($\frac{\partial \mathbf{K}_{mm}}{\partial r}$ and $\frac{\partial \mathbf{M}_{mm}}{\partial r}$), and the derivatives of the transformation matrix $\frac{\partial \mathbf{t}}{\partial r}$. $\frac{\partial \mathbf{K}_{mm}}{\partial r}$ and $\frac{\partial \mathbf{M}_{mm}}{\partial r}$

are constant values. The pivot task to achieve the response sensitivities is to search the accurate derivative of the transformation matrix $\frac{\partial \mathbf{t}}{\partial r}$.

Since the transformation matrix $\mathbf{t}$ is determined by $\mathbf{M}_d$ in Section 2 (Eq. (17b) and Eq. (20)), the derivative matrix $\frac{\partial \mathbf{t}}{\partial r}$ can be achieved based on the following:

$$\frac{\partial \mathbf{M}_d}{\partial r} = \frac{\partial \mathbf{M}_{mm}}{\partial r} + \mathbf{M}_{ms}\frac{\partial \mathbf{t}}{\partial r} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\frac{\partial \mathbf{t}}{\partial r} \tag{32}$$

$$\frac{\partial \mathbf{t}}{\partial r} = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}\right)\left(-\mathbf{M}_d^{-1}\frac{\partial \mathbf{M}_d}{\partial r}\mathbf{M}_d^{-1}\mathbf{K}_G + \mathbf{M}_d^{-1}\frac{\partial \mathbf{K}_G}{\partial r}\right) + \mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\frac{\partial \mathbf{t}}{\partial r}\mathbf{M}_d^{-1}\mathbf{K}_G \tag{33}$$

Eq. (33) implies that $\frac{\partial \mathbf{t}}{\partial r}$ depends on the variables $\mathbf{t}$, $\mathbf{M}_d$ and $\frac{\partial \mathbf{M}_d}{\partial r}$. The calculation of matrices $\mathbf{t}$ and $\mathbf{M}_d$ was explained in Section 2. The following procedure is conducted to calculate $\frac{\partial \mathbf{t}}{\partial r}$ and the response sensitivities.

1) $\frac{\partial \mathbf{t}}{\partial r}$ and $\frac{\partial \mathbf{M}_d}{\partial r}$ are initialized as

$$\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[0]} = \frac{\partial \mathbf{t}_G}{\partial r} = \frac{\partial\left(-\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}\right)}{\partial r} = \mathbf{0} \tag{34}$$

$$\left[\frac{\partial \mathbf{M}_d}{\partial r}\right]^{[0]} = \frac{\partial \mathbf{M}_{mm}}{\partial r} + \mathbf{M}_{ms}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[0]} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[0]} = \frac{\partial \mathbf{M}_{mm}}{\partial r} \tag{35}$$

2) The derivative of the transformation matrix $\frac{\partial \mathbf{t}}{\partial r}$ is updated iteratively. In the kth (k=1, 2, 3...) iteration,

$$\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k]} = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{[k]}\right)\left(-\left(\mathbf{M}_d^{[k]}\right)^{-1}\left[\frac{\partial \mathbf{M}_d}{\partial r}\right]^{[k-1]}\left(\mathbf{M}_d^{[k]}\right)^{-1}\mathbf{K}_G + \left(\mathbf{M}_d^{[k]}\right)^{-1}\frac{\partial \mathbf{K}_G}{\partial r}\right) + \mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k-1]}\left(\mathbf{M}_d^{[k]}\right)^{-1}\mathbf{K}_G \tag{36}$$

$$\left[\frac{\partial \mathbf{M}_d}{\partial r}\right]^{[k]} = \frac{\partial \mathbf{M}_{mm}}{\partial r} + \mathbf{M}_{ms}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k]} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k]} \tag{37}$$

3) The iterations are terminated when the differences of the eigenvalues $\lambda^{[k]} = eig\left(\left[\frac{\partial \mathbf{M}_d}{\partial r}\right]^{[k]} \mathbf{K}_G\right)$ from two consecutive iterations are less than the predefined tolerance.

$$error = \left|\frac{\lambda^{[k]} - \lambda^{[k-1]}}{\lambda^{[k-1]}}\right| < tol \tag{38}$$

The derivative of the transformation matrix $\frac{\partial \mathbf{t}}{\partial r}$ can be obtained by $\frac{\partial \mathbf{t}}{\partial r} = \left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k]}$.

4) The derivatives of the condensed stiffness and mass matrices ($\frac{\partial \mathbf{K}_R}{\partial r}$ and $\frac{\partial \mathbf{M}_R}{\partial r}$) are calculated according to Eq. (30) and Eq. (31), respectively.

5) The structural responses gained in Section 2 are used here to form the effective force. According to Eq. (27), the derivatives of the acceleration, velocity and displacement ($\frac{\partial \ddot{\mathbf{x}}_m(\tau)}{\partial r}$, $\frac{\partial \dot{\mathbf{x}}_m(\tau)}{\partial r}$ and $\frac{\partial \mathbf{x}_m(\tau)}{\partial r}$) of the master DOFs are solved by the Newmark method [3, 29].

6) The structural response sensitivities of the whole DOFs are recovered from the response sensitivities of the master DOFs by matrix **T** as

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{t} \end{bmatrix}, \frac{\partial \mathbf{T}}{\partial r} = \begin{bmatrix} \mathbf{0} \\ \frac{\partial \mathbf{t}}{\partial r} \end{bmatrix} \tag{39a}$$

$$\frac{\partial \ddot{\mathbf{x}}(\tau)}{\partial r} = \frac{\partial\left[\mathbf{T}\ddot{\mathbf{x}}_m(\tau)\right]}{\partial r} = \frac{\partial \mathbf{T}}{\partial r}\ddot{\mathbf{x}}_m + \mathbf{T}\frac{\partial \ddot{\mathbf{x}}_m(\tau)}{\partial r} \tag{39b}$$

$$\frac{\partial \dot{\mathbf{x}}(\tau)}{\partial r} = \frac{\partial\left[\mathbf{T}\dot{\mathbf{x}}_m(\tau)\right]}{\partial r} = \frac{\partial \mathbf{T}}{\partial r}\dot{\mathbf{x}}_m + \mathbf{T}\frac{\partial \dot{\mathbf{x}}_m(\tau)}{\partial r} \tag{39c}$$

$$\frac{\partial \mathbf{x}(\tau)}{\partial r} = \frac{\partial\left[\mathbf{T}\mathbf{x}_m(\tau)\right]}{\partial r} = \frac{\partial \mathbf{T}}{\partial r}\mathbf{x}_m + \mathbf{T}\frac{\partial \mathbf{x}_m(\tau)}{\partial r} \tag{39d}$$

Usually, the response sensitivities are required together with the structural responses. The structural responses and response sensitivities can be calculated simultaneously by combining their iterative schemes together.

In this paper, by selecting the DOFs of the concerned element as master DOFs, $\frac{\partial \mathbf{M}_d}{\partial r}$ is calculated by the derivatives of the stiffness and mass matrices of the master DOFs ($\frac{\partial \mathbf{K}_{mm}}{\partial r}$ and $\frac{\partial \mathbf{M}_{mm}}{\partial r}$), while the derivatives related to the slave DOFs are zeros (as Eq.(29)). Based on the simplified condensed vibration equation, $\frac{\partial \mathbf{t}}{\partial r}$ is calculated from $\frac{\partial \mathbf{M}_d}{\partial r}$, avoiding the repeated calculation of $\frac{\partial \mathbf{K}_R}{\partial r}$ and $\frac{\partial \mathbf{M}_R}{\partial r}$.

4. The computational speed of the proposed method

Computational speed is a necessary consideration for the dynamic analysis of large-scale structures. One criterion commonly used to estimate the computational speed is the number of multiplications consumed in the whole process, which is represented by the multiplication count (MC) [5]. In this study, the assembled stiffness and mass matrices are carefully banded and sparse to reduce the computational cost. Here, the half-bandwidths of the global stiffness and mass matrices are assumed to be b ($b_{\mathbf{K}}$=$b_{\mathbf{M}}$=b).

The structural responses and response sensitivities are calculated by both the traditional full model method and the proposed dynamic condensation method for

comparison. First, employing the traditional Newmark method on the full model, the calculation of the structural responses consumes [29]

$$\mathrm{MC}_{response} = \left(Nb^2 + 10Nb\right) + n\left(6Nb + 11N\right) \tag{40}$$

where n is the number of the time steps of excitation. The first term denotes the MC consumed in the initialization of the Newmark method, and the second term is consumed by the n time steps to achieve the structural responses.

Next, the response sensitivities are calculated by the traditional full model method. The structural responses are taken as a constant to form the effective force in the Newmark method [29]. The MC is computed as

$$\mathrm{MC}_{sensitivity} = \left(4Nb + 8nNb\right) + \left[\left(Nb^2 + 10Nb\right) + n\left(6Nb + 11N\right)\right] \tag{41}$$

where the first term is contributed by the process to form the effective force. The second term represents the MC cost in calculating the response sensitivities with the Newmark method.

Alternatively, the MC consumed by the structural responses and response sensitivities with the proposed dynamic condensation method is estimated. If the proposed method achieves the condensed model after p iterations, then the MC is [5]

$$\begin{aligned}\mathrm{MC}_{R_{response}} = &\left(n_s b^2/2 + 4n_s n_m b + 2n_m^2 b + n_s n_m^2/2\right) + p\left(4n_s n_m b + 2n_s n_m^2\right) \\ &+ \left(4n_m^2 b + n_s n_m^2 + Nnn_m\right) + n\left(3n_m^2 + 11n_m\right)\end{aligned} \tag{42}$$

where the first term represents the MC cost in forming $\mathbf{t}_G$, $\mathbf{M}_G$ and $\mathbf{K}_G$. The second term represents the MC cost of the p iterations used to calculate $\mathbf{t}$. The third term

represents the MC cost in forming the condensed matrices ($\mathbf{M}_R$, $\mathbf{K}_R$, $\mathbf{C}_R$ and $\mathbf{F}_R$). The fourth term is contributed by the process of solving the condensed vibration equation using the Newmark method.

Similarly, the structural response sensitivities are calculated using the proposed dynamic condensation method, and the total MC for the response sensitivities is

$$\mathrm{MC}_{Rsensitivity}=p\left(4n_s n_m^2+4n_s n_m b\right)+\left(4n_s n_m^2+8n_s n_m b+2n_m^2 b\right)+4nn_m^2+n\left(10n_m^2+22n_m\right) \tag{43}$$

where the first term, multiplied by p iterations, is used to calculate the derivative of the transformation matrix ($\frac{\partial \mathbf{t}}{\partial r}$). The second term denotes the MC consumed to calculate the derivatives of the condensed matrices. The third term consists of the MC consumed in forming the effective load according to Eq. (27). The last term is the MC cost to calculate the response sensitivities of n time steps with the Newmark method.

5. Case study 1: a frame structure

An eight-storey steel frame (Figure 1) is employed as an illustration to investigate the procedure and convergence of the proposed condensation method in calculation of structural responses and response sensitivities. The frame is modelled by 160 Euler-Bernoulli beam elements and 140 nodes. The elements and nodes are numbered in Figure 1. Each node has 3 DOFs, and there are 408 DOFs in total. The chosen material constants of each element are as follows: bending rigidity (EI) =170×10^6 Nm^2, axial rigidity (EA) =2500×10^6 N, mass per unit length (ρA) =110 kg/m, and Poisson's ratio υ=0.3. The Rayleigh damping coefficients with respect to the mass matrix and stiffness matrix are 0.6247 and 0.0039, respectively. The frame structure is excited at Node 92 in horizontal direction, and the excitation is F=10sin20t+5cos15t

(kN)($0 \le t \le 5$ s). The excitation is discretized into 2501 time steps with each lasting 0.002 s. The structural responses and response sensitivities of the frame with respect to the bending rigidities of two elements (r_1 and r_2 in Figure 1(b)) will be calculated in this example.

For comparison, the structural responses and response sensitivities are calculated by the proposed condensation method, the Guyan condensation method and the traditional full model method. The results from the traditional full model method are regarded as the exact ones.

First, the proposed condensation method is employed to calculate the structural responses and response sensitivities. As the structural responses are usually required together with the response sensitivities, this example combines the iterative schemes for $\mathbf{t}$ and $\frac{\partial \mathbf{t}}{\partial r}$ to achieve the structural responses and the response sensitivities simultaneously. Employing the proposed condensation method, the displacements, velocities and accelerations of the frame and their derivatives with respect to r_1 are calculated by the following procedure.

1) Nodes 40, 41, 45, 49, 92, 93, 97, and 101 are selected as the master nodes (Figure 1(a)), which are uniformly distributed among the structure. In addition, to calculate the response sensitivities with respect to the bending rigidity of r_1 (Element 104), Node 109 is attached to Element 104 and is thus chosen as the master node as well.

2) Some constant values are calculated to avoid the repeated calculation in later procedures, for example, $\mathbf{K}_{ss}^{-1}$, $\mathbf{K}_{ss}^{-1}\mathbf{M}_{ms}^{\mathrm{T}}$, $\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}$, $\mathbf{t}_G = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}$, $\mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}$, $\mathbf{K}_G = \mathbf{K}_{mm} + \mathbf{K}_{ms}\mathbf{t}_G$, $\frac{\partial \mathbf{M}_{mm}}{\partial r}$, $\frac{\partial \mathbf{K}_{mm}}{\partial r}$ and so forth.

3) The initial values of $\mathbf{t}$ and $\mathbf{M}_d$ and their derivatives with respect to r ($\frac{\partial \mathbf{t}}{\partial r}$ and $\frac{\partial \mathbf{M}_d}{\partial r}$) are calculated from Guyan static condensation, as follows:

$$\mathbf{t}^{[0]} = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}} \tag{44}$$

$$\mathbf{M}_d^{[0]} = \mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}_G + \left(\mathbf{M}_{ms}\mathbf{t}_G\right)^{\mathrm{T}} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\mathbf{t}_G \tag{45}$$

$$\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[0]} = \frac{\partial \mathbf{t}_G}{\partial r} = \frac{\partial\left(-\mathbf{K}_{ss}^{-1}\mathbf{K}_{ms}^{\mathrm{T}}\right)}{\partial r} = \mathbf{0} \tag{46}$$

$$\left[\frac{\partial \mathbf{M}_d}{\partial r}\right]^{[0]} = \frac{\partial \mathbf{M}_{mm}}{\partial r} \tag{47}$$

4) The transformation matrix $\mathbf{t}$ and its derivative $\frac{\partial \mathbf{t}}{\partial r}$ are updated iteratively. In the kth (k=1, 2, 3...) iteration,

$$\mathbf{t}^{[k]} = \mathbf{t}_G + \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{[k-1]}\right)\left(\mathbf{M}_d^{[k-1]}\right)^{-1}\mathbf{K}_G \tag{48}$$

$$\mathbf{M}_d^{[k]} = \mathbf{M}_{mm} + \mathbf{M}_{ms}\mathbf{t}^{[k]} + \mathbf{t}_G^{\mathrm{T}}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{[k]}\right) \tag{49}$$

$$\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k]} = \mathbf{K}_{ss}^{-1}\left(\mathbf{M}_{ms}^{\mathrm{T}} + \mathbf{M}_{ss}\mathbf{t}^{[k]}\right)\left(-\left(\mathbf{M}_d^{[k]}\right)^{-1}\left[\frac{\partial \mathbf{M}_d}{\partial r}\right]^{[k-1]}\left(\mathbf{M}_d^{[k]}\right)^{-1}\mathbf{K}_G + \left(\mathbf{M}_d^{[k]}\right)^{-1}\frac{\partial \mathbf{K}_G}{\partial r}\right) + \mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k-1]}\left(\mathbf{M}_d^{[k]}\right)^{-1}\mathbf{K}_G \tag{50}$$

$$\left[\frac{\partial \mathbf{M}_d}{\partial r}\right]^{[k]} = \frac{\partial \mathbf{M}_{mm}}{\partial r} + \mathbf{M}_{ms}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k]} + \mathbf{t}_G^{\mathrm{T}}\mathbf{M}_{ss}\left[\frac{\partial \mathbf{t}}{\partial r}\right]^{[k]} \tag{51}$$

The iterations are terminated when the differences of the lowest five eigenvalues $\lambda^{[k]} = eig\left(\left(\mathbf{M}_d^{[k]}\right)^{-1}\mathbf{K}_G\right)$ from two consecutive iterations are less than 1×10^{-5}. For this frame structure, $\mathbf{t}$ and $\frac{\partial \mathbf{t}}{\partial r}$ converge to the predefined tolerance (1×10^{-5}) after six iterations.

5) According to the updated $\mathbf{t}$, the condensed matrices ($\mathbf{M}_R$, $\mathbf{K}_R$, $\mathbf{C}_R$ and $\mathbf{F}_R$) are computed according to Eqs. (25a), (25b), (25c) and (25d). The condensed matrices ($\mathbf{M}_R$, $\mathbf{K}_R$ and $\mathbf{C}_R$) take the size of 24×24, which is much smaller than the original system matrices with a size of 408×408. Afterwards, the displacements, velocities and

accelerations of the master DOFs are calculated from the 24×24 condensed vibration equation (Eq. (4)) with the Newmark method [3, 29]. The displacements, velocities and accelerations of the whole DOFs are recovered from the responses of the master DOFs according to Eq. (26).

6) Based on the updated derivative $\frac{\partial \mathbf{t}}{\partial r}$, the derivatives of the condensed mass matrix $\frac{\partial \mathbf{M}_R}{\partial r}$ and stiffness matrix $\frac{\partial \mathbf{K}_R}{\partial r}$ are calculated by Eqs. (30) and (31). Finally, the derivatives of acceleration, velocity and displacement ($\frac{\partial \ddot{\mathbf{x}}_m(\tau)}{\partial r}$, $\frac{\partial \dot{\mathbf{x}}_m(\tau)}{\partial r}$ and $\frac{\partial \mathbf{x}_m(\tau)}{\partial r}$) of the master DOFs are calculated from the condensed vibration equation (Eq. (27)).

Afterwards, the structural responses and response sensitivities of the frame are calculated by the Guyan condensation method [20, 24]. In the Guyan condensed model, $\mathbf{t}=\mathbf{t}_G$, $\mathbf{K}_R=\mathbf{K}_G$ and $\mathbf{M}_R=\mathbf{M}_G$ are used to calculate the structural responses, and $\frac{\partial \mathbf{t}}{\partial r}=\frac{\partial \mathbf{t}_G}{\partial r}$, $\frac{\partial \mathbf{K}_R}{\partial r}=\frac{\partial \mathbf{K}_G}{\partial r}$ and $\frac{\partial \mathbf{M}_R}{\partial r}=\frac{\partial \mathbf{M}_G}{\partial r}$ are used to calculate the response sensitivities. They are equivalent to the initial step of the proposed condensation method. In addition, the structural responses and response sensitivities are calculated from the original full model with the Newmark method, which are taken as the exact values for comparison.

The displacement, velocity and acceleration of Node 40 in horizontal direction (X direction) by the above three methods are displayed in Figure 2. Clearly, the curve obtained from the full model is nearly overlapped with the proposed condensed model, implying that the proposed condensation method is accurate to calculate the structural responses of a structure. In contrast, the structural responses gained by the Guyan

condensed model appear as an obvious discrepancy from the exact responses of the frame.

Without losing generality, Table 1 lists the average relative errors of the structural responses at randomly selected five nodes, and the average relative error is estimated by

$$\mathrm{diff}\left(x(\tau), x_R(\tau)\right) = \frac{\mathrm{average}(|x_R(\tau) - x(\tau)|)}{\mathrm{average}(|x(\tau)|)} \tag{52}$$

where $x(\tau)$ represents the exact structural responses, and $x_R(\tau)$ represents the structural responses of the condensed model. Among the five nodes, Nodes 40, 41 and 101 are the master nodes, and Nodes 43 and 103 are the slave nodes. Table 1 shows that the relative differences of the proposed condensed model and the original full model are approximately 10^{-5}. The relative errors of responses on the two slave nodes are also approximately 10^{-5}, which proves that the dynamic transformation matrix **t** is a constant matrix irrelevant to the external forces. However, the relative errors from the Guyan condensed model are greater than 1% and even reach 10%, which demonstrates that the Guyan condensed model is not accurate enough to estimate the structural responses.

The response sensitivities of Node 40 with respect to elemental parameter r_1 by the three methods are compared in Figure 3. It is obvious that the curves from the proposed condensed model are almost identical to the real curves, whereas the curves gained by the Guyan condensed model are sharply separated.

In addition, the structural response sensitivities of Nodes 40, 41, 45, 49, 92, 93, 97 and 101 with respect to r_1 are calculated, and Table 2 lists the differences between the full model and the condensed model in terms of the following equation:

$$\text{diff}\left(\frac{\partial x(\tau)}{\partial r_1}, \frac{\partial x_R(\tau)}{\partial r_1}\right) = \frac{\text{average}\left(\left|\frac{\partial x_R(\tau)}{\partial r_1} - \frac{\partial x(\tau)}{\partial r_1}\right|\right)}{\text{average}\left(\left|\frac{\partial x_R(\tau)}{\partial r_1}\right|\right)} \tag{53}$$

where $\frac{\partial x(\tau)}{\partial r_1}$ represents the response derivatives of the full model, and $\frac{\partial x_R(\tau)}{\partial r_1}$ represents the response derivatives of the condensed model. Table 2 shows that the relative errors of the response derivatives by the proposed condensed model are approximately 10^{-5}. The proposed condensation method is sufficiently accurate to calculate the response sensitivities. However, the relative errors of the Guyan condensed model almost reach 10%. The Guyan condensed model is insufficiently accurate to calculate the response sensitivities.

Without loss of generality, the response derivatives with respect to another randomly selected beam element r_2 (Element 153) are also calculated. Node 137, attached to element r_2, is chosen as the additional master node. The relative differences of response derivatives with respect to r_2 in Table 3 shows that the relative errors of the proposed condensed model are primarily approximately 10^{-5}, whereas the relative errors of Guyan condensed model sometimes reach 10%. It again shows that the proposed condensation method is accurate in calculation of structural responses and response sensitivities.

In this paper, if the tolerance is set to be 1×10^{-5} [5], then the relative errors of structural responses and response sensitivities are expected to be approximately 10^{-5}. A smaller tolerance will lead to a higher precision, but more iterations are required. For different structures or different cases, a trial and error analysis is helpful to balance the accuracy and efficiency.

In this example, the size of the full model is 408×408, whereas the size of the condensed model is 24×24. Although the proposed condensation method adds a small amount of computation time to achieve $\mathbf{t}$ and $\frac{\partial \mathbf{t}}{\partial r}$ by an iterative scheme, it is negligible compared to the large number of time steps in calculation of structural responses and response sensitivities. Investigation of efficiency will be illustrated by a cantilever plate in the following case study.

6. Case study 2: a cantilever plate

A cantilever plate (Figure 4) is illustrated to investigate the computational efficiency of the present method to expand the theory to the field of relatively large structures [23]. The material properties of the plate are chosen as follows: a Young's modulus E=206 GPa, a mass density ρ=7800 kg/m^3 and a Poisson ratio υ=0.3. The Rayleigh damping coefficients with respect to the mass matrix and stiffness matrix are 0.6247 and 0.0039, respectively. The dimensions of the plate are 4000 mm × 2000 mm × 10 mm. The plate is modelled by 40×20=800 shell elements, and each element is 100 mm ×100 mm ×10 mm in size. There are 861 nodes and 4920 DOFs (each node has 6 DOFs) in total. The plate is excited at Node 861 in the Z direction, and the excitation is F=-10sin20t-5cos15t (N) ($0\leq t\leq5$ s). The excitation is discretized into 2501 time steps, with each time step equal to 0.002 s, to calculate the structural responses and

response sensitivities with respect to the bending rigidity of a randomly-selected element *r*. To reduce the bandwidth of the large system matrices, the nodes are carefully numbered, and the bandwidth of the global stiffness matrix is b=128.

Nodes 11, 21, 221, 231, 431, 441, 641, 651, 851, and 861 are chosen as the master nodes to calculate the structural responses of the plate. Nodes 432, 452 and 453, each attached to Element *r*, are chosen as the additional master nodes to calculate the response sensitivities with respect to *r*.

For comparison, the structural responses and response sensitivities are calculated by the proposed method, the IRS method [21, 22], Guyan condensed model [20, 24] and the full model [3, 29]. In addition, the results from the full model are regarded as exact ones. The tolerance of convergence is set to 10^{-5} for the proposed method and the IRS method. The proposed method converges to the predefined tolerance within seven iterations, while the IRS method takes twelve iterations to reach the same precision. For brevity, the acceleration and acceleration derivatives of Node 21 in the Z direction with respect to *r* are listed in Figures 5 and 6, respectively. According to Figures 5 and 6, the acceleration and acceleration derivative curves obtained using the proposed method are nearly overlapped to those obtained using the full model method. Compared to the IRS method, the acceleration derivative curves gained by the proposed method are closer to the real ones. Therefore, the proposed method is more accurate than the IRS method in calculation of response sensitivities. In contrast, the acceleration and acceleration curves gained by the Guyan condensation are quite separate from the exact ones, implying that the Guyan condensation method is not

accurate enough regarding the calculation of structural responses and response sensitivities.

To investigate the computational efficiency of the proposed method, the MC and the running time using an ordinary personal computer of 4.00 GHz CPU and 16 GB memory are listed in Table 4. “Model reduction” records the model reduction process in forming the condensed system, and “Newmark” represents the process of calculating the responses or response sensitivities using the Newmark method.

The proposed condensation method costs 0.483 seconds to achieve the condensed model. The size of the vibration equation is reduced from 4920×4920 to 60×60. As the vibration equation of the condensed model has a much smaller size than that of the full model, calculation of structural responses using the condensed model is completed faster than the calculation using the traditional full model. In particular, the proposed method takes only approximately 0.069 seconds to calculate the structural responses, whereas the full model takes 6.303 seconds, i.e., the calculation time proposed model approximately 1% of that consumed by the full model. The proposed method takes only approximately 0.129 seconds to calculate the response sensitivities, whereas the full model takes 6.512 seconds, i.e., the time consumed by the proposed method is 2% of that consumed by the full model. Notably, there are only 2501 time steps considered in this case study to calculate the responses and response sensitivities. The saving of computational time can be more significant when a long-time excitation and a large number of time steps are used to calculate the structural responses and response sensitivities.

Because the size of the vibration equation of the proposed method and the IRS method are the same, the operating time in the calculation of the responses and response sensitivities are nearly identical, as shown in Table 4. The proposed method converges to the accurate model faster than the IRS method. In particular, the IRS method takes 0.877 seconds to form **t** with twelve iterations, and the proposed method takes 0.483 seconds with seven iterations. The IRS method costs 0.773 seconds to achieve $\frac{\partial \mathbf{t}}{\partial r}$, and the proposed method costs 0.597 seconds. Although the Guyan condensation method achieves the condensed model faster than the two dynamic condensation methods, the precision of Guyan static condensed model is not satisfactory according to Figures 5 and 6.

MC is another basic index that can be used to evaluate the efficiency of the proposed method, thereby avoiding the random errors in running time caused by different operating environments. It proves again that the proposed condensation method requires fewer multiplication counts than the IRS method to achieve the condensed model, and the MC of the proposed method is approximately 60% of the MC of the IRS method. The computational saving can be more significant when applied to a large-scale structure.

The precision and efficiency of the dynamic condensation methods are inevitably influenced by the number of master DOFs selected [26, 27]. To investigate the effects of the number of master DOFs to the efficiency of the proposed method, the operating time is estimated, respectively, with 48, 60 and 120 master DOFs included. The master DOFs are uniformly distributed on the plate, and the criteria of convergence are set to be the same.

Table 5 shows that when the number of the master DOFs increases, the condensation method converges to the predefined tolerance within fewer iterations. However, each iteration consumes a higher amount of computational time, and the computation time of the structural responses and response sensitivities using the Newmark method becomes longer because an increase of the number of master DOFs increases the size of the condensed vibration equation. In contrast, a decrease of the number of master DOFs will reduce the size of the condensed vibration equation and thus speed up the calculation of structural responses and response sensitivities; however, more iterations are required to achieve the condensed model. Therefore, it is wise to select rational master DOFs in practical engineering [26, 27]. In this example, a selection of 60 master DOFs is proved to be more efficient than that of 48 master DOFs and 120 master DOFs.

The advantages in efficiency and accuracy of the proposed condensation method were proved by the relatively large structure of 4920 DOFs and 800 elemental parameters. The excitation lasts 5 seconds, and 2501 time steps are used to calculate the responses and response sensitivities. Since model condensation is inherently superior to cope with large structures, the proposed method will be more promising when dealing with the practical large-scale structures for calculation of long-time responses and response sensitivities.

7. Conclusions

This paper proposed a simplified dynamic condensation method to calculate the structural responses and response sensitivities. In this method, the global vibration

equation is transformed into a condensed one by a simplified scheme. The transformation matrix **t** is determined by $\mathbf{M}_d$, avoiding the repeated calculation of $\mathbf{K}_R$ and $\mathbf{M}_R$. By selecting the DOFs of concerned elements as master DOFs, the perturbation of an elemental parameter are localized in the derivatives of master DOFs, and the response sensitivity is quickly calculated by the condensed model.

Applications to two numerical examples demonstrate that the proposed method can accurately calculate the responses and response sensitivities by a simplified scheme. The proposed condensation method is efficient in terms of the multiplication count and running time, and its precision is very high. Compared to the IRS method, the proposed condensation method converges faster with fewer iterations, thus saving more computational resources. Compared to the Guyan static condensation method, although the dynamic condensation methods add a small amount of computation in achieving the condensed model, it is far more accurate than the Guyan static model. In addition, the increase of the number of master DOFs enhances the size of the vibration equation and thus extends the computation time for structural responses and response sensitivities, although it is helpful to accelerate the convergence of the process of dynamic condensation. A trial and error analysis is helpful to balance the accuracy and efficiency. The applications of the proposed method to the frame and plate structures revealed its superiority in accuracy and efficiency to other methods; the proposed method is more promising for use in practical large-scale structures for which large system matrices are formed to calculate the long-time responses or response sensitivities.

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