https://dx.doi.org/10.1016/j.apm.2018.12.010

On the nano-structural dependence of nonlocal dynamics and its relationship to the upper limit of nonlocal scale parameter

C. Li[1], **S.K. Lai**[2,3,*], **X. Yang**[2]

[1]Department of Vehicle Engineering, School of Rail Transportation, Soochow University, Suzhou 215131, Jiangsu, P.R. China

[2]Department of Civil and Environmental Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, P.R. China

[3]The Hong Kong Polytechnic University Shenzhen Research Institute, Shenzhen, P.R. China

Abstract: Nonlocal elasticity theory is one of the most popular theoretical approaches to investigate the intrinsic scale effect of nano-materials/structures. The coupling of an internal characteristic length and a material parameter can be regarded as a nonlocal scale parameter in nano-meters. The range of this non-dimensional scale parameter is from zero up to different values previously. There is no doubt that the zero nonlocal scale parameter corresponds to a situation without any nonlocal effect. However, the determination of a peak value for the scale parameter is still uncertain. In fact, we frequently ask a simple but unresolved question, i.e., how strong is the nonlocal scale effect? This question is equivalent to what the maximum value of the nonlocal scale parameter is, since it was introduced to characterize the scale effect theoretically. Until now, various maximum values have been selected without rigorous verifications. In this paper, the nano-structural dependence of nonlocal dynamical behavior is investigated to present the existence of an upper limit for the scale parameter. Through three typical examples, the size-dependent behavior of nonlocal dynamics for various nano-structures is analyzed. The upper limit of the scale parameter can be determined accordingly. It is shown that an interval for the scale parameter in the illustrative examples can be found on the basis of the nonlocal softening physical mechanism, in which the equivalent stiffness of nano-structures is weakened than that predicted by the classical continuum theory. The present study contributes to a fuzzy zone in nonlocal elasticity where people are puzzled over the question how to select the upper limit of the nonlocal scale parameter. It is not only beneficial to the refinement of nonlocal theory, and also useful for the exploration of

*Corresponding author. E-mail: sk.lai@polyu.edu.hk (S.K. Lai)

similar theories in nano-mechanics.

Keywords: Nonlocal theory, Scale parameter, Nonlocal scale effect, Structural dependence, Softening model

1. Introduction

Since the discovery of carbon nano-tubes, the mechanical and electrical properties of nano-materials/structures have attracted considerable interests, resulting in the advancement of multi-disciplinary topics in nano-mechanics [1-4]. There are several theoretical methods to deal with problems in nano-mechanics, of which the nonlocal theory has been widely applied to reveal different characteristics from the classical results. The nonlocal theory was originated from the micro-polar and micro-morphic theories, and it was firstly proposed by Eringen and Edelen [5, 6] in 1970s. As one of the generalized continuum mechanics theories, the nonlocal theory considers the long-range force between atoms/molecules. It is assumed that the stress at a specific point is a function of strains at all points in the continuum. In order to reflect contributions of the strains at other points to the stress at the specific point, the nonlocal theoretical constitutive relations introduce a relevant kernel function, which are associated with the scale effect and the distance between other points and the specific point. It is easy to quantify the Euclidean distance, but the difficulty is to consider the nonlocal scale effect. The issue was solved by introducing an internal characteristic length in the nonlocal theory. Because of the kernel function and the internal characteristic length, the nonlocal theory can capture the nonlocal scale effect that is neglected in the classical continuum theory. The results based on the nonlocal theory are in excellent agreement with those obtained from molecular dynamics simulations and experiments. In particular, the nonlocal theory can be transformed to the classical one in accordance with the assumption of the long wavelength limit, while to atomic lattice dynamics by using the assumption of the short wavelength limit.

The original form of the nonlocal theory was expressed by an integro-partial-differential equation [5, 6]. Due to the difficulties of spatial integrals in the nonlocal integro-partial-differential constitution, Eringen [7] transformed it to an equivalent nonlocal differential form using Green’s function and proposed a popular singular partial-differential constitutive relation for a special class of physically admissible kernels. The differential form is suitable for one/two/three-dimensional nano-structures through a unified constitutive relation expressed in terms of tensors. It is found that the results of screw dislocation and Rayleigh surface waves based on the differential constitution were verified by experimental observations and atomic lattice dynamics [7]. Due to a great deal of simplicity, it is more convenient to apply the nonlocal differential form to formulate the equilibrium equations or equations of motion and analyze its mechanical characteristics at a nano-scale. Consequently, the nonlocal theory expressed in a differential form has been employed extensively in predicting the static and dynamical properties of nano-materials/structures [8-20]. Using a perturbation technique, Li et al. [8] studied the dynamic and stability behaviors for the transverse free vibration of nano-beams subjected to a variable axial load. Arash and Wang [12] conducted a review study on the application of nonlocal elastic models for carbon nano-tubes and graphene sheets. Rahmani and Pedram [13] investigated the free vibration of functionally graded nano-beams using the nonlocal differential constitution and the Timoshenko beam model to reveal the size effect. Ansari et al. [14] presented the vibration analysis of piezoelectric Timoshenko nano-beams in conjunction with the von Kármán geometric nonlinearity. Xu et al. [18] proposed a boundary value problem of Euler–Bernoulli beams based on the weighted residual method, and studied the boundary conditions of the nonlocal strain gradient model. Subsequently, they also presented closed-form solutions for the bending and buckling analysis of nonlocal strain gradient beams. Furthermore, Ebrahimi and Barati [20] examined the vibration behavior of functionally graded nano-beams that can be acted as a component in smart piezoelectric actuators, where the magneto-electro-thermo-elastic coupling effect was taken into consideration.

As aforementioned, the widespread use of the nonlocal theory makes it as a research hotspot in nano-mechanics. It should be emphasized that, two kinds of nonlocal elasticity models were formulated along the development of such a theory, i.e., the nonlocal softening model [9, 11, 13-17, 20] and the nonlocal hardening model [8, 10, 21-23]. The nonlocal softening model can be derived from the nonlocal differential constitutive relation and the classical equilibrium equation or equation of motion. In other words, the commonly used governing equations for classical macro-scale materials/structures using the classical continuum theory are still valid at nano-scales, but those classical equations should be modified appropriately based on the connotation of the nonlocal theory. Therefore, the classical elasticity analysis for nano-structures is regarded as the first step in the nonlocal softening model. Subsequently, the nonlocal differential intension is implanted into the classical framework. On the other hand, the nonlocal hardening model is established based on the nonlocal variational principle or the long-range interaction among atoms. The nonlocal stresses can be determined directly from the nonlocal differential constitutive equations or the physically-based nonlocal models. Various nonlocal internal forces in new form can also be calculated. Moreover, the nonlocal strain potential energy, external potential energy (or external work) and/or kinetic energy of nano-structures are derived. The principle of virtual work or the Hamilton principle can then be employed to obtain the nonlocal governing equation. Nevertheless, it is strange to find that some reversed conclusions are established from the nonlocal softening and hardening models. There are many research papers to support the correctness of each model. The controversial argument is related to the nano-structural stiffness predicted by the nonlocal theory, it is reduced or strengthened by comparing with those predicted via the classical continuum theory. Generally, it is reduced in the nonlocal softening model while strengthened in the hardening model. Hence, the nonlocal deformation is higher and the nonlocal natural frequencies (or wave velocities) are lower than that of the classical counterparts in the softening model, but an opposite conclusion can be achieved in the hardening model. Recently, Li et al. [24, 25] considered this paradox and proved that both nonlocal softening and hardening models are correct, it depends on different types of surface stress effects, namely the long-range attractive (compressive surface stress) or repulsive (tensile surface stress) interactions.

Hitherto, there is still an unsolved question about the extent of the nonlocal scale effect in nano-materials/structures, or the existence of an upper limit in the nonlocal scale parameter, i.e., its magnitude. This is an urgent problem because the nonlocal scale parameter can act as a connector between the classical theory and the nonlocal theory, but different ranges of the nonlocal scale parameter can be seen in the literature without reasonable explanation. Many researchers are puzzled as to the selection of the nonlocal scale parameter in nanometers. In order to avoid such an unclear issue, the physical nonlocal scale parameter e_0a was hidden via a dimensionless quantity $\tau = e_0a/l$ in most research studies, where l is an external characteristic length scale (e.g., wavelength, crack length), but subsequently, the dimensionless quantity τ was only chosen roughly. Here, we list some of them as examples in the literature: the dimensionless quantity τ was used from 0 to 0.02 in Lim [22]; from 0 to 0.05 in Li et al. [15]; from 0 to 0.06 in Wang et al. [26]; from 0 to 0.1 in Ansari et al. [14], Liu et al. [16], and Yu and Lim [27]; from 0 to 0.15 in Wang et al. [28]; from 0 to 0.2 in Xu et al. [18], Lim [21], Lim et al. [29], and Li et al. [30]; from 0 to 0.3 in Yang and Lim [31]; from 0 to 0.4 in Wang and Duan [32]; from 0 to 0.6 in Lu [33]; from 0 to 0.7 in Wang et al. [34]; from 0 to 0.8 in Guo and Yang [35], and Murmu and Pradhan [36]; from 0 to 1.0 in Murmu et al. [37] and Lu et al. [38,39] etc.

In addition, the magnitude of the nonlocal material parameter e_0 was endowed with different values, even for the same carbon material (cf. [32, 40, 41]). Definitely, the nonlocal scale parameter should be a constant for a given material, this is because the nonlocal material parameter e_0 and the internal characteristic length a are related to material properties. However, when we consider a common structure, e.g., a nano-beam (maybe a carbon nano-tube or metal nano-rod or other one-dimensional nano-materials) or a nano-plate (maybe a graphene sheet or boron-nitride sheet or other two-dimensional nano-materials), it is required to provide a variation interval of the nonlocal scale parameter in order to fit for each material. Currently, no one holds that the nonlocal scale parameter can be chosen freely, or no one takes an infinite nonlocal scale parameter to carry out the analysis of nano-structures. Most researchers believe that there exists an upper limit of the nonlocal scale parameter (unit: *nm*), but the selection of this upper limit is an uncertain topic. In addition to the nonlocal theory, other approaches including molecular dynamics simulation and experimental tests can only be used to obtain a constant value for a given material. It is impracticable to repeat the work for all materials due to the experimental difficulty and measurement errors. As a result, the use of molecular dynamics simulation and experiments is incapable of determining the range or upper limit of the nonlocal scale parameter. It is also hard to continue research studies on the variation of physical quantities with respect to the nonlocal scale parameter. Hence, it is important to re-consider some analytical methods to solve this question theoretically. In this study, we attempt to determine an upper limit for the nonlocal scale parameter through a dynamic analysis for various nano-structures, including the axial vibration of nano-rods, transverse vibration of nano-beams, and free vibration of nano-plates. To this end, the nonlocal softening model is adopted, and then the nonlocal scale parameter emerges in the framework of the softening physical mechanism. The study can also provide useful guidelines for the advancement of the nonlocal theory and its potential applications in nano-mechanics.

2 Nonlocal differential constitutive relation

In generalized continuum mechanics, the nonlocal elasticity theory can be used to predict the mechanical properties of nano-materials/structures by introducing the size-dependent constitutive parameters to consider the intrinsic scale effect. The fundamental constitutive equation of the nonlocal theory in differential form is given by [7]

$$\left[1-(e_0 a)^2\nabla^2\right]\mathbf{t}=\boldsymbol{\sigma} \tag{1}$$

where $\mathbf{t}$ is the nonlocal stress tensor, $\boldsymbol{\sigma}$ is the classical (local) stress tensor determined by the classical continuum mechanics theory, ∇^2 is the Laplacian operator, e_0 is a material parameter and a is an internal characteristic length. The product of e_0a was introduced as the nonlocal scale parameter by Eringen, aimed at forming a nonlocal kernel function and measuring the contributions of the strains at other points to the stress at a specific point. With increasing the distance between other points and the specific point, the influences of the strains at other points weaken. In fact, the purpose of the nonlocal scale parameter is to describe and determine a domain in which the scale effect must be taken into consideration, while it can be neglected outside the domain. For classical bulk materials/structures at macro-scale, the nonlocal scale effect can be neglected. However, the scale effect should be considered for nano-materials/structures when the external characteristic scale is at the same level with the internal characteristic scale. In such a situation, the nonlocal scale parameter plays a significant role since it represents and quantifies the scale effect. The main purpose of this paper is to determine the nonlocal scale parameter. Additionally, the nonlocal differential constitutive relation (1) can be reduced to the classical one when e_0a approaches to zero, and so the nonlocal stress is identical to the classical stress under the condition $e_0a = 0$. Actually, the classical theory is one of the simplified versions in generalized continuum mechanics and it can be treated as a special case of the nonlocal theory.

The differential constitutive equation has been extensively applied to study the size-dependent mechanical behaviors of nano-structures, including static deformation, buckling, free and forced vibrations, wave propagation, dislocation and damage mechanics. Indeed, nano-materials can be modeled by various forms, as nano-wires, nano-rods, nano-tubes, nano-beams and nano-plates/shells. The present work is concerned with the upper limit of the nonlocal scale parameter. For this purpose, we investigate the scale range in dynamical behaviors by taking three different nano-structures for illustration. Different nano-structural dynamical characteristics as well as their common physical mechanisms are taken into consideration. The nonlocal dynamics of different nano-structures may differ from each other, implying the structural dependence of nonlocal dynamical behaviors. Nevertheless, the effect of dynamic behavior should be attributed to the factor of different structural characteristics, rather than the nonlocal scale effect. Consider various nano-structures made of the same material, the nonlocal scale effect has a commonality. This is the inspiration of the present study, in which the upper limit of the nonlocal scale parameter can be determined according to the nonlocal softening mechanism.

3 Vibration of nano-rods and nano-beams

3.1 Axial vibration of nano-rods

In order to show the characteristics of nonlocal softening model, the derivation of equation of motion for axial vibration of nano-rods based on the nonlocal differential constitutive relation and the classical equation of motion is provided as follows. For a uniform, axially vibrating nano-rod with mass density ρ and cross-sectional area *A*, we consider an element *dx* of the nano-rod. The internal forces (i.e. axial force) on both sides of the element are N and $N+\partial N/\partial x\,dx$, respectively. Hence the classical equation of motion in axial direction can be written as

$$\rho A dx \frac{\partial^2 u}{\partial t^2} = \left(N + \frac{\partial N}{\partial x} dx \right) - N \tag{2}$$

where *u* is the axial displacement, *x* is the axial coordinate, and *t* is time. In fact, Eq. (2) can also be treated as an equilibrium equation with inertial force $-\rho A dx\,\partial^2 u/\partial t^2$ according to Dynamic-Static Method.

Considering the one-dimensional expression of Eq. (1) for nano-rods, we multiply *A* on both sides of the one-dimensional nonlocal constitutive equation and then obtain

$$N - (e_0 a)^2 \frac{\partial^2 N}{\partial x^2} = EA\varepsilon \tag{3}$$

where the classical relation between axial force and stress $N=\sigma A$ and classical relation between stress and strain $\sigma=E\varepsilon$ are adopted, in which *E* is the Young's modulus and ε is strain. The first partial derivative of Eq. (3) with respect to *x* is given by

$$\frac{\partial N}{\partial x} - (e_0 a)^2 \frac{\partial^3 N}{\partial x^3} = EA \frac{\partial^2 u}{\partial x^2} \tag{4}$$

where the classical relation between strain and displacement (geometric equation) $\varepsilon = \partial u/\partial x$ is adopted.

From Eq. (2) we can obtain

$$\frac{\partial N}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2} \tag{5a}$$

$$\frac{\partial^3 N}{\partial x^3} = \rho A \frac{\partial^4 u}{\partial x^2 \partial t^2} \tag{5b}$$

Substitution of Eqs. (5a) and (5b) into Eq. (4) yields the nonlocal softening based governing equation of motion for a nano-rod as

$$\rho \frac{\partial^2 u}{\partial t^2} - \rho\left(e_0 a\right)^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} = E \frac{\partial^2 u}{\partial x^2} \tag{6}$$

It is found that the nonlocal softening model is developed by combining the nonlocal constitutive relation including nonlocal scale parameter with the classical equation of motion or equilibrium equation including various classical relations. It should be noted that the same result with Eq. (6) was derived by Aydogdu [42]. The solution to Eq. (6) can be obtained via the method of separation of variables as

$$u_n\left(x,t\right) = \left[C_{1n} \sin\left(\sqrt{\frac{\rho\omega_n^2}{E-\rho\left(e_0 a\right)^2 \omega_n^2}} x\right) + C_{2n} \cos\left(\sqrt{\frac{\rho\omega_n^2}{E-\rho\left(e_0 a\right)^2 \omega_n^2}} x\right)\right] e^{i\omega_n t} \tag{7}$$

where ω_n (n=1,2,3,…) are circular frequencies for the axial vibration of nano-rods with mode numbers n, and C_{1n} and C_{2n} are undetermined coefficients.

If the axial displacement related to the circular frequency of each mode is obtained, the general solution for the axial displacement of nano-rods can be achieved after the superposition of each mode. Considering the fixed-fixed boundary constraints at both ends of the nano-rod, we have

$$u\big|_{x=0,\,L} = 0 \tag{8}$$

where L is length of the nano-rod. The circular frequencies can thus be determined analytically by combining Eq. (7) with (8), as

$$\omega_n = \sqrt{\frac{n^2\pi^2 E}{\rho L^2 + n^2\pi^2\rho\left(e_0 a\right)^2}} \tag{9}$$

It is implied that the circular frequencies decrease with an increase in the nonlocal scale parameter, and the scale parameter may increase all the time mathematically. By choosing the parameters L=10 nm, E=1.06 TPa, ρ=2250 kg/m^3, we can plot the effect of the circular frequency against the nonlocal scale parameter numerically in Fig. 1. It is observed that the circular frequencies of an axially vibrating nano-rod decrease slowly at the beginning of increasing the nonlocal scale parameter. Subsequently, the frequencies decreases rapidly when the nonlocal scale parameter reaches a certain value (e.g. e_0a>0.2nm). Although the frequency curves decline all the time, they remain smooth. Based on this case, it is hard to conclude that there exists an upper limit of the nonlocal scale parameter for the axial vibration of nano-rods, because an infinity value is meaningless from a physical perspective.

The procedure of how to derive the governing equation using the nonlocal softening model is shown above. However, it should be mentioned that the classical equilibrium equation or equation of motion is also available for the nonlocal hardening model. Hence not only the softening model but also the hardening model can be derived from the classical equations. The differences are the expressions of relevant physical quantities used in two nonlocal models. Still considering the axial vibration of nano-rods, the expressions of nonlocal stress in the softening and hardening models are, respectively [21]

$$t_{xx} = E\sum_{m=0}^{\infty}\left(e_0 a\right)^{2m} \varepsilon_{xx}^{\langle 2m\rangle} \tag{10a}$$

$$t_{xx} = E\sum_{m=0}^{\infty}\left(e_0 a\right)^{2m} \varepsilon_{xx}^{\langle 2m\rangle} - 2E\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(e_0 a\right)^{2(m+n)} \varepsilon_{xx}^{\langle 2(m+n)\rangle} \tag{10b}$$

where the superscript <2m> denotes $(2m)^{\text{th}}$-order derivative with respect to axial coordinate. The classical stress $\sigma_{xx}=E\varepsilon_{xx}$ is recovered in case of vanishing the nonlocal scale parameter in Eqs. (10a) and (10b).

Another example for torsion of nano-rods, the dimensionless twisting moment based on the nonlocal softening model is expressed as [23]

$$\bar{T} = \sum_{n=1}^{\infty}\tau^{2n-2}\theta^{\langle 2n-1\rangle} \tag{11a}$$

where τ is the non-dimensional nonlocal scale parameter, θ is the torsional angle. However, in the nonlocal hardening model it should be replaced by [23]

$$\bar{T} = -\sum_{n=0}^{\infty}\left(2n-1\right)\tau^{2n}\theta^{\langle 2n+1\rangle} \tag{11b}$$

which was derived through the variational principle [23]. For both models, the same equilibrium equation $\bar{T}^{\langle 1\rangle} = -\bar{t}_m$ can be used, where $\bar{t}_m$ is a distributed torsional load per unit length. Nevertheless, the expression of twisting moment $\bar{T}$ for nonlocal hardening model is different from that of the softening model.

3.2 Transverse vibration of nano-beams

Based on the nonlocal softening model, we can further investigate the transverse vibration of nano-beams. Firstly the classical equation of motion for transverse vibration of nano-beams is given by

$$\frac{\partial^2 M}{\partial x^2} - \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{12}$$

where M is the bending moment, w is the transverse displacement, A is the rectangular cross sectional area of the nano-beam (= breadth (b) × height (h)). On the other hand, still considering the one-dimensional expression of Eq. (1) for nano-beams, we multiply A on both sides of the one-dimensional nonlocal constitutive equation and then obtain

$$M - (e_0 a)^2 \frac{\partial^2 M}{\partial x^2} = -EI \frac{\partial^2 w}{\partial x^2} \tag{13}$$

where $I = \int_A y^2 dA$ is the area moment of inertia, $M = \int_A y t_{xx} dA$, and $\sigma = E\varepsilon = -Ey \frac{\partial^2 w}{\partial x^2}$.

From Eq. (12) we get

$$\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 w}{\partial t^2} \tag{14a}$$

$$\frac{\partial^4 M}{\partial x^4} = \rho A \frac{\partial^4 w}{\partial x^2 \partial t^2} \tag{14b}$$

From Eq. (13) we get

$$\frac{\partial^2 M}{\partial x^2} - (e_0 a)^2 \frac{\partial^4 M}{\partial x^4} = -EI \frac{\partial^4 w}{\partial x^4} \tag{15}$$

Substituting Eqs. (14a) and (14b) into (15), one derives the following equation of motion of a nano-beam as

$$\rho A \frac{\partial^2 w}{\partial t^2} - \rho A (e_0 a)^2 \frac{\partial^4 w}{\partial x^2 \partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0 \tag{16}$$

It is also found that the governing equation (16) is constructed based on the one-dimensional nonlocal differential constitutive relation and classical equations, which accords with the basic derivation procedure of nonlocal softening model. It should be noted that the similar governing equation was derived by Lim, Li and Yu [43]. In this case, a simply supported boundary condition is applied at both ends of the nano-beam, that is

$$w\big|_{x=0,\,L} = 0; \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0,\,L} = 0 \tag{17}$$

In order to solve Eq. (16) subjected to the boundary condition (17), the following displacement form is assumed as

$$w_n\left(x,t\right)=W_n\sin\frac{n\pi x}{L}e^{i\omega_n t} \tag{18}$$

where n=1,2,3,…, and it satisfies the boundary conditions above. The general form of the transverse displacement can be obtained by superposing all the vibration modes and circular frequencies, given by

$$w\left(x,t\right)=\sum_{n=1}^{\infty}W_n\sin\frac{n\pi x}{L}e^{i\omega_n t} \tag{19}$$

Substituting the trial solution of Eq. (18) into Eq. (16) yields

$$\omega_n=\sqrt{\frac{EI}{\rho A}\frac{\left(\frac{n\pi}{L}\right)^4}{\left[1+\left(e_0a\right)^2\left(\frac{n\pi}{L}\right)^2\right]}} \tag{20}$$

In order to demonstrate the relationship between the circular frequency and the nonlocal scale parameter for a rectangular nano-beam, the following parameters are selected: E=1.06 TPa, ρ=2250 kg/m^3 and L=10 nm. The dimensions of the rectangular section are b=1.5 nm, h=0.34 nm, A=0.51 nm^2, and I=4.913×10^{-3} nm^4. The numerical results are presented in Fig. 2.

The effect shown in Fig. 2 is similar to the axial vibration of nano-rods in Fig. 1. The circular frequencies decrease with increasing the nonlocal scale parameter for a simply supported nano-beam. However, the slope of the frequency curves or the frequency decrease rate is different from that of a nano-rod. The circular frequencies decreases rapidly at an early stage of increasing the nonlocal scale parameter from zero. After that, the frequency decrease rate slows down gradually. In additional, from Eq. (20), it seems that the nonlocal scale parameter may increase continuously in theory, similar to that of the axial vibration of nano-rods. Nevertheless, it is difficult to understand why the nonlocal scale parameter can be chosen arbitrarily. What is the physical meaning of an infinite nonlocal scale parameter, and how to explain the principle for selection of the nonlocal scale parameter? We believe there is an upper bound for the nonlocal scale parameter in the vibration of nano-rods and nano-beams but not shown in those examples. Therefore, we consider the free vibration of nano-plates in the next section, since there is no essential difference in the nonlocal softening dynamics of nano-rods, nano-beams and nano-plates. Although the nonlocal dynamical behavior reveal the nano-structural dependence due to different structural properties, the mechanism of the nonlocal scale effect appeared in nano-rods and nano-beams should coincide with that of nano-plates. This is because different nano-structures with the same material can be converted to each other. For example, a carbon nano-tube and a graphene nano-ribbon can be produced from a graphene sheet as shown in Fig. 3, where the carbon nano-tube, graphene nano-ribbon and graphene sheet can be modeled as structures of a nano-rod, a nano-beam and a nano-plate, respectively. The nonlocal scale effect of carbon nano-tube and graphene nano-ribbon should be qualitatively consistent with that of graphene sheet because the nonlocal scale effect is reflected by the nonlocal scale parameter e_0a, which completely depends on the properties of specific materials.

As a matter of fact, we can explain the reason why the upper limit of the nonlocal scale parameter disappears in the previous examples. In the abovementioned calculations, it is convenient to insert Eq. (18) rather than the summation Eq. (19) into Eq. (16), resulting in a simple expression (20). However, if we consider the nonlocal differential constitutive relation (1) and transform it into the nonlocal differential equation for a bending moment, we can acquire the bending moment by solving the nonlocal differential equation. Then, substituting the nonlocal bending moment into the classical equation of motion yields

$$\rho A\frac{\partial^2 w}{\partial t^2}-\rho A\left(e_0a\right)^2\frac{\partial^4 w}{\partial x^2\partial t^2}+EI\sum_{m=0}^{\infty}\left(e_0a\right)^{2m}\frac{\partial^{2m+4}w}{\partial x^{2m+4}}=0 \tag{21}$$

Equation (21) includes higher-order nonlocal terms. In fact, this is the exact version of the nonlocal governing equation for nano-beams. Some different results can be obtained from Eq. (21), and the upper limit of the nonlocal scale parameter may arise in the solution procedures. Similarly, the modified governing equation for the axial vibration of nano-rods can be given by

$$\rho\frac{\partial^2 u}{\partial t^2}-\rho\left(e_0a\right)^2\frac{\partial^4 u}{\partial x^2\partial t^2}=E\sum_{m=0}^{\infty}\left(e_0a\right)^{2m}\frac{\partial^{2m+2}u}{\partial x^{2m+2}} \tag{22}$$

In Section 4, the free vibration of nano-plates will be discussed. The higher-order governing equation that is similar to Eqs. (21) and (22) is adopted and solved in detail. The upper limit of the nonlocal scale parameter will be duly determined.

4 Upper limit of nonlocal scale parameter

4.1 Free vibration of nano-plates

In order to determine the upper limit of the nonlocal scale parameter, we consider the free vibration of nano-plates. As shown in Fig. 4, L_a, L_b and h are length, width and thickness of a nano-plate, respectively. Consider the nonlocal two-dimensional differential constitutive relations, one obtains

$$\sigma_{xx}-(e_0a)^2\nabla^2\sigma_{xx}=\frac{E}{1-\mu^2}(\varepsilon_{xx}+\mu\varepsilon_{yy})$$
$$\sigma_{yy}-(e_0a)^2\nabla^2\sigma_{yy}=\frac{E}{1-\mu^2}(\varepsilon_{yy}+\mu\varepsilon_{xx}) \tag{23}$$
$$\tau_{xy}-(e_0a)^2\nabla^2\tau_{xy}=\frac{E}{2(1+\mu)}\gamma_{xy}$$

where σ and τ are stresses, ε and γ are strains, and μ is the Poisson's ratio. According to the geometric equations of a plane problem in the classical elasticity (a part of the classical continuum theory), we can derive the nonlocal constitutions for nano-plates by ignoring the in-plane displacements at the neutral plane in x and y directions, as follows

$$\sigma_{xx}-(e_0a)^2\nabla^2\sigma_{xx}=\frac{-Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)$$
$$\sigma_{yy}-(e_0a)^2\nabla^2\sigma_{yy}=\frac{-Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right) \tag{24}$$
$$\tau_{xy}-(e_0a)^2\nabla^2\tau_{xy}=\frac{-Ez}{1+\mu}\frac{\partial^2 w}{\partial x\partial y}$$

Besides, the correlation between the nonlocal bending moment and the displacement can be expressed as [36]

$$M_{xx}-(e_0a)^2\nabla^2M_{xx}=-D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)$$
$$M_{yy}-(e_0a)^2\nabla^2M_{yy}=-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right) \tag{25}$$
$$M_{xy}-(e_0a)^2\nabla^2M_{xy}=-D(1-\mu)\frac{\partial^2 w}{\partial x\partial y}$$

where $D=Eh^3/12(1-\mu^2)$ is the bending rigidity defined in the classical elasticity. The nonlocal bending moment relations in Eq. (25) can be reducible to the classical counterparts when the nonlocal scale parameter is set to zero. The components of the nonlocal bending moment per unit length can be obtained by solving Eq. (25), given by

$$M_{xx}=-D\sum_{k=0}^{\infty}(e_0a)^{2k}\nabla^{2k}\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)$$
$$M_{yy}=-D\sum_{k=0}^{\infty}(e_0a)^{2k}\nabla^{2k}\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right) \tag{26}$$
$$M_{xy}=-D(1-\mu)\sum_{k=0}^{\infty}(e_0a)^{2k}\nabla^{2k}\frac{\partial^2 w}{\partial x\partial y}$$

According to the classical elasticity, the equilibrium equation for a thin-plate in terms of the internal force (bending moment) can be expressed as

$$\frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_{yy}}{\partial y^2}+q=0 \tag{27}$$

where q is the transverse load per unit area. Substituting Eq. (26) into Eq. (27) yields the nonlocal governing differential equation for the free vibration of nano-plates [44]

$$D\sum_{k=0}^{N}\left(e_0a\right)^{2k}\nabla^{2k}\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=\frac{1}{12}\rho h^3\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)-\rho h\frac{\partial^2 w}{\partial t^2} \tag{28}$$

where N=1,2,3,… ∞. Note that inertia forces are contained in the transverse load by considering the translation and rotation motions. It is worth to mention that only the first few items of the infinite series in Eq. (28) are taken into account in most previous studies (e.g. [22, 45]), other higher-order terms can be neglected. It is not clear whether the higher-order items should be abandoned. Unlike the previous studies, we consider all the higher-order terms of Eq. (28) in the present work. For simplicity, a simply supported condition is applied to all plate edges as an example, we can write the boundary conditions as

$$\begin{aligned} w\big|_{x=0,\,L_a}&=0; \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0,\,L_a}=0\\ w\big|_{y=0,\,L_b}&=0; \quad \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,\,L_b}=0 \end{aligned} \tag{29}$$

Assume that the displacement of the nano-plate is expressed as

$$w(x,y,t)=W_{mn}\sin\frac{m\pi x}{L_a}\sin\frac{n\pi y}{L_b}e^{i\omega_{mn}t} \tag{30}$$

where m, n=1,2,3,… are half-wave numbers. It is obvious the assumed displacement function can satisfy the boundary conditions in Eq. (29).

Making use of the Rayleigh-Ritz method, the circular frequencies for the free vibration of simply supported nano-plates can be obtained as

$$\omega_{mn}=\sqrt{\frac{D\left\{\sum_{k=0}^{N}(-1)^k\left(e_0a\right)^{2k}\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]^k\right\}\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]^2}{\frac{\rho h^3}{12}\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]+\rho h}} \tag{31}$$

We found that there is an upper limit of the nonlocal scale parameter in the free vibration of nano-plates. For this purpose, the infinite series in Eq. (31) is considered gradually. When N=0, the nonlocal circular frequencies become the classical counterparts, and it is independent of the nonlocal scale parameter. When N=1, it must satisfy

$$1-(e_0a)^2\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]>0 \tag{32}$$

Hence, one obtains

$$e_0a<\sqrt{\frac{L_a^2L_b^2}{(m\pi L_b)^2+(n\pi L_a)^2}} \tag{33}$$

When N=2, we get

$$1-(e_0a)^2\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]+(e_0a)^4\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]^2\geq 0 \tag{34}$$

The admissibility of Eq. (34) is satisfied naturally for selecting the nonlocal scale parameter arbitrarily.

Furthermore, it is easily proved that the requirement of the nonlocal scale parameter in the case of N=2s+1 is identical to the case with N=1, while the case of N=2s is the same as N=2, i.e., no upper limit of e_0a is required when N=2s in Eq. (31). It seems the required condition of the nonlocal scale parameter is just summarized in Eq. (33), since Eq. (34) makes no demand. Under this circumstance, the ratio of the geometrical sequence in brace of Eq. (31) is less than 1. Consequently, the circular frequencies can be written as

$$\omega_{mn}=\sqrt{\frac{D\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]^2}{\left\{\frac{\rho h^3}{12}\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]+\rho h\right\}\left\{1+(e_0a)^2\left[\left(\frac{m\pi}{L_a}\right)^2+\left(\frac{n\pi}{L_b}\right)^2\right]\right\}}} \tag{35}$$

In order to reveal the nonlocal scale effect on the free vibration of nano-plates, Fig. 5 is plotted according to Eq. (35). Note that the upper limit of the nonlocal scale parameter can be calculated from Eq. (33) with the following parameters: $L_a=L_b=10$ nm, h=0.34 nm, E=1.06 TPa, μ=0.25, ρ=2250 kg/m^3 [46]. It shows that the circular frequencies decrease slowly at the beginning of increasing the nonlocal scale parameter. When the nonlocal scale parameter arrives at a certain value, the decrease rate of the circular frequency curves becomes faster. There is a common upper limit for the nonlocal scale parameter in Fig. 5 and it is the minimum one determined from Eq. (33).

In Fig. 5, the analysis is not exactly correct. Indeed, the nonlocal scale parameter in Eq. (34) cannot be chosen freely, although the tenability of the inequality (34) is always valid with the arbitrary parameter e_0a mathematically. There is a sudden increase for the circular frequencies caused by arbitrarily choosing the nonlocal scale parameter, thereby causing a contradictory to the nonlocal softening model. We can explain it as follows. By selecting different terms in the infinite series of Eq. (31), we show some numerical examples for the effect of the nonlocal scale parameter on vibration frequencies, where the parameters in [46] are still adopted. It is noted that the Galerkin method was used to solve the natural frequencies of nano-plates in [46]. When different terms of the infinite series are taken (e.g., N=1, 2 and 20), the first four order modes are presented in Figs. 6 and 7 and Table 1. As introduced earlier, there are two nonlocal elasticity models. In this study, we use the nonlocal softening model, the equivalent stiffness of nano-structures should decrease as compared with the stiffness predicted directly by the classical continuum theory. In other words, the vibration frequencies based on the nonlocal theory are smaller than that of the classical theory.

Using Eq. (33) and the parameters listed above [46], we can calculate the limit values of the nonlocal scale parameter e_0a as $10\big/\sqrt{2}\pi$ nm for $m=n=1$ (ω_{11}), $10\big/\sqrt{5}\pi$ nm for m=1 and n=2, or m=2 and n=1 (ω_{12} or ω_{21}), $5\big/\sqrt{2}\pi$ nm for $m=n=2$ (ω_{22}), respectively. Hence, the common upper limit of the nonlocal scale parameter for the first four order modes with N=1 is $5\big/\sqrt{2}\pi$ nm. Accordingly, the effect of the nonlocal scale parameter on the first four order modes is shown in Fig. 6 for N=1, where the upper limit of the nonlocal scale parameter is utilized for the range of the horizontal axis. It is seen that the nonlocal circular frequencies ($e_0a>0$) are smaller than of the classical ones ($e_0a=0$). This is consistent with the viewpoint of the nonlocal softening model. As aforementioned, the case of odd number terms N=2s+1 is the same as N=1. In the present work, we only consider the first four order modes, which can reflect some fundamental and remarkable findings.

Unlike the cases of N=1 and N=2s+1, the calculation results of N=2 show that the nonlocal circular frequencies decrease with increasing the nonlocal scale parameter, and so the nonlocal frequencies are smaller than the corresponding classical counterparts involving $e_0a=0$ when the nonlocal scale parameter is not too large, as illustrated in Figs. 7(a)-7(c). When the nonlocal scale parameter exceeds a certain critical value, a reversed relationship between the circular frequency and the nonlocal scale parameter occurs, i.e., the nonlocal circular frequencies increase with an increase in the nonlocal scale parameter, as illustrated in Fig. 7(a) where only ω_{11} with a relative larger range of nonlocal scale parameter is taken as an example. Obviously, it opposes the mechanism of the nonlocal softening model adopted in this work. Consequently, the upper limit of the nonlocal scale parameter can be determined accordingly, that is, the critical value of the nonlocal scale parameter that causes an opposite performance of the circular frequencies is the upper limit of that nonlocal scale parameter. Accordingly, although the nonlocal scale parameter seems to be chosen freely when N=2 from Eq. (34), the upper limits of the nonlocal scale parameter in the first four order modes should be 1.6nm, 1.0nm, 0.8nm, respectively, in Figs. 7(a)-7(c), in order to conform the physical implication of the nonlocal softening phenomenon. Hence the common upper limit of the nonlocal scale parameter in these cases is 0.8nm. In addition, the nonlocal material parameter e_0 can also be determined by considering the magnitude of the internal characteristic length a. For a carbon material (the nano-plate model corresponds to a graphene sheet), a=0.142nm and hence e_0=5.63 is obtained. Because the actual value is unknown, e_0 was usually assumed as a range of 0~14 [36, 40] or 0~19 [41] in numerical examples. The present value e_0=5.63 falls within these intervals.

The results for N=20 are presented in Table 1. The first four order modes decrease with increasing the nonlocal scale parameter first, but the trend changes and the circular frequencies increase dramatically. For example, when the nonlocal scale parameter increases from 1.0 nm to 2.2 nm, the first four order modes surge from 0.14THz (ω_{12}) and 0.21THz (ω_{22}) to 874.45THz (ω_{12}) and even 1.63×10^5THz (ω_{22}). There is a huge mutation on the order of magnitude of circular frequencies inconceivably. On one hand, the nonlocal softening model requires the reduction of the frequency or the stiffness. On the other hand, it is difficult to explain the great leap of circular frequencies from a physical sense. Therefore, the common upper limit of the nonlocal scale parameter for the first four order modes is 1.0nm when N=20. A similar conclusion can be achieved for even number terms N=2s. Although specific values of the upper limit may be different in even numbers, the change is relatively small. That means the upper limit of the scale parameter e_0a or the material parameter e_0 is related to the selected number of terms N and the frequency orders m and n, but such effects do not affect the qualitative conclusion of this paper. After determining the upper limit of the scale parameter, we can conclude the circular frequencies reduce with increases of the scale parameter, which coincides with the point of nonlocal softening model. In addition, the degree of change is not identical for different modes. For example, with an increase of e_0a from 0 to 0.8nm in Table 1, the decrease percentages are 5.77%, 12.82% and 18.49% for ω_{11}, ω_{12} (ω_{21}) and ω_{22}, respectively. Consequently, the rate of fall expands for higher modes. In fact, the higher circular frequencies correspond to shorter wavelengths, which results in higher energy. The energy loss is more significant for higher energy in vibration due to various physical factors including both nonlocal and classical contributions. Hence the nonlocal scale parameter makes the higher circular frequencies decline more obviously.

We can also calculate the natural frequencies with N=0, which are f_{11}=433.89/2π GHz, f_{21}=f_{12}=1083.19/2π GHz, f_{22}=1730.65/2π GHz. These are exactly the classical results for the free vibration of simply supported plates without a nonlocal scale effect using the parameters listed above. They are in good agreement with the previous studies [46, 47]. However, there are no research studies concerned with the upper limit of the nonlocal scale parameter previously, and the similar form of Eq. (35) was derived and utilized in most work. As noted earlier, Eq. (35) is actually not correct because the upper limit of the nonlocal scale parameter is not considered and involved appropriately. Besides, the classical natural frequencies f_{mn} are identical to the results under the condition of e_0a=0 in Figs. 6 and 7 and Table 1. This validates the present model and calculation. Compared with the previous studies, the computational process is much easier in the present work.

4.2 Selection of nonlocal scale parameter in nano-structures

From the above results, we can select an upper limit for the nonlocal scale parameter. For various nano-structures, the nonlocal scale parameter indicates different effects on structural dynamical behavior. Hence it appears the structure-dependent property of the nonlocal dynamics. For the axial vibration of nano-rods and the transverse vibration of nano-beams, the circular frequencies always decrease with increasing the nonlocal scale parameter, but the decrease rates are different. While for the free vibration of nano-plates, the frequencies decrease first and then increase sharply with increasing the nonlocal scale parameter. The behavior of nonlocal dynamics may be different for different nano-structures, but the trend of the nonlocal scale effect must be the same due to the same concept of the nonlocal theory. Therefore, there exists an upper bound of the nonlocal scale parameter according to the physical implication of the nonlocal softening model. The upper limits should be common for various nano-structures because the nonlocal scale effect in nano-mechanics is identical. The upper limit and the range of the nonlocal scale parameter for nano-plates are available to nano-rods and nano-beams, although the frequency functions of nano-rods and nano-beams do not contain a jump phenomenon with respect to the scale parameter in the previous models. In summary, we can take 0~0.8nm as a range of e_0a while studying the nonlocal scale effect in dynamical behavior of nano-structures. The physical meaning of this range is that one has to account for the scale effect if the structural external characteristic length scale is at the same level as the nonlocal scale parameter, i.e., $l \approx Ne_0a \approx 0.8N$ where N<10.

In fact, the present results can be confirmed by the previous studies, Lim [22], Li [23] and Liu et al. [44] derived the nonlocal stress from the nonlocal differential constitutive relation in Eq. (1) for nano-beams, nano-rods and nano-plates, respectively. The unified form of solutions can be written as $\mathbf{t} = \mathbf{C} : \sum_{n=0}^{\infty} \left(e_0 a\right)^{2n} \nabla^{2n} \boldsymbol{\varepsilon}$, where $\mathbf{C}$ is the elastic coefficient tensor and $\boldsymbol{\varepsilon}$ is the strain tensor. The boundness of the nonlocal scale parameter can also be implied from the solution in order to ensure the convergence and finity of the nonlocal stress. Wang et al. [48] concluded the nonlocal scale effect is related to surface effect for vibrating nano-plates. The wavelengths in x and y directions are $\lambda_x = 2L_a/m$ and $\lambda_y = 2L_b/n$, respectively. An equivalent wavelength is defined as $\lambda = \lambda_x \eta / \sqrt{\eta^2+1}$ where $\eta = \lambda_y/\lambda_x$. After that, a critical equivalent wavelength $\lambda_{cr} = 19.12\left(e_0 a\right)$ for significant nonlocal scale effect is determined. Because the equivalent wavelength cannot be lower than its critical value, i.e. $\lambda \geq \lambda_{cr}$, otherwise the scale effect is so weak that one can ignore it. The range of $e_0 a$ is in the order of magnitude 0~0.74nm for the fundamental frequency via $\lambda \geq \lambda_{cr}$. Although the nonlocal scale parameter was not involved in the work by Wang et al. [48], the upper limit of the nonlocal scale parameter derived from [48] indirectly is basically in accordance with the results suggested in the present study.

Liang and Han [40] investigated the nonlocal scale parameter for graphene sheets and proposed a formulation $(e_0a)^2=R^2/8$ where R is the radius of nonlocal influence domain (circle). Until now, the area of nonlocal influence domain is still not clear. Because the nonlocal scale effect stems from the long-range interactions between atoms, we can consider how long the distance from other points to a specified point (i.e. center of the nonlocal influence circle) is, with significant long-range interactions. Edelen [6] pointed out that the long-range interactions for electrostatic force between non-adjacent atoms in crystal materials can arrive at the order of magnitude of tenfold lattice distance. At a nano-scale, such an acting distance can be considered to be of a long-range. Therefore, if we choose the radius of nonlocal influence circle R=15a at most, otherwise it no longer belongs to the nonlocal long-range interaction after the radius exceeds 15a, we can deduce the result from the existing literature: the range of e_0a is in the order of magnitude 0~0.75nm. This result also supports the claim of this study.

5 Conclusions

The main purpose of this article is to discuss the upper limit of the nonlocal scale parameter. The nano-structural dependence of nonlocal dynamic behavior is demonstrated. It is found that the vibration behavior of nano-rods, nano-beams and nano-plates due to their structural characteristics is different from each other. The nonlocal scale effect should be coincident qualitatively as these nonlocal nano-structural models have a common consideration of long-range interactions between atoms. Based on the nonlocal partial-differential constitutive relations, we investigate three models, i.e., the axial vibration of nano-rods, transverse vibration of nano-beams and free vibration of nano-plates, in accordance with the nonlocal softening elasticity model. Furthermore, not only we formulated the new governing equations for nano-rods and nano-beams, and also proposed a simpler solution method for nano-plates. The upper limit of the nonlocal scale parameter is thus determined in order to obey the nonlocal softening mechanism. The physical meaning of the scale upper limit is explained and it is conducive to estimating the structural external characteristic scale at which the nonlocal scale effect should be considered. From the present study, we can take 0~0.8nm as a range of e_0a while studying the nonlocal scale effect in dynamical behavior of nano-structures. The results are compared with the existing literature to confirm its validity. The puzzled question is solved accordingly, and the present work provides a basis for the choice of a nonlocal scale range when studying the mechanical properties of nano-materials/structures by means of the nonlocal theory.

Acknowledgements

The work described in this paper was fully supported by the National Natural Science Foundation of China (Grant No. 11602210), the open project of State Key Laboratory of Mechanics and Control of Mechanical Structures (Nanjing University of Aeronautics and Astronautics) (Grant No. MCMS-0418G01), and the Matching Grant from the Hong Kong Polytechnic University (Project No. 4-BCDS).

References

[1] G.M. Odegard, T.S. Gates, L.M. Nicholson, K.E. Wise, Equivalent-continuum modeling of nano-structured materials, Compos. Sci. Technol. 62 (2002) 1869–1880.

[2] L.F. Wang, H.Y. Hu, Flexural wave propagation in single-walled carbon nanotubes, Phys. Rev. B 71 (2005) 195412.

[3] J.W. Yan, K.M. Liew, L.H. He, A higher-order gradient theory for modeling of the vibration behavior of single-wall carbon nanocones, Appl. Math. Model. 38 (2014) 2946–2960.

[4] J.W. Yan, K.M. Liew, Predicting elastic properties of single-walled boron nitride nanotubes and nanocones using an atomistic-continuum approach, Compos. Struct. 125 (2015) 489–498.

[5] A.C. Eringen, D.G.B. Edelen, On nonlocal elasticity, Int. J. Eng. Sci. 10 (1972) 233–248.

[6] D.G.B. Edelen, Nonlocal field theories, In: Eringe (Ed.), Continuum Physics (IV), Academic Press, New York, 1976, pp. 75–204.

[7] A.C. Eringen, On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves, J. Appl. Phys. 54 (1983) 4703–4710.

[8] C. Li, C.W. Lim, J.L. Yu, Dynamics and stability of transverse vibrations of nonlocal nanobeams with a variable axial load, Smart Mater. Struct. 20 (2011) 015023.

[9] A. Assadi, B. Farshi, Stability analysis of graphene based laminated composite sheets under non-uniform inplane loading by nonlocal elasticity, Appl. Math. Model. 35 (2011) 4541–4549.

[10] Z. Huang, Nonlocal effects of longitudinal vibration in nanorod with internal long-range interactions, Int. J. Solids Struct. 49 (2012) 2150–2154.

[11] Ö. Civalek, C. Demir, Bending analysis of microtubules using nonlocal Euler–Bernoulli beam theory, Appl. Math. Model. 35 (2011) 2053–2067.

[12] B. Arash, Q. Wang, A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes, Comp. Mater. Sci. 51 (2012) 303–313.

[13] O. Rahmani, O. Pedram, Analysis and modeling the size effect on vibration of functionally graded nanobeams based on nonlocal Timoshenko beam theory, Int. J. Eng. Sci. 77 (2014) 55–70.

[14] R. Ansari, M.F. Oskouie, R. Gholami, F. Sadeghi, Thermo-electro-mechanical vibration of postbuckled piezoelectric Timoshenko nanobeams based on the nonlocal elasticity theory, Compos. Part B-Eng. 89 (2016) 316–327.

[15] X. Li, L. Li, Y. Hu, Z. Ding, W. Deng, Bending, buckling and vibration of axially functionally graded beams based on nonlocal strain gradient theory, Compos. Struct. 165 (2017) 250–265.

[16] J.J. Liu, C. Li, X.L. Fan, L.H. Tong, Transverse free vibration and stability of

axially moving nanoplates based on nonlocal elasticity theory, Appl. Math. Model. 45 (2017) 65–84.

[17] C. Li, J.J. Liu, M. Cheng, X.L. Fan, Nonlocal vibrations and stabilities in parametric resonance of axially moving viscoelastic piezoelectric nanoplate subjected to thermo-electro-mechanical forces, Compos. Part B-Eng. 116 (2017) 153–169.

[18] X.J. Xu, X.C. Wang, M.L. Zheng, Z. Ma, Bending and buckling of nonlocal strain gradient elastic beams, Compos. Struct. 160 (2017) 366–377.

[19] S.A. Faghidian, On non-linear flexure of beams based on non-local elasticity theory, Int. J. Eng. Sci. 124 (2018) 49–63.

[20] F. Ebrahimi, M.R. Barati, Vibration analysis of smart piezoelectrically actuated nanobeams subjected to magneto-electrical field in thermal environment, J. Vib. Control 24 (2018) 549–564.

[21] C.W. Lim, Is a nanorod (or nanotube) with a lower Young's modulus stiffer? Is not Young's modulus a stiffness indicator? Sci. China Phys. Mech. 53 (2010) 712–724.

[22] C.W. Lim, On the truth of nanoscale for nanobeams based on nonlocalelastic stress field theory: equilibrium, governing equation and static deflection, Appl. Math. Mech. 31 (2010) 37–54.

[23] C. Li, A nonlocal analytical approach for torsion of cylindrical nanostructures and the existence of higher-order stress and geometric boundaries, Compos. Struct. 118 (2014) 607–621.

[24] C. Li, Torsional vibration of carbon nanotubes: comparison of two nonlocal models and a semi-continuum model, Int. J. Mech. Sci. 82 (2014) 25–31.

[25] C. Li, S. Li, L.Q. Yao, Z.K. Zhu, Nonlocal theoretical approaches and atomistic simulations for longitudinal free vibration of nanorods/nanotubes and verification of different nonlocal models, Appl. Math. Model. 39 (2015) 4570–4585.

[26] C.M. Wang, S. Kitipornchai, C.W. Lim, M. Eisenberger, Beam bending solutions based on nonlocal Timoshenko beam theory, J. Eng. Mech.-ASCE 134 (2008) 475–481.

[27] Y.M. Yu, C.W. Lim, Nonlinear constitutive model for axisymmetric bending of annular graphene-like nanoplate with gradient elasticity enhancement effects, J. Eng. Mech.-ASCE 139 (2014) 1025–1035.

[28] J. Wang, H. Shen, B. Zhang, J. Liu, Y. Zhang, Complex modal analysis of transverse free vibrations for axially moving nanobeams based on the nonlocal strain gradient theory, Physica E 101 (2018) 85–93.

[29] C.W. Lim, G. Zhang, J.N. Reddy, A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation, J. Mech. Phys. Solids 78 (2015) 298–313.

[30] X.F. Li, G.J. Tang, Z.B. Shen, K.Y. Lee, Size-dependent resonance frequencies

of longitudinal vibration of a nonlocal Love nanobar with a tip nanoparticle, Math. Mech. Solids 22 (2017) 1529–1542.

[31] Y. Yang, C.W. Lim, Non-classical stiffness strengthening size effects for free vibration of a nonlocal nanostructure, Int. J. Mech. Sci. 54 (2012) 57–68.

[32] C.M. Wang, W.H. Duan, Free vibration of nanorings/arches based on nonlocal elasticity, J. Appl. Phys.104 (2008) 014303.

[33] P. Lu, Dynamic analysis of axially prestressed micro/nanobeam structures based on nonlocal beam theory, J. Appl. Phys. 101 (2007) 073504.

[34] C.M. Wang, Y.Y. Zhang, X.Q. He, Vibration of nonlocal Timoshenko beams, Nanotechnology 18 (2007) 105401.

[35] S.Q. Guo, S.P. Yang, Axial vibration analysis of nanocones based on nonlocal elasticity theory, Acta Mech. Sin. 28 (2012) 801–807.

[36] T. Murmu, S.C. Pradhan, Vibration analysis of nano-single-layered graphene sheets embedded in elastic medium based on nonlocal elasticity theory, J. Appl. Phys. 105 (2009) 064319.

[37] T. Murmu, J. Sienz, S. Adhikari, C. Arnold, Nonlocal buckling of double-nanoplate-systems under biaxial compression, Compos. Part B-Eng. 44 (2013) 84–94.

[38] P. Lu, H.P. Lee, C. Lu, P.Q. Zhang, Dynamic properties of flexural beams using a nonlocal elasticity model, J. Appl. Phys. 99 (2006) 073510.

[39] P. Lu, H.P. Lee, C. Lu, P.Q. Zhang, Application of nonlocal beam models for carbon nanotubes, Int. J. Solids Struct. 44 (2007) 5289–5300.

[40] Y. Liang, Q. Han, Prediction of the nonlocal scaling parameter for graphene sheet, Eur. J. Mech.-A/Solid. 45 (2014) 153–160.

[41] W.H. Duan, C.M. Wang, Y.Y. Zhang, Calibration of nonlocal scaling effect parameter for free vibration of carbon nanotubes by molecular dynamics, J. Appl. Phys. 101 (2007) 024305

[42] M. Aydogdu, Axial vibration of the nanorods with the nonlocal continuum rod model, Physica E 41 (2009) 861–864.

[43] C.W. Lim, C. Li, J.L. Yu, Free vibration of pre-tensioned nanobeams based on nonlocal stress theory, J. Zhejiang Univ.-Sci. A 11 (2010) 34–42.

[44] J.J. Liu, L. Chen, F. Xie, X.L. Fan, C. Li, On bending, buckling and vibration of graphene nanosheets based on the nonlocal theory, Smart Struct. Syst. 17 (2016) 257–274.

[45] A. Anjomshoa, A.R. Shahidi, B. Hassani, E. Jomehzadeh, Finite element buckling analysis of multi-layered graphene sheets on elastic substrate based on nonlocal elasticity theory, Appl. Math. Model. 38 (2014) 5934–5955.

[46] Z.B. Shen, H.L. Tang, D.K. Li, G.J. Tang, Vibration of single-layered graphene sheet-based nanomechanical sensor via nonlocal Kirchhoff plate theory, Comp. Mater. Sci. 61 (2012) 200–205.

[47] S.C. Pradhan, A. Kumar, Vibration analysis of orthotropic graphene sheets using

nonlocal elasticity theory and differential quadrature method, Compos. Struct. 93 (2011) 774–779.

[48] C.Y. Wang, T. Murmu, S. Adhikari, Mechanisms of nonlocal effect on the vibration of nanoplates, Appl. Phys. Lett. 98 (2011) 153101.

Caption of Figures

Caption of Table

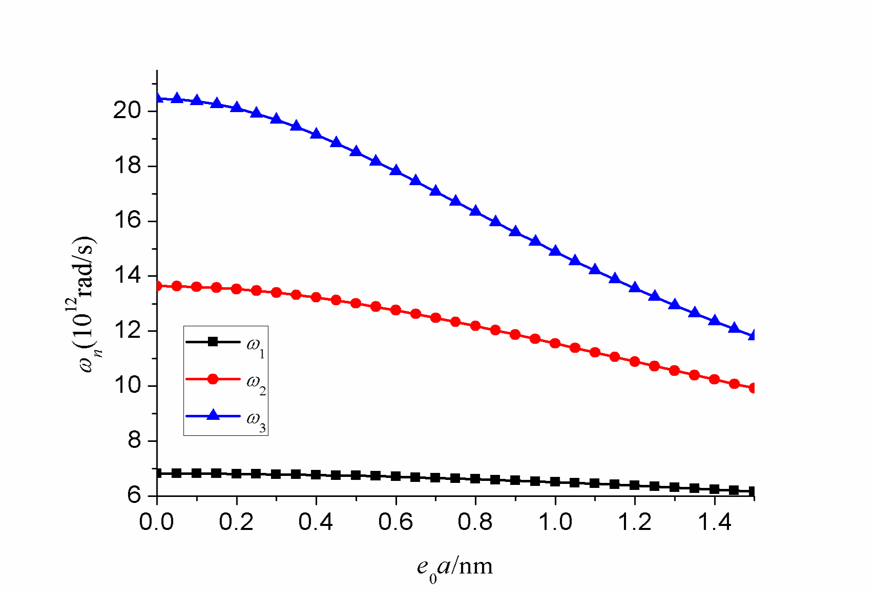


Fig. 1 Effect of nonlocal scale parameter on circular frequency for clamped nano-rods

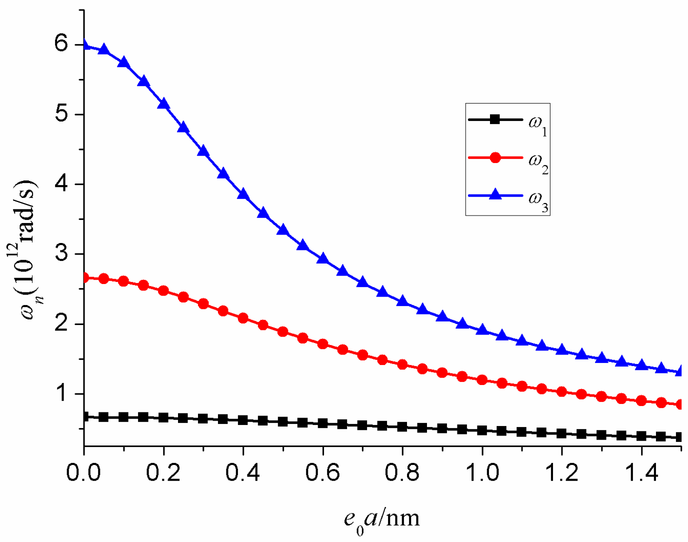


Fig. 2 Effect of nonlocal scale parameter on circular frequency for simply supported nano-beams

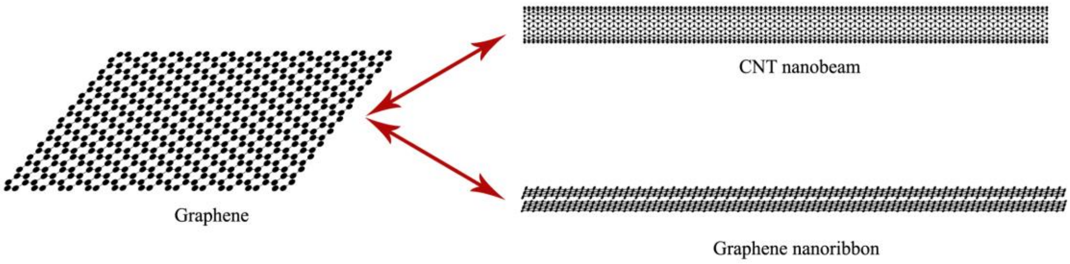


Fig. 3 Conversion between graphene sheet, carbon nanotube and graphene nano-ribbon

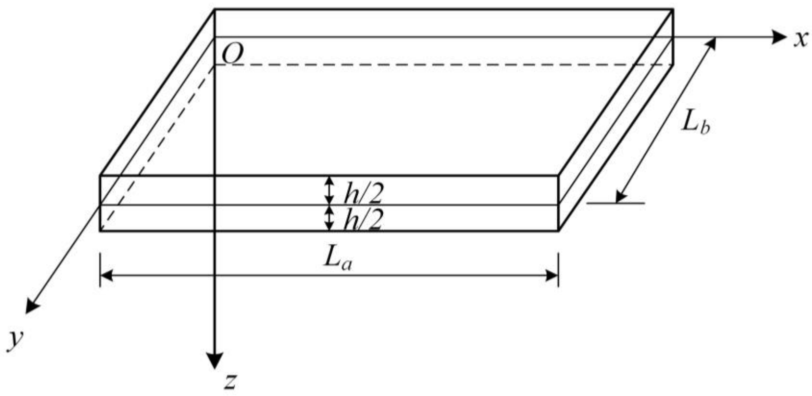


Fig. 4 Schematic of a nano-plate

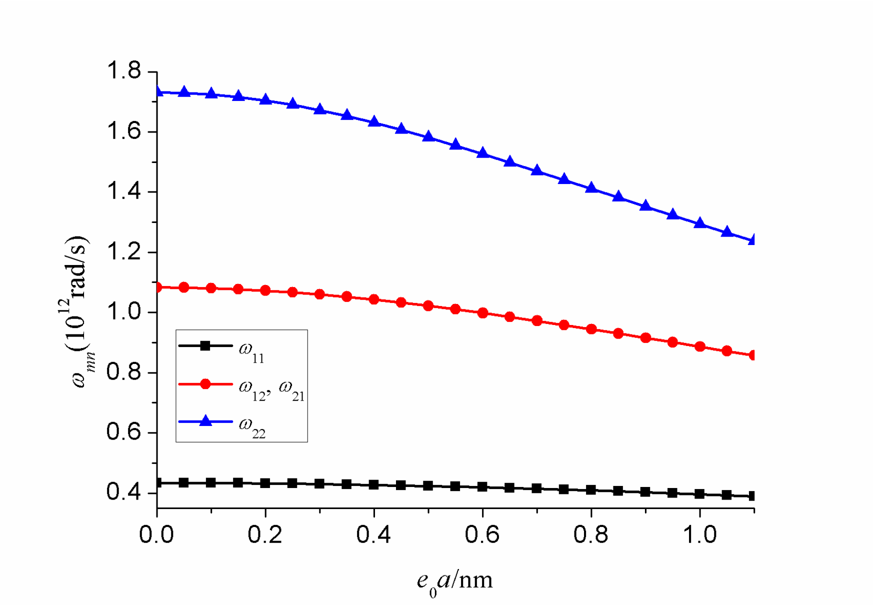


Fig. 5 Effect of nonlocal scale parameter on circular frequency for simply supported nano-plates

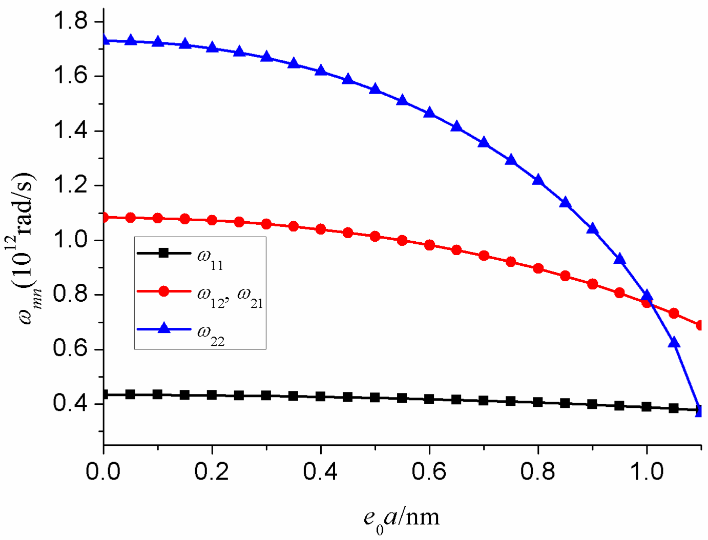
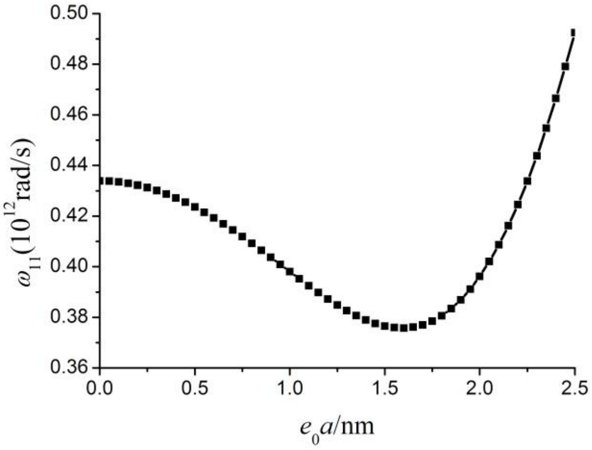
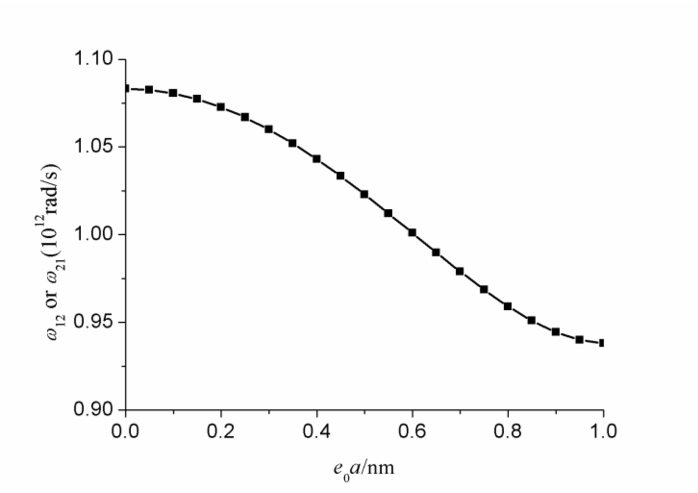


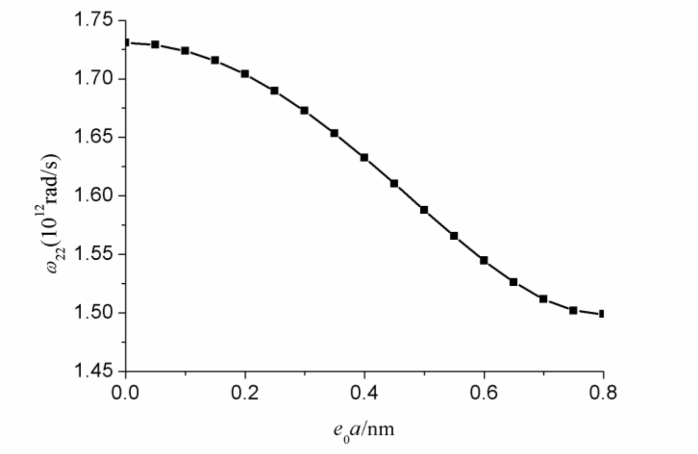
Fig. 6 Variation of circular frequencies with respect to nonlocal scale parameter for *N*=1



(a)



(b)



(c)

Fig. 7 Variation of circular frequencies with respect to nonlocal scale parameter for N=2

Table 1 Variation of circular frequencies with respect to nonlocal scale parameter for N=20

e_0a (nm)	**ω_{11} (10^{12}rad/s)**	**ω_{12} or ω_{21} (10^{12}rad/s)**	**ω_{22} (10^{12}rad/s)**
0	0.43389	1.08319	1.73065
0.2	0.43219	1.07265	1.70395
0.4	0.42720	1.04280	1.63071
0.6	0.41925	0.99815	1.52716
0.8	0.40884	0.94429	1.41057
1.0	0.39652	0.88635	1.29822
1.2	0.38288	0.82850	4.70856
1.4	0.36843	0.94479	106.27974
1.6	0.35364	8.40973	1611.47657
1.8	0.33887	92.76365	17616.03573
2.0	0.32548	792.50489	148927.27052
2.2	0.36496	5494.30464	1.02349×10^6