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An iterative reduced-order substructuring approach to the calculation of eigensolutions and eigensensitivities

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ABSTRACT

Substructuring methods are efficient to estimate some lowest eigensolutions and eigensensitivities of large-scale structural systems by representing the global eigenequation with small-sized substructural eigenmodes. Inclusion of more substructural eigenmodes improves the accuracy of eigensolutions and eigensensitivities, whereas decreases the computational efficiency adversely. This paper proposes a new iterative reduced-order substructuring method to calculate the eigensolutions and eigensensitivities of the global structure. A modal transformation matrix, relating the higher modes to the lower modes, is derived to transform the original frequency-dependent matrices of each substructure into frequency-independent ones. A simplified reduced-order eigenequation is then obtained through a few iterations performed on the modal transformation matrix and mass matrix. The eigensolutions and eigensensitivities of the global structure are calculated accurately with a small number of substructural eigenmodes retained, avoiding the inclusion of numerous substructural eigenmodes. Applications of the proposed method to a numerical frame and a practical large-scale structure demonstrate that the eigensolutions and eigensensitivities of the global structure can be calculated accurately with only a small number of substructural eigenmodes and a few iterations.

Keywords: Substructuring method; Model reduction; Eigensolution; Eigensensitivity

1. Introduction

Eigensolutions and eigensensitivities are widely used in model updating, damage detection, optimization design and so on. For example, in vibration-based model updating, the finite element (FE) model is iteratively modified to ensure its vibration properties reproduce the measured counterparts in an optimal way [1-7]. The eigensolutions are used to form the objective function and the eigensensitivities are used to indicate a rapid searching direction for the optimization algorithm. To accurately reflect the physical characteristics of a large-scale structure, a complex FE model with numerous nodes, elements, degrees of freedom (DOFs) and elemental parameters is required. Calculation of eigensolutions and eigensensitivities on such complex model is time-consuming.

Substructuring methods are efficient to calculate the eigensolutions and eigensensitivities of large-scale structures. The methods partition the global structure into several substructures to calculate the substructural solutions independently, which are then assembled to recover the vibration characteristics of the global structure [8, 9]. The substructuring methods are efficient in three main respects. First, the global structure is divided into several smaller substructures, and it is quicker and easier to analyze the small substructural system matrices. Second, the substructures are studied independently. The substructuring methods allow for the specific substructures (local area) to be analyzed, without analyzing the entire structure as a whole. Third, the substructuring method will be more efficient if combined with the model reduction [10] or parallel computation technique [11, 12].

Since Hurty [13] first proposed the dynamic substructuring method with a fixed interface condition, many substructuring methods have been developed to calculate structural eigensolutions and eigensensitivities. For example, Craig and Bampton [14] simplified the fixed interface method without separating the boundary forces into statically determinate and indeterminate ones, which is successful due to its simplicity, precision and robustness. Rixen [15] proposed an efficient dual Craig-Bampton method with free interface condition. It assembled the substructures using dual assembly of the interface forces and defines a transformation matrix to reduce the primal assembled system. Kim et al. [16, 17] later improved its accuracy with enhanced transformation matrix by compensating the higher order effects of substructural modes with residual flexibility. Kron [18] proposed an efficient free-interface substructuring method to deal with large-scale structures. Weng et al. [19, 20] later improved the Kron's method for fast computation of eigensolutions and eigensensitivities.

In most of the existing substructuring methods, some lowest eigenmodes (master modes) of each substructure are retained to recover the vibration characteristics of the global structure, and the higher modes (slave modes) are removed or compensated by a static item for the efficiency [9, 19]. More substructural master modes need to be retained to search for more accurate results, which conversely cut down the computational efficiency sharply. Weng et al. [21] proposed an iterative substructuring method to achieve more accurate eigensolutions and eigensensitivities with only a few master modes retained. However, it is inefficient as the system matrices are frequency-dependent and thus the eigensolutions and eigensensitivities have to be calculated mode by mode.

This paper proposes an iterative reduced-order substructuring method to calculate the eigensolutions and eigensensitivities efficiently and accurately. A modal transformation matrix is derived to relate the master modes to the slave modes. The system matrices, which are frequency-dependent in the original eigenequation, are transformed into the frequency-independent ones. The modal transformation matrix is obtained with an iterative process and it has a much smaller order than the iterative variables employed in the previous iterative substructuring method [21], which improves the computational efficiency significantly. In consequence, the eigensolutions and eigensensitivities of required modes are calculated simultaneously based on a small number of substructural eigenmodes. The computational accuracy and efficiency of the proposed method are verified using a numerical frame model and a practical large-scale structure.

2. Proposed approach to the calculation of eigensolutions

In the substructuring methods, the global structure is first partitioned into *NS* manageable substructures. Each substructure is treated as an independent structure and its eigensolutions are calculated from the following eigenequation

$$\mathbf{K}^{(j)}\mathbf{\Phi}^{(j)} = \mathbf{\Lambda}^{(j)}\mathbf{M}^{(j)}\mathbf{\Phi}^{(j)} \tag{1}$$

where $\mathbf{K}^{(j)}$ and $\mathbf{M}^{(j)}$ are the $n^{(j)}\times n^{(j)}$ stiffness and mass matrices of the jth (j=1, 2, …, *NS*) substructure, respectively. $n^{(j)}$ is the number of DOFs of the jth substructure. $\mathbf{\Lambda}^{(j)} = \mathrm{Diag}\left(\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_{n^{(j)}}^{(j)}\right)$ encloses all eigenvalues of the jth substructure, and

$\boldsymbol{\Phi}^{(j)}=\left[\boldsymbol{\phi}_1^{(j)},\boldsymbol{\phi}_2^{(j)},\ldots,\boldsymbol{\phi}_{n^{(j)}}^{(j)}\right]$ are the corresponding mass-normalized mode shapes. $\boldsymbol{\Lambda}^{(j)}$ and $\boldsymbol{\Phi}^{(j)}$ follow the orthogonal conditions of

$$\left[\boldsymbol{\Phi}^{(j)}\right]^{\mathrm{T}}\mathbf{K}^{(j)}\boldsymbol{\Phi}^{(j)}=\boldsymbol{\Lambda}^{(j)} \tag{2}$$

$$\left[\boldsymbol{\Phi}^{(j)}\right]^{\mathrm{T}}\mathbf{M}^{(j)}\boldsymbol{\Phi}^{(j)}=\mathbf{I}_{n^{(j)}} \tag{3}$$

According to Kron's substructuring method, the *NS* independent substructures are assembled to form the global structure by exerting internal connection forces on the interfaces of the adjacent substructures as [18, 21]

$$\begin{bmatrix}\overline{\boldsymbol{\Lambda}}-\lambda_i\mathbf{I} & -\boldsymbol{\Gamma}\\ -\boldsymbol{\Gamma}^{\mathrm{T}} & \mathbf{0}\end{bmatrix}\begin{Bmatrix}\mathbf{z}\\ \boldsymbol{\tau}\end{Bmatrix}_i=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\end{Bmatrix} \tag{4}$$

where $\boldsymbol{\Gamma}=\left(\mathbf{C}\overline{\boldsymbol{\Phi}}\right)^{\mathrm{T}}$. $\overline{\boldsymbol{\Lambda}}=\mathrm{Diag}\left(\boldsymbol{\Lambda}^{(1)},\boldsymbol{\Lambda}^{(2)},\cdots,\boldsymbol{\Lambda}^{(NS)}\right)$ and $\overline{\boldsymbol{\Phi}}=\mathrm{Diag}\left(\boldsymbol{\Phi}^{(1)},\boldsymbol{\Phi}^{(2)},\cdots,\boldsymbol{\Phi}^{(NS)}\right)$ are the diagonally assembled eigenvalues and mode shapes of all substructures respectively. λ_i is the *i*th eigenvalue of the global structure. $\mathbf{z}_i$ is the mode participation factor, which recovers the mode shapes of the assembled global structure by $\tilde{\boldsymbol{\phi}}_i=\overline{\boldsymbol{\Phi}}\mathbf{z}_i$. $\mathbf{C}$ indicates the connection relationship between the interface DOFs of the adjacent substructures. Hereinafter, the matrices with an overbar imply the variables diagonally assembled from the substructures. The assembled eigenvalues $\overline{\boldsymbol{\Lambda}}$ and mode shapes $\overline{\boldsymbol{\Phi}}$ satisfy the orthogonal conditions of

$$\overline{\boldsymbol{\Phi}}^{\mathrm{T}}\overline{\mathbf{K}}\overline{\boldsymbol{\Phi}}=\overline{\boldsymbol{\Lambda}} \tag{5}$$

$$\overline{\boldsymbol{\Phi}}^{\mathrm{T}}\overline{\mathbf{M}}\overline{\boldsymbol{\Phi}}=\mathbf{I} \tag{6}$$

where $\overline{\mathbf{K}}$=Diag($\mathbf{K}^{(1)}$, $\mathbf{K}^{(2)}$, …, $\mathbf{K}^{(NS)}$) and $\overline{\mathbf{M}}$=Diag($\mathbf{M}^{(1)}$, $\mathbf{M}^{(2)}$, …, $\mathbf{M}^{(NS)}$) are respectively the diagonally assembled stiffness and mass matrices of all substructures. It is time consuming to calculate the complete eigensolutions of all substructures according to Eq. (4). As a solution, a modal truncation technique is introduced. In this paper, only the master modes of each

substructure are retained to compute the eigensolutions of the global structure, and the contribution of the discarded slave modes is compensated by an iterative process performed on the master modes.

The $n^{(j)}$ complete modes of the jth substructure is divided into $m^{(j)}$ master modes and $s^{(j)}$ slave modes ($n^{(j)}=m^{(j)}+s^{(j)}$). Eq. (4) is thus rewritten as [21]

$$\begin{bmatrix} \bar{\mathbf{\Lambda}}^m-\lambda_i\mathbf{I}^m & \mathbf{0} & -\mathbf{\Gamma}^m \\ \mathbf{0} & \bar{\mathbf{\Lambda}}^s-\lambda_i\mathbf{I}^s & -\mathbf{\Gamma}^s \\ -\left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} & -\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{z}^m \\ \mathbf{z}^s \\ \boldsymbol{\tau} \end{Bmatrix}_i = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{7}$$

where $\mathbf{\Gamma}^m=\left(\mathbf{C}\bar{\mathbf{\Phi}}^m\right)^{\mathrm{T}}$ and $\mathbf{\Gamma}^s=\left(\mathbf{C}\bar{\mathbf{\Phi}}^s\right)^{\mathrm{T}}$. $\bar{\mathbf{\Lambda}}^m$ and $\bar{\mathbf{\Phi}}^m$ are respectively the diagonally assembled master eigenvalues and eigenvectors of the substructures. $\bar{\mathbf{\Lambda}}^s$ and $\bar{\mathbf{\Phi}}^s$ are the diagonally assembled slave eigenvalues and eigenvectors. Hereinafter, the superscripts 'm' and 's' denote the variables associated with the master and slave modes respectively. $\bar{\mathbf{\Lambda}}^m$, $\bar{\mathbf{\Phi}}^m$, $\bar{\mathbf{\Lambda}}^s$ and $\bar{\mathbf{\Phi}}^s$ take the form of

$$\begin{aligned}
\bar{\mathbf{\Lambda}}^m &= \mathrm{Diag}\left(\left[\mathbf{\Lambda}^m\right]^{(1)},\cdots,\left[\mathbf{\Lambda}^m\right]^{(j)},\cdots,\left[\mathbf{\Lambda}^m\right]^{(NS)}\right),\ \left[\mathbf{\Lambda}^m\right]^{(j)}=\mathrm{Diag}\left(\lambda_1^{(j)},\lambda_2^{(j)},\cdots,\lambda_{m^{(j)}}^{(j)}\right) \\
\bar{\mathbf{\Phi}}^m &= \mathrm{Diag}\left(\left[\mathbf{\Phi}^m\right]^{(1)},\cdots,\left[\mathbf{\Phi}^m\right]^{(j)},\cdots,\left[\mathbf{\Phi}^m\right]^{(NS)}\right),\ \left[\mathbf{\Phi}^m\right]^{(j)}=\left[\boldsymbol{\phi}_1^{(j)},\boldsymbol{\phi}_2^{(j)},\cdots,\boldsymbol{\phi}_{m^{(j)}}^{(j)}\right] \\
\bar{\mathbf{\Lambda}}^s &= \mathrm{Diag}\left(\left[\mathbf{\Lambda}^s\right]^{(1)},\cdots,\left[\mathbf{\Lambda}^s\right]^{(j)},\cdots,\left[\mathbf{\Lambda}^s\right]^{(NS)}\right),\ \left[\mathbf{\Lambda}^s\right]^{(j)}=\mathrm{Diag}\left(\lambda_{m^{(j)}+1}^{(j)},\lambda_{m^{(j)}+2}^{(j)},\cdots,\lambda_{m^{(j)}+s^{(j)}}^{(j)}\right) \\
\bar{\mathbf{\Phi}}^s &= \mathrm{Diag}\left(\left[\mathbf{\Phi}^s\right]^{(1)},\cdots,\left[\mathbf{\Phi}^s\right]^{(j)},\cdots,\left[\mathbf{\Phi}^s\right]^{(NS)}\right),\ \left[\mathbf{\Phi}^s\right]^{(j)}=\left[\boldsymbol{\phi}_{m^{(j)}+1}^{(j)},\boldsymbol{\phi}_{m^{(j)}+2}^{(j)},\cdots,\boldsymbol{\phi}_{m^{(j)}+s^{(j)}}^{(j)}\right]
\end{aligned} \tag{8}$$

The second line of Eq. (7) gives

$$\mathbf{z}_i^s=\left(\bar{\mathbf{\Lambda}}^s-\lambda_i\mathbf{I}^s\right)^{-1}\mathbf{\Gamma}^s\boldsymbol{\tau}_i=\mathbf{t}_i\boldsymbol{\tau}_i \tag{9}$$

where

$$\mathbf{t}_i = \left(\overline{\mathbf{\Lambda}}^s - \lambda_i \mathbf{I}^s\right)^{-1} \mathbf{\Gamma}^s \tag{10}$$

$\mathbf{t}_i$ is rewritten as

$$\mathbf{t}_i = \left[\overline{\mathbf{\Lambda}}^s\right]^{-1} \mathbf{\Gamma}^s + \lambda_i \left[\overline{\mathbf{\Lambda}}^s\right]^{-1} \mathbf{t}_i \tag{11}$$

Substitution of Eq. (9) into the first and third line of Eq. (7) gives

$$\begin{bmatrix} \overline{\mathbf{\Lambda}}^m - \lambda_i \mathbf{I}^m & -\mathbf{\Gamma}^m \\ -\left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} & -\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i \end{bmatrix} \begin{Bmatrix} \mathbf{z}^m \\ \boldsymbol{\tau} \end{Bmatrix}_i = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{12}$$

According to the second line of Eq. (12), $\boldsymbol{\tau}_i$ is expressed by $\mathbf{z}_i^m$ as

$$\boldsymbol{\tau}_i = -\left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m \tag{13}$$

Substitute Eq. (13) into Eq. (9), $\mathbf{z}_i^s$ is thus expressed by $\mathbf{z}_i^m$ as

$$\mathbf{z}_i^s = -\mathbf{t}_i \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m \tag{14}$$

Substitute Eq. (13) into the first line of Eq. (12), one can obtain

$$\left(\overline{\mathbf{\Lambda}}^m + \mathbf{\Gamma}^m \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} - \lambda_i \mathbf{I}^m\right) \mathbf{z}_i^m = \mathbf{0} \tag{15}$$

Eq. (15) is the reduced eigenequation of Kron's substructuring method with the order of n_m ($n_m = \sum_{j=1}^{NS} m^{(j)}$), which is equal to the number of master modes of all substructures. $\mathbf{\Gamma}^m \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}}$ is a frequency-dependent matrix and consequently Eq. (15) cannot be solved directly. In our previous iterative substructuring method [21], an iterative process based on Eq. (11) is performed to solve Eq. (15), which is precise but inefficient as the eigensolutions and eigensensitivities have to be calculated mode by mode. In this study, a simplified process will be derived to transform the frequency-dependent matrix $\mathbf{\Gamma}^m \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}}$ into a frequency-independent one. This ensures the eigensolutions and eigensensitivities to be calculated efficiently for all modes simultaneously.

According to Eq. (11), the frequency-dependent item $\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}$ is expanded as

$$\begin{aligned}\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}&=\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\boldsymbol{\Gamma}^s+\lambda_i\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\mathbf{t}_i\right)^{-1}\\&=\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\boldsymbol{\Gamma}^s\right)^{-1}-\lambda_i\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\boldsymbol{\Gamma}^s\right)^{-1}\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\mathbf{t}_i\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}\end{aligned} \tag{16}$$

Considering the orthogonal condition of $\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}\bar{\mathbf{M}}\bar{\boldsymbol{\Phi}}^s=\mathbf{I}^s$ and $\boldsymbol{\Gamma}^s=\left(\mathbf{C}\bar{\boldsymbol{\Phi}}^s\right)^{\mathrm{T}}$, Eq. (16) is rewritten as

$$\begin{aligned}\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}&=\left(\mathbf{C}\bar{\boldsymbol{\Phi}}^s\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\right)^{-1}-\lambda_i\left(\mathbf{C}\bar{\boldsymbol{\Phi}}^s\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\boldsymbol{\Phi}}^s\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}\bar{\mathbf{M}}\bar{\boldsymbol{\Phi}}^s\mathbf{t}_i\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}\\&=\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}-\lambda_i\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}\bar{\mathbf{M}}\bar{\boldsymbol{\Phi}}^s\mathbf{t}_i\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}\end{aligned} \tag{17}$$

where $\bar{\mathbf{F}}=\bar{\boldsymbol{\Phi}}^s\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}$ is the residual flexibility of the assembled structures. It is calculated by [21]

$$\bar{\boldsymbol{\Phi}}^s\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}=\mathrm{Diag}\left(\left[\mathbf{K}^{(1)}\right]^{-1}-\left[\boldsymbol{\Phi}^m\right]^{(1)}\left(\left[\boldsymbol{\Lambda}^m\right]^{(1)}\right)^{-1}\left[\left[\boldsymbol{\Phi}^m\right]^{(1)}\right]^{\mathrm{T}},\ldots,\left[\mathbf{K}^{(N_S)}\right]^{-1}-\left[\boldsymbol{\Phi}^m\right]^{(NS)}\left(\left[\boldsymbol{\Lambda}^m\right]^{(NS)}\right)^{-1}\left[\left[\boldsymbol{\Phi}^m\right]^{(NS)}\right]^{\mathrm{T}}\right) \tag{18}$$

As a consequence, the frequency-dependent item $\boldsymbol{\Gamma}^m\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}}\mathbf{z}_i^m$ in Eq. (15) is rewritten as

$$\boldsymbol{\Gamma}^m\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}}\mathbf{z}_i^m=\boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}}\mathbf{z}_i^m-\lambda_i\boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}\bar{\mathbf{M}}\boldsymbol{\eta}_i\mathbf{z}_i^m \tag{19}$$

where

$$\boldsymbol{\eta}_i=\bar{\boldsymbol{\Phi}}^s\mathbf{t}_i\left(\left[\boldsymbol{\Gamma}^s\right]^{\mathrm{T}}\mathbf{t}_i\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}} \tag{20}$$

It is evident from Eq. (14) that $\boldsymbol{\eta}_i\mathbf{z}_i^m=-\bar{\boldsymbol{\Phi}}^s\mathbf{z}_i^s$. Here a modal transformation matrix $\mathbf{T}$ is introduced to represent the contribution of the slave modes in terms of master modes. It satisfies [22]

$$\mathbf{T}\mathbf{z}_i^m = -\bar{\mathbf{\Phi}}^s \mathbf{z}_i^s = \boldsymbol{\eta}_i \mathbf{z}_i^m \tag{21}$$

Accordingly, the entire n_m master modes satisfy

$$\mathbf{T}\mathbf{Z}^m = \left[\mathbf{T}\mathbf{z}_1^m \quad \mathbf{T}\mathbf{z}_2^m \quad \cdots \quad \mathbf{T}\mathbf{z}_{n_m}^m\right] = \left[\boldsymbol{\eta}_1 \mathbf{z}_1^m \quad \boldsymbol{\eta}_2 \mathbf{z}_2^m \quad \cdots \quad \boldsymbol{\eta}_m \mathbf{z}_{n_m}^m\right] \tag{22}$$

where $\mathbf{Z}^m = \left[\mathbf{z}_1^m, \mathbf{z}_2^m, \ldots, \mathbf{z}_{n_m}^m\right]$ encloses the mode participation factors $\mathbf{z}^m$ of the entire n_m substructural master modes. As $\mathbf{z}_i^m$ is an orthogonal basis in modal space with the size of $n_m \times 1$, $\mathbf{Z}^m$ is a full-rank matrix with the size of $n_m \times n_m$. Accordingly, $\mathbf{T}$ is associated with all substructural master modes as

$$\mathbf{T} = \left[\boldsymbol{\eta}_1 \mathbf{z}_1^m \quad \boldsymbol{\eta}_2 \mathbf{z}_2^m \quad \cdots \quad \boldsymbol{\eta}_{n_m} \mathbf{z}_{n_m}^m\right]\left[\mathbf{Z}^m\right]^{-1} \tag{23}$$

In Eqs. (21)-(23), $\boldsymbol{\eta}_i \mathbf{z}_i^m$ is replaced by $\mathbf{T}\mathbf{z}_i^m$. The transformation matrix $\mathbf{T}$ is associated with all the substructural master modes rather than a specific mode, which is helpful for all modes to be calculated simultaneously in later study.

Substituting Eqs. (11) and (20) into Eq. (21) leads to

$$\begin{aligned}
\mathbf{T}\mathbf{z}_i^m &= \bar{\mathbf{\Phi}}^s \mathbf{t}_i \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m \\
&= \bar{\mathbf{\Phi}}^s \left(\left[\bar{\mathbf{\Lambda}}^s\right]^{-1} \mathbf{\Gamma}^s + \lambda_i \left[\bar{\mathbf{\Lambda}}^s\right]^{-1} \mathbf{t}_i\right)\left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m \\
&= \bar{\mathbf{\Phi}}^s \left[\bar{\mathbf{\Lambda}}^s\right]^{-1} \left[\bar{\mathbf{\Phi}}^s\right]^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m + \lambda_i \bar{\mathbf{\Phi}}^s \left[\bar{\mathbf{\Lambda}}^s\right]^{-1} \left[\bar{\mathbf{\Phi}}^s\right]^{\mathrm{T}} \bar{\mathbf{M}} \bar{\mathbf{\Phi}}^s \mathbf{t}_i \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m \\
&= \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}} \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m + \lambda_i \bar{\mathbf{F}} \bar{\mathbf{M}} \mathbf{T} \mathbf{z}_i^m
\end{aligned} \tag{24}$$

According to Eq. (17), the first item $\bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}} \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m$ in Eq. (24) is expressed as

$$\begin{aligned}
\bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}} \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m &= \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}} \left(\mathbf{C} \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}}\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m - \lambda_i \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}} \left(\mathbf{C} \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}}\right)^{-1} \mathbf{C} \bar{\mathbf{F}} \bar{\mathbf{M}} \bar{\mathbf{\Phi}}^s \mathbf{t}_i \left(\left[\mathbf{\Gamma}^s\right]^{\mathrm{T}} \mathbf{t}_i\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m \\
&= \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}} \left(\mathbf{C} \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}}\right)^{-1} \left[\mathbf{\Gamma}^m\right]^{\mathrm{T}} \mathbf{z}_i^m - \lambda_i \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}} \left(\mathbf{C} \bar{\mathbf{F}} \mathbf{C}^{\mathrm{T}}\right)^{-1} \mathbf{C} \bar{\mathbf{F}} \bar{\mathbf{M}} \mathbf{T} \mathbf{z}_i^m
\end{aligned} \tag{25}$$

Consequently, Eq. (24) is rewritten as

$$\mathbf{T}\mathbf{z}_i^m = \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}}\mathbf{z}_i^m + \lambda_i\left(\bar{\mathbf{F}} - \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}\right)\bar{\mathbf{M}}\mathbf{T}\mathbf{z}_i^m = \left(\mathbf{T}_C + \lambda_i\mathbf{S}\bar{\mathbf{M}}\mathbf{T}\right)\mathbf{z}_i^m \tag{26}$$

where the constant matrices $\mathbf{T}_C = \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}}$ and $\mathbf{S} = \bar{\mathbf{F}} - \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}$.

Based on Eqs. (19) and (21), the original eigenequation Eq. (15) is simplified into

$$\left(\bar{\boldsymbol{\Lambda}}^m + \boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}} - \lambda_i\left(\mathbf{I}^m + \boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}\bar{\mathbf{M}}\mathbf{T}\right)\right)\mathbf{z}_i^m = \mathbf{0} \tag{27}$$

As $\bar{\mathbf{F}}\bar{\mathbf{K}}\bar{\mathbf{F}} = \bar{\boldsymbol{\Phi}}^s\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left(\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}\bar{\mathbf{K}}\bar{\boldsymbol{\Phi}}^s\right)\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}} = \bar{\boldsymbol{\Phi}}^s\left[\bar{\boldsymbol{\Lambda}}^s\right]^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}} = \bar{\mathbf{F}}$, it is easily found that

$$\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{T}_C = \boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}\bar{\mathbf{K}}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}} = \boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}} \tag{28}$$

Accordingly, Eq. (15) is further simplified into

$$\left(\mathbf{K}_C - \lambda_i\mathbf{M}_D\right)\mathbf{z}_i^m = \mathbf{0} \tag{29}$$

where

$$\mathbf{K}_C = \bar{\boldsymbol{\Lambda}}^m + \boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\left[\boldsymbol{\Gamma}^m\right]^{\mathrm{T}} = \bar{\boldsymbol{\Lambda}}^m + \mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{K}}\mathbf{T}_C \tag{30}$$

$$\mathbf{M}_D = \mathbf{I}^m + \boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}\bar{\mathbf{M}}\mathbf{T} = \mathbf{I}^m + \mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{T} \tag{31}$$

$\mathbf{K}_C$ is a symmetric matrix, which is calculated from the constant value $\mathbf{T}_C$. In consequence, only $\mathbf{M}_D$ is unknown and required to solve the new eigenequation (Eq. (29)). In this paper, the original eigenequation (Eq. (15)) is simplified into a new one (Eq. (29)). Eq. (29) has the same order as Eq. (15) of n_m, but its system matrix $\mathbf{M}_D$ is frequency-independent. This ensures the eigensolutions of all required modes to be solved simultaneously. It is noted in Eq. (31) that $\mathbf{M}_D$ is relevant to the modal transformation matrix $\mathbf{T}$. An iterative process will be proposed later to search for the accurate value of $\mathbf{T}$.

Eq. (29) implies $\lambda_i\mathbf{z}_i^m = \mathbf{M}_D^{-1}\mathbf{K}_C\mathbf{z}_i^m$. Accordingly, Eq. (26) is therefore simplified into

$$\mathbf{T}\mathbf{z}_i^m = \left(\mathbf{T}_C + \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\mathbf{K}_C\right)\mathbf{z}_i^m \tag{32}$$

Eq. (32) is satisfied for i=1, 2, …, n_m, and $\mathbf{T}$ is therefore expressed as

$$\mathbf{T} = \mathbf{T}_C + \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\mathbf{K}_C \tag{33}$$

Consequently, the modal transformation matrix $\mathbf{T}$ is calculated iteratively based on Eqs. (31) and (33). $\mathbf{T}$ has the size of $n_m \times n_m$, which is much smaller than the iterative variable $\mathbf{t}_i$ utilized in the previous iterative substructuring method [21] of $\bar{N} \times n_m$ ($\bar{N} = \sum_{j=1}^{NS} n^{(j)}$ is the number of the total DOFs of the assembled structures). The order of the iterative variables in the previous method is reduced greatly, which is significant to improve the computational efficiency. Once the modal transformation matrix $\mathbf{T}$ is available, λ_i and $\mathbf{z}_i^m$ are calculated from Eq. (29). Based on Eq. (21), the eigenvector of the assembled global structure is expressed in modal coordinates as

$$\tilde{\phi}_i = \bar{\mathbf{\Phi}}^m \mathbf{z}_i^m + \bar{\mathbf{\Phi}}^s \mathbf{z}_i^s = \left(\bar{\mathbf{\Phi}}^m - \mathbf{T}\right)\mathbf{z}_i^m \tag{34}$$

Finally, the eigenvector of the global structure ϕ_i is extracted from $\tilde{\phi}_i$ by merging the interface DOFs.

Actually, the proposed iterative procedures to calculate the eigensolutions have some similarities to that of the dynamic condensation methods [23-30]. They reduce the size of the global structure to improve its analysis efficiency in different aspects. The dynamic condensation approach reduces the eigenequation of the global structure in the physical coordinate while the proposed method in the modal coordinate.

3. Proposed approach to the calculation of eigensensitivities

In this section, the eigensensitivities will be calculated by the proposed substructuring method. The first order eigenvalue and eigenvector derivatives with respect to an elemental parameter r located in the Rth substructure will be derived for illustration.

3.1 Eigenvalue derivative

Eq. (29) is differentiated with respect to r as

$$\left(\mathbf{K}_C-\lambda_i\mathbf{M}_D\right)\frac{\partial\mathbf{z}_i^m}{\partial r}+\left(\frac{\partial\mathbf{K}_C}{\partial r}-\lambda_i\frac{\partial\mathbf{M}_D}{\partial r}-\frac{\partial\lambda_i}{\partial r}\mathbf{M}_D\right)\mathbf{z}_i^m=\mathbf{0} \tag{35}$$

Pre-multiply $\left\{\mathbf{z}_i^m\right\}^{\mathrm{T}}$ on both sides of Eq. (35), and consider Eq. (29) and the orthogonal condition $\left\{\mathbf{z}_i^m\right\}^{\mathrm{T}}\mathbf{M}_D\mathbf{z}_i^m=1$, one can obtain

$$\frac{\partial\lambda_i}{\partial r}=\left\{\mathbf{z}_i^m\right\}^{\mathrm{T}}\left(\frac{\partial\mathbf{K}_C}{\partial r}-\lambda_i\frac{\partial\mathbf{M}_D}{\partial r}\right)\mathbf{z}_i^m \tag{36}$$

Differentiating Eqs. (30) and (31) with respect to r, $\frac{\partial\mathbf{K}_C}{\partial r}$ and $\frac{\partial\mathbf{M}_D}{\partial r}$ are calculated by

$$\frac{\partial\mathbf{K}_C}{\partial r}=\frac{\partial\bar{\boldsymbol{\Lambda}}_m}{\partial r}+\frac{\partial\mathbf{T}_C^{\mathrm{T}}}{\partial r}\bar{\mathbf{K}}\mathbf{T}_C+\mathbf{T}_C^{\mathrm{T}}\frac{\partial\bar{\mathbf{K}}}{\partial r}\mathbf{T}_C+\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{K}}\frac{\partial\mathbf{T}_C}{\partial r} \tag{37}$$

$$\frac{\partial\mathbf{M}_D}{\partial r}=\frac{\partial\mathbf{T}_C^{\mathrm{T}}}{\partial r}\bar{\mathbf{M}}\mathbf{T}+\mathbf{T}_C^{\mathrm{T}}\frac{\partial\bar{\mathbf{M}}}{\partial r}\mathbf{T}+\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\frac{\partial\mathbf{T}}{\partial r} \tag{38}$$

where

$$\frac{\partial\mathbf{T}_C}{\partial r}=\frac{\partial\left[\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\boldsymbol{\Phi}}^m\right]}{\partial r}=\frac{\partial\bar{\mathbf{F}}}{\partial r}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\boldsymbol{\Phi}}^m+\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\left(\frac{\partial\bar{\boldsymbol{\Phi}}^m}{\partial r}-\frac{\partial\bar{\mathbf{F}}}{\partial r}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\boldsymbol{\Phi}}^m\right) \tag{39}$$

Since the variables $\mathbf{z}_i^m$, λ_i, $\bar{\mathbf{K}}$, $\bar{\mathbf{M}}$, $\bar{\mathbf{F}}$, $\mathbf{T}_C$, $\mathbf{T}$, $\mathbf{C}$, $\bar{\boldsymbol{\Lambda}}^m$ and $\bar{\boldsymbol{\Phi}}^m$ are available in the calculation of eigensolutions, they are reused here directly. Each substructure is treated as an independent structure, $\frac{\partial\bar{\mathbf{K}}}{\partial r}$, $\frac{\partial\bar{\mathbf{M}}}{\partial r}$, $\frac{\partial\bar{\mathbf{F}}}{\partial r}$, $\frac{\partial\bar{\boldsymbol{\Lambda}}^m}{\partial r}$ and $\frac{\partial\bar{\boldsymbol{\Phi}}^m}{\partial r}$ are therefore zeros except for the specific substructure containing r. If r locates in the Rth substructure, these matrices take the form of

$$\frac{\partial \bar{\mathbf{K}}}{\partial r}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial \mathbf{K}^{(R)}}{\partial r} & \\ & & \mathbf{0}\end{bmatrix},\ \frac{\partial \bar{\mathbf{M}}}{\partial r}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial \mathbf{M}^{(R)}}{\partial r} & \\ & & \mathbf{0}\end{bmatrix},\ \frac{\partial \bar{\mathbf{F}}}{\partial r}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial \mathbf{F}^{(R)}}{\partial r} & \\ & & \mathbf{0}\end{bmatrix},$$

$$\frac{\partial \bar{\mathbf{\Lambda}}^{m}}{\partial r}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial \left[\mathbf{\Lambda}^{m}\right]^{(R)}}{\partial r} & \\ & & \mathbf{0}\end{bmatrix},\ \frac{\partial \bar{\mathbf{\Phi}}^{m}}{\partial r}=\begin{bmatrix}\mathbf{0} & & \\ & \dfrac{\partial \left[\mathbf{\Phi}^{m}\right]^{(R)}}{\partial r} & \\ & & \mathbf{0}\end{bmatrix} \tag{40}$$

$\frac{\partial \mathbf{K}^{(R)}}{\partial r}$ and $\frac{\partial \mathbf{M}^{(R)}}{\partial r}$ are respectively the elemental stiffness and mass matrices with respect to the designed parameter r. $\frac{\partial \mathbf{F}^{(R)}}{\partial r}$ is the derivative of the residual flexibility of the Rth substructure with respect to r, which is calculated by

$$\frac{\partial \mathbf{F}^{(R)}}{\partial r}=\frac{\partial\left(\left[\mathbf{K}^{(R)}\right]^{-1}-\left[\mathbf{\Phi}^{m}\right]^{(R)}\left(\left[\mathbf{\Lambda}^{m}\right]^{(R)}\right)^{-1}\left[\left[\mathbf{\Phi}^{m}\right]^{(R)}\right]^{\mathrm{T}}\right)}{\partial r}=-\left[\mathbf{K}^{(R)}\right]^{-1}\frac{\partial \mathbf{K}^{(R)}}{\partial r}\left[\mathbf{K}^{(R)}\right]^{-1}-\frac{\partial\left(\left[\mathbf{\Phi}^{m}\right]^{(R)}\left(\left[\mathbf{\Lambda}^{m}\right]^{(R)}\right)^{-1}\left[\left[\mathbf{\Phi}^{m}\right]^{(R)}\right]^{\mathrm{T}}\right)}{\partial r} \tag{41}$$

$\frac{\partial \left[\mathbf{\Lambda}^{m}\right]^{(R)}}{\partial r}$ and $\frac{\partial \left[\mathbf{\Phi}^{m}\right]^{(R)}}{\partial r}$ can be calculated by employing traditional methods such as Rogers' method [31] or Nelson's method [32] to the Rth substructure. Therefore, only $\frac{\partial \mathbf{T}}{\partial r}$ is required in Eq. (38) to calculate $\frac{\partial \mathbf{M}_D}{\partial r}$ and solve Eq. (36). $\mathbf{T}$ is available with an iterative algorithm based on Eqs. (31) and (33) in the previous section. Similarly, $\frac{\partial \mathbf{T}}{\partial r}$ is available with an iterative scheme according to the derivatives of Eqs. (31) and (33) with respect to r as

$$\frac{\partial \mathbf{M}_D}{\partial r}=\frac{\partial \mathbf{T}_C^{\mathrm{T}}}{\partial r}\bar{\mathbf{M}}\mathbf{T}+\mathbf{T}_C^{\mathrm{T}}\frac{\partial \bar{\mathbf{M}}}{\partial r}\mathbf{T}+\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\frac{\partial \mathbf{T}}{\partial r} \tag{42}$$

$$\frac{\partial \mathbf{T}}{\partial r}=\frac{\partial \mathbf{T}_C}{\partial r}+\left(\frac{\partial \mathbf{S}}{\partial r}\bar{\mathbf{M}}\mathbf{T}+\mathbf{S}\frac{\partial \bar{\mathbf{M}}}{\partial r}\mathbf{T}+\mathbf{S}\bar{\mathbf{M}}\frac{\partial \mathbf{T}}{\partial r}\right)\mathbf{M}_D^{-1}\mathbf{K}_C+\mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\left(\frac{\partial \mathbf{K}_C}{\partial r}-\frac{\partial \mathbf{M}_D}{\partial r}\mathbf{M}_D^{-1}\mathbf{K}_C\right) \tag{43}$$

where

$$\frac{\partial \mathbf{S}}{\partial r}=\frac{\partial\left[\overline{\mathbf{F}}-\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\overline{\mathbf{F}}\right]}{\partial r}$$
$$=\frac{\partial\overline{\mathbf{F}}}{\partial r}-\frac{\partial\overline{\mathbf{F}}}{\partial r}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\overline{\mathbf{F}}+\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\frac{\partial\overline{\mathbf{F}}}{\partial r}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\overline{\mathbf{F}}-\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\overline{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\frac{\partial\overline{\mathbf{F}}}{\partial r} \tag{44}$$

Superior to our previous iterative substructuring method [21] which calculates the eigenvalue derivative mode by mode, the eigenvalue derivative of all the required modes can be calculated simultaneously from Eq. (36) in the proposed method once $\frac{\partial \mathbf{T}}{\partial r}$ is available. In addition, the iterative variable $\frac{\partial \mathbf{T}}{\partial r}$ employed in the proposed method has a size of $n_m \times n_m$, much smaller than $\frac{\partial \mathbf{t}_i}{\partial r}$ of $\bar{N}\times n_m$ used in the previous method. It is predictable that the proposed method is much more computationally efficient than the previous method for eigenvalue derivatives.

3.2 Eigenvector derivative

Differentiating Eq. (34) with respect to the elemental parameter r, one can obtain the eigenvector derivatives of the assembled global structure as

$$\frac{\partial \tilde{\phi}_i}{\partial r}=\left(\frac{\partial\overline{\mathbf{\Phi}}^m}{\partial r}-\frac{\partial \mathbf{T}}{\partial r}\right)\mathbf{z}_i^m+\left(\overline{\mathbf{\Phi}}^m-\mathbf{T}\right)\frac{\partial \mathbf{z}_i^m}{\partial r} \tag{45}$$

In Eq. (45), $\frac{\partial\overline{\mathbf{\Phi}}^m}{\partial r}$ and $\frac{\partial \mathbf{T}}{\partial r}$ have been obtained in the calculation of eigenvalue derivative. $\overline{\mathbf{\Phi}}^m$, $\mathbf{T}$ and $\mathbf{z}_i^m$ are available when computing eigensolutions in Section 2. Therefore, the pivot task is to compute $\frac{\partial \mathbf{z}_i^m}{\partial r}$.

$\frac{\partial \mathbf{z}_i^m}{\partial r}$ is decomposed into a particular item and a general item as [21, 32]

$$\frac{\partial \mathbf{z}_i^m}{\partial r} = \mathbf{v}_i + c_i \mathbf{z}_i^m \tag{46}$$

where $\mathbf{v}_i$ is a particular item and c_i is a participation factor. Substituting Eq. (46) into Eq. (35) and considering $\left(\mathbf{K}_C - \lambda_i \mathbf{M}_D\right)\mathbf{z}_i^m = \mathbf{0}$ (Eq. (29)) gives

$$\left(\mathbf{K}_C - \lambda_i \mathbf{M}_D\right)\mathbf{v}_i = -\left(\frac{\partial \mathbf{K}_C}{\partial r} - \lambda_i \frac{\partial \mathbf{M}_D}{\partial r} - \frac{\partial \lambda_i}{\partial r}\mathbf{M}_D\right)\mathbf{z}_i^m \tag{47}$$

Since $\mathbf{K}_C$, $\mathbf{M}_D$, $\frac{\partial \mathbf{K}_C}{\partial r}$, $\frac{\partial \mathbf{M}_D}{\partial r}$, λ_i, $\frac{\partial \lambda_i}{\partial r}$ and $\mathbf{z}_i^m$ are available beforehand in the calculation of eigensolutions and eigenvalue derivatives, $\mathbf{v}_i$ can be solved from Eq. (47) directly.

The new eigenequation Eq. (29) satisfies the orthogonal condition of

$$\left\{\mathbf{z}_i^m\right\}^{\mathrm{T}} \mathbf{M}_D \mathbf{z}_i^m = 1 \tag{48}$$

Differentiating Eq. (48) with respect to r gives

$$\frac{\partial \left\{\mathbf{z}_i^m\right\}^{\mathrm{T}}}{\partial r}\mathbf{M}_D \mathbf{z}_i^m + \left\{\mathbf{z}_i^m\right\}^{\mathrm{T}} \frac{\partial \mathbf{M}_D}{\partial r}\mathbf{z}_i^m + \left\{\mathbf{z}_i^m\right\}^{\mathrm{T}} \mathbf{M}_D \frac{\partial \mathbf{z}_i^m}{\partial r} = 0 \tag{49}$$

Substitute Eq. (46) into Eq. (49) and the participation factor c_i is thus calculated by

$$c_i = -\frac{1}{2}\left(\mathbf{v}_i^{\mathrm{T}} \mathbf{M}_D \mathbf{z}_i^m + \left\{\mathbf{z}_i^m\right\}^{\mathrm{T}} \frac{\partial \mathbf{M}_D}{\partial r}\mathbf{z}_i^m + \left\{\mathbf{z}_i^m\right\}^{\mathrm{T}} \mathbf{M}_D \mathbf{v}_i\right) \tag{50}$$

Given the vector $\mathbf{v}_i$ in Eq. (47) and the participation factor c_i in Eq. (50), $\frac{\partial \mathbf{z}_i^m}{\partial r}$ is computed according to Eq. (46). The eigenvector derivative of the assembled global structure $\frac{\partial \tilde{\phi}_i}{\partial r}$ is calculated from Eq. (45). Finally, the eigenvector derivative of the global structure $\frac{\partial \phi_i}{\partial r}$ can be extracted from $\frac{\partial \tilde{\phi}_i}{\partial r}$ by merging the interface DOFs.

4. Computational operation

The detailed computational procedures of the proposed method will be illustrated in this section. As the eigensensitivities are usually required together with the eigensolutions, for brevity, this part combines the procedures to calculate the eigensolutions and eigensensitivities simultaneously. The detailed steps are described as follows:

(1) The global structure is partitioned into *NS* substructures. Each substructure is treated as an independent structure to calculate its first $m^{(j)}$ eigensolutions by

$$\mathbf{K}^{(j)}\left[\mathbf{\Phi}^{m}\right]^{(j)}=\left[\mathbf{\Lambda}^{m}\right]^{(j)}\mathbf{M}^{(j)}\left[\mathbf{\Phi}^{m}\right]^{(j)} \tag{51}$$

The first $m^{(j)}$ eigensensitivities of the jth substructure $\frac{\partial\left[\mathbf{\Lambda}^{m}\right]^{(j)}}{\partial r}$ and $\frac{\partial\left[\mathbf{\Phi}^{m}\right]^{(j)}}{\partial r}$ are calculated with Nelson's method [32]. Some constant matrices are then assembled from those of the substructures, i.e. $\bar{\mathbf{K}}$, $\bar{\mathbf{M}}$, $\bar{\mathbf{\Lambda}}^{m}$, $\bar{\mathbf{\Phi}}^{m}$, $\frac{\partial\bar{\mathbf{K}}}{\partial r}$, $\frac{\partial\bar{\mathbf{M}}}{\partial r}$, $\frac{\partial\bar{\mathbf{\Lambda}}^{m}}{\partial r}$ and $\frac{\partial\bar{\mathbf{\Phi}}^{m}}{\partial r}$.

(2) Some intermediate variables are calculated to avoid repeated computation in later steps, such as $\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}$, $\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}$, $\mathbf{C}\bar{\mathbf{\Phi}}^{m}$, $\mathbf{T}_C$, $\mathbf{K}_C$, $\frac{\partial\bar{\mathbf{F}}}{\partial r}\mathbf{C}^{\mathrm{T}}$, $\mathbf{C}\frac{\partial\bar{\mathbf{\Phi}}^{m}}{\partial r}$, $\frac{\partial\mathbf{T}_C}{\partial r}$, $\frac{\partial\mathbf{K}_C}{\partial r}$, etc.

(3) The modal transformation matrix $\mathbf{T}$ and its derivative $\frac{\partial\mathbf{T}}{\partial r}$ are initiated with $\mathbf{T}^{[0]}=\mathbf{0}$ and $\left(\frac{\partial\mathbf{T}}{\partial r}\right)^{[0]}=\mathbf{0}$ according to the commonly used static substructuring method [6, 18, 19]. $\mathbf{T}$ and $\frac{\partial\mathbf{T}}{\partial r}$ are calculated with an iterative process. In the kth (k=1, 2, 3…) iteration,

$$\mathbf{M}_D^{[k]}=\mathbf{I}^{m}+\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{T}^{[k-1]} \tag{52}$$

$$\mathbf{T}^{[k]}=\mathbf{T}_C+\mathbf{S}\bar{\mathbf{M}}\mathbf{T}^{[k-1]}\left(\mathbf{M}_D^{[k]}\right)^{-1}\mathbf{K}_C \tag{53}$$

$$\left(\frac{\partial\mathbf{M}_D}{\partial r}\right)^{[k]}=\frac{\partial\mathbf{T}_C^{\mathrm{T}}}{\partial r}\bar{\mathbf{M}}\mathbf{T}^{[k]}+\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\left(\frac{\partial\mathbf{T}}{\partial r}\right)^{[k-1]} \tag{54}$$

$$\left(\frac{\partial \mathbf{T}}{\partial r}\right)^{[k]}=\frac{\partial \mathbf{T}_C}{\partial r}+\left[\frac{\partial \mathbf{S}}{\partial r}\bar{\mathbf{M}}\mathbf{T}^{[k]}+\mathbf{S}\bar{\mathbf{M}}\left(\frac{\partial \mathbf{T}}{\partial r}\right)^{[k-1]}\right]\left(\mathbf{M}_D^{[k]}\right)^{-1}\mathbf{K}_C+\mathbf{S}\bar{\mathbf{M}}\mathbf{T}^{[k]}\left(\mathbf{M}_D^{[k]}\right)^{-1}\left[\frac{\partial \mathbf{K}_C}{\partial r}-\left(\frac{\partial \mathbf{M}_D}{\partial r}\right)^{[k]}\left(\mathbf{M}_D^{[k]}\right)^{-1}\mathbf{K}_C\right] \tag{55}$$

This iterative steps are proved to be convergent, with the detailed proof illustrated in Appendix A.

(4) The iterations are terminated once the relative differences of the interested eigenvalues $\lambda^{[k]}=\text{eig}((\mathbf{M}_D^{[k]})^{-1}\mathbf{K}_C)$ from two consecutive iterations are less than the required tolerance

$$\left|\frac{\lambda^{[k]}-\lambda^{[k-1]}}{\lambda^{[k-1]}}\right|<Tol \tag{56}$$

(5) According to the updated $\mathbf{M}_D$ and $\mathbf{T}$, the eigenvalues λ and their corresponding mode participation factors $\mathbf{z}^m$ are calculated simultaneously for all interested modes by Eq. (29). The corresponding eigenvectors are then calculated by Eq. (34).

(6) According to the updated $\frac{\partial \mathbf{M}_D}{\partial r}$ and $\frac{\partial \mathbf{T}}{\partial r}$, the eigenvalue derivative $\frac{\partial \lambda}{\partial r}$ is calculated simultaneously for all interested modes by Eq. (36). $\frac{\partial \mathbf{z}^m}{\partial r}$ is computed following the procedures described in Subsection 3.2. The eigenvector derivative is finally computed by Eq. (45).

It is essential to evaluate the efficiency of different numerical methods, especially for large-scale structures. One basic index is the number of multiplication involved in computer operation, which is known as multiplication counts (MC). It is effective to avoid the random errors in CPU time caused by different operating environments. The detailed computational methods of MC for the proposed method and the traditional method performed on the global structure are introduced in Appendix B. It is noted that the MC consumed by the traditional

method is dominated by the total DOFs of the structure N and the bandwidth b, while the MC consumed by the proposed method is dominated by the number of substructural DOFs $n^{(j)}$ and its bandwidth $b^{(j)}$. As the substructure has much less DOFs than the global structure, it is predictable that the MC consumed by the proposed method is much less than the traditional method.

5. Case study 1: a numerical example

The proposed substructuring method is first applied to a frame model. As shown in Fig. (1), the frame comprises 69 nodes, 76 two-dimensional beam elements and 195 DOFs in total. The lengths of the horizontal, vertical and diagonal bars are 4.0 m, 3.0 m and 5.0 m, respectively. The properties of the elements are: the cross sectional area $A = 2.4\times10^{-4}$ m^2, moment of inertia of the section $I = 9.0\times10^{-9}$ m^4, Young's modulus $E = 210$ GPa, mass density $\rho = 7850$ kg/m^3 and Poisson's ratio $v = 0.3$. The eigensensitivities with respect to the bending rigidities of two randomly selected elements r_1 and r_2 (Elements 58 and 8 in Fig. (1a)) will be investigated.

The global structure is partitioned into three substructures (Fig. (1b)). After partition, the first, second and third substructure have 60, 93 and 60 DOFs respectively. The first 12 modes of each substructure are retained to calculate the first 10 eigensolutions and eigensensitivities with respect to r_1, following the procedures of the proposed method described in Section 4. In this example, 5 iterations are required for the relative errors of the first 10 natural frequencies to reach the predefined tolerance of 1×10^{-8}.

To evaluate the accuracy of the proposed iterative substructuring method, the traditional method performed on the global model is also used to calculate the eigensolutions and eigensensitivities of the frame, which are treated as exact results. In the traditional global method, the eigensolutions of the global structure are calculated using the Lanczos method [33] and the eigensensitivities are computed with the Nelson's method [32]. To verify the convergence of the proposed method in calculation of eigensolutions, the relative errors of natural frequencies are recorded in each iteration. For brevity, the convergences of natural frequencies of three representative modes (1^{st}, 6^{th} and 10^{th} modes) are drawn in Fig. (2). The eigensolutions and eigensensitivities are also calculated by the widely used static substructuring method [6, 18, 19], where the eigenequation is formed from the master modes and compensated by a static residual flexibility. It is equivalent to the initial step of the proposed method without iteration performed. In the static method, the modal transformation matrix $\mathbf{T}$ and its derivative matrix $\frac{\partial \mathbf{T}}{\partial r}$ are initiated with zeros.

It is noted in Fig. (2) that, employing the static method, the relative errors of natural frequencies of the 1^{st}, 6^{th} and 10^{th} modes are about 1.0×10^{-3}, 2.5×10^{-3} and 3.8×10^{-3} respectively. After 5 iterations performed, the relative error curves decline sharply. The 1^{st}, 6^{th} and 10^{th} natural frequencies reach an extremely high precision, with the relative errors of about 9×10^{-9}, 8×10^{-9} and 5×10^{-8}. Therefore, the proposed iterative substructuring method converges to an accurate result of eigensolutions with only 5 iterations performed and 12 master modes included in each substructure.

Table 1 compares the first 10 eigensolutions calculated by the static method and the proposed method. The relative errors of the conventional static method are about 10^{-4} to 10^{-3}. Using the proposed method with 5 iterations performed, the relative errors drop to the order of 10^{-10} to 10^{-8}. Moreover, the Modal Assurance Criterion (MAC) values [19], which indicate the similarity between the calculated eigenvectors and the exact ones, all reach 1.0000 for the proposed method. It again proves that the proposed method achieves very accurate eigensolutions with only 5 iterations and 12 master modes retained in each substructure.

Following the steps in Section 4, the eigenvalue and eigenvector derivatives of the first 10 modes with respect to r_1 are calculated. Table 2 compares the eigensensitivities calculated by the static method and the proposed method. The similarity of eigenvector derivative (SED) obtained by the proposed iterative substructuring method ($\frac{\partial \phi_S}{\partial r}$) and the traditional global method ($\frac{\partial \phi_G}{\partial r}$) is defined in a similar way to MAC values as

$$\mathrm{SED}\left(\frac{\partial \phi_G}{\partial r}, \frac{\partial \phi_S}{\partial r}\right) = \frac{\left(\frac{\partial \phi_G^{\mathrm{T}}}{\partial r}\frac{\partial \phi_S}{\partial r}\right)^2}{\left(\frac{\partial \phi_G^{\mathrm{T}}}{\partial r}\frac{\partial \phi_G}{\partial r}\right)\left(\frac{\partial \phi_S^{\mathrm{T}}}{\partial r}\frac{\partial \phi_S}{\partial r}\right)} \tag{57}$$

Similar to the MAC values, an SED value of 1 indicates the two eigenvector derivatives are identical while the SED value of 0 implies they are orthogonal. It is noted that the SED values of some modes (7^{th}, 8^{th} and 9^{th}) are less than 0.95 in the initial step and then are improved to 0.99 or above after 5 iterations. Moreover, the relative errors of the eigenvalue derivatives calculated by the static method are more than 0.5% for most modes and some are even significantly much larger than 1%, which are then improved greatly to about 1×10^{-4} or less for

all modes when 5 iterations are performed in the proposed method.

Without losing generality, the eigensensitivities with respect to the bending rigidity of another randomly selected element r_2 in the free substructure (the second substructure in Fig. (1)) are also computed, with the first 10 eigenvalue and eigenvector derivatives listed in Table 3. It shows again that the accuracy of the eigensensitivities is improved significantly with the proposed method with only 5 iterations performed and 12 master modes included in each substructure.

6. Case study 2: a practical large-scale structure

The proposed method is then applied to the main building of Wuhan Yangtze River Navigation Center to investigate its computational efficiency. This high-rise structure (see Fig. 3a), located in Wuhan, P.R. China, is 334.6 m tall. It comprises a frame-core tube structure at the bottom and a steel frame on the top. The outer frame is a square with 50.1 m in length and width. The structure is modelled by 3950 nodes, 9112 elements, and 23364 DOFs in total (Fig. 3b). The bandwidth of the FE model is 461. The bending rigidity of a randomly selected shear wall element is used as the designed parameter r to calculate the eigensensitivity. The global structure is divided into 9 substructures along the vertical direction as in Fig. 3c. The number of nodes, elements and interface tearing nodes are listed in Table 4.

Using the proposed iterative substructuring method, 30 master modes of each substructure are

retained to calculate the first 10 eigensolutions and eigensensitivities of the global structure. The convergence criterion is set to 1×10^{-6}, and 3 iterations are therefore performed for the proposed method to reach this criterion. Again, the eigensolutions and eigensensitivities calculated by the global method are taken as the exact results. As a comparison, the eigensolutions and eigensensitivities are also calculated by the conventional iterative substructuring (CIS) method presented by the authors [21]. This method presents an iterative process based on $\mathbf{t}_i$ (Eq. (11)) to solve the original frequency-dependent eigenequation (Eq. (15)). It calculates the required eigensolutions and eigensensitivities iteratively mode by mode. In this example, 2 or 3 iterations are required for each mode to reach the same convergence criterion. As a consequence, 25 and 23 iterations are employed in total for the eigensolutions and eigensensitivities, respectively.

Table 5 and Table 6 compare the first 10 eigensolutions and eigensensitivities calculated with the above mentioned methods. It is seen that the proposed method and the CIS method can both achieve very accurate eigensolutions and eigensensitivities. Nevertheless, only 3 iterations are employed in the proposed method whereas 25 and 23 iterations are required in the CIS method to gain a similar high precision of eigensolutions and eigensensitivities, respectively. This decreases the computational efficiency sharply, which will be discussed later.

The computational time consumed in a desktop computer with 4.00 GHz CPU and 16 GB RAM by the above mentioned methods is compared in Table 7. The MATLAB platform is used to implement these methods, which allows for the system matrices to be stored and operated in

sparse matrices. In Table 7, it takes 20.78 s and 9.03 s for the global method to compute the required eigensolutions and eigensensitivities respectively, while only 4.49 s and 5.49 s are consumed in the proposed method correspondingly. Therefore, the proposed method is much more efficient than the global method in calculation of eigensolutions and eigensensitivities. This is because the system matrices (mass and stiffness matrices) with the consideration of sparse nature have the size of 23364×461 in the global method, which is still much larger than the proposed method of 270×270.

The CIS method presented by the authors [21] is also compared in Table 7. Although the CIS method can achieve a similar precision to the proposed method as seen in Table 5 and Table 6, it requires 25 and 23 iterations respectively for the eigensolutions and eigensensitivities. As a consequence, 145.43 s and 202.96 s are consumed in total for eigensolutions and eigensensitivities, much longer than those of the proposed method. Therefore, the proposed iterative substructuring method can improve the computational efficiency of the CIS method significantly. In addition, the CIS method takes 5.795 s and 8.689 s in each iteration for the eigensolutions and eigensensitivities respectively, while the proposed method consumes only 0.98 s and 0.66 s accordingly. It is noted that the iterative variables $\mathbf{T}$ and $\frac{\partial \mathbf{T}}{\partial r}$ employed in the proposed method has a size of 270×270, much smaller than $\mathbf{t}_i$ and $\frac{\partial \mathbf{t}_i}{\partial r}$ (25812×270) in the CIS method. The order of the iterative variables in the CIS method is reduced greatly. It is reasonable that the proposed method saves plenty of computation time in each iteration.

Moreover, the computational efficiency of the frequently used static substructuring method [6,

18, 19] is also compared in Table 7. It is equivalent to the proposed method without iteration. To reach the same precision as the proposed method, 300 master modes have to be retained in each substructure. Consequently, 18.35 s and 11.07 s are consumed for eigensolutions and eigensensitivities. While employing the proposed method with only 3 iterations performed, 30 master modes of each substructure is sufficient, resulting in only 4.49s and 5.49 s for eigensolutions and eigensensitivities. Therefore, superior to the static method that retains a large number of master modes to ensure accurate eigensolutions and eigensensitivities, the proposed method is much more efficient with only few iterations and much less master modes. It should be noted that the size of the system matrices ($\mathbf{K}_C$, $\mathbf{M}_D$ etc) in the static method is 2700×2700, much larger than the proposed method of 270×270. It is reasonable that the static method consumes much more time to gain accurate eigensolutions and eigensensitivities.

Besides the CPU time, Table 7 also compares the efficiency of different methods in terms of MC. It is obvious that proposed method consumes much fewer multiplication counts than other methods in the calculation of eigensolutions and eigensensitivities. It proves again that the proposed method is much more efficient than other commonly used methods. It is noted that the relative ratios of the CPU time consumed by different methods are close to those of the MC counterparts. The result of the CPU time is consistent with MC. It is reliable to evaluate the efficiency of different methods with CPU time in this example.

7. Conclusions

This paper proposes an iterative reduced-order substructuring method to calculate the eigensolutions and eigensensitivities of the global structure. The present method is advantageous and efficient in two aspects. First, the original frequency-dependent system matrices of substructures become frequency-independent using the modal transformation matrix. This enables the eigensolutions and eigensensitivities of all modes to be calculated simultaneously. Second, the modal transformation matrix, estimated with an iterative process, has a much smaller size than the iterative variables used in the previous iterative substructuring method, which improves the computational efficiency significantly.

Applications to a numerical frame and a practical super-tall building demonstrate that the proposed method is able to obtain accurate eigensolutions and eigensensitivities simultaneously and efficiently with a few iterations and a small number of master modes in each substructure, as compared with the global method and the previously developed substructuring approaches.

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Appendix A. Convergence of the proposed method

It is very difficult to prove the convergence of eigensolutions and eigensensitivities in a direct mathematical way, which is a shortcoming of model reduction methods. This section will prove the proposed iterative scheme for $\mathbf{T}^{[k]}$ and $\left(\frac{\partial \mathbf{T}}{\partial r}\right)^{[k]}$ converge to $\mathbf{T}$ and $\frac{\partial \mathbf{T}}{\partial r}$ respectively, in a similar way to that given in Ref. [24-26].

An error matrix is defined to estimate the error of $\mathbf{T}^{[k]}$ as

$$\mathbf{E}^{[k]} = \mathbf{T}^{[k]} - \mathbf{T} = \mathbf{S}\bar{\mathbf{M}}\left[\mathbf{T}^{[k-1]}\left(\mathbf{M}_D^{[k]}\right)^{-1} - \mathbf{T}\mathbf{M}_D^{-1}\right]\mathbf{K}_C \tag{A.1}$$

Assume $\mathbf{T}^{[k\text{-}1]}=\mathbf{T}+\mathbf{E}^{[k\text{-}1]}$ and $\mathbf{E}^{[k]}$ is small, $\left(\mathbf{M}_D^{[k]}\right)^{-1}$ is expanded as the first order Taylor series:

$$\begin{aligned}\left(\mathbf{M}_D^{[k]}\right)^{-1} &= \left(\mathbf{I}^m + \mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\left(\mathbf{T}+\mathbf{E}^{[k-1]}\right)\right)^{-1} \\ &= \left(\mathbf{M}_D + \mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\right)^{-1} \\ &\approx \mathbf{M}_D^{-1} - \mathbf{M}_D^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{M}_D^{-1}\end{aligned} \tag{A.2}$$

Substitute Eq. (A.2) into Eq. (A.1) and neglect the higher order of the error matrix $\mathbf{E}^{[k\text{-}1]}$, one can obtain

$$\mathbf{E}^{[k]} = \mathbf{S}\bar{\mathbf{M}}\left(\mathbf{I} - \mathbf{T}\mathbf{M}_D^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\right)\mathbf{E}^{[k-1]}\mathbf{M}_D^{-1}\mathbf{K}_C \tag{A.3}$$

Eq. (29) implies $\lambda_i\mathbf{z}_i^m = \mathbf{M}_D^{-1}\mathbf{K}_C\mathbf{z}_i^m$. Accordingly, post-multiplying Eq. (A.3) with $\mathbf{z}_i^m$ leads to

$$\mathbf{E}^{[k]}\mathbf{z}_i^m = \mathbf{S}\bar{\mathbf{M}}\left(\mathbf{I} - \mathbf{T}\mathbf{M}_D^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\right)\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i = \mathbf{S}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i - \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i \tag{A.4}$$

The first item of Eq. (A.4) is expanded as

$$\mathbf{S}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i = \left[\mathbf{I} - \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\right]\bar{\mathbf{F}}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i \tag{A.5}$$

The second item of Eq. (A.4) is expanded as

$$\mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i = \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\left[\boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\bar{\mathbf{F}}\right]\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i \tag{A.6}$$

Based on Eqs. (A.5) and (A.6), Eq. (A.4) is rewritten as

$$\mathbf{E}^{[k]}\mathbf{z}_i^m = \left[\mathbf{I} - \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C} - \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\right]\bar{\mathbf{F}}\bar{\mathbf{M}}\mathbf{E}^{[k-1]}\mathbf{z}_i^m\lambda_i \tag{A.7}$$

The error vector $\mathbf{E}^{[k-1]}\mathbf{z}_i^m$ can be expressed as the linear combination of the full eigenvector matrix as

$$\mathbf{E}^{[k-1]}\mathbf{z}_i^m = \bar{\boldsymbol{\Phi}}\{q\} \tag{A.8}$$

where q is a coefficient vector that encloses the contribution of each eigenvector. $\bar{\boldsymbol{\Phi}}$ is divided into n_m master eigenvectors and n_s slave eigenvectors (n_s=$\bar{N}$-n_m). Accordingly, Eq. (A.8) is rewritten as

$$\mathbf{E}^{[k-1]}\mathbf{z}_i^m = \left[\bar{\boldsymbol{\Phi}}^m \quad \bar{\boldsymbol{\Phi}}^s\right]\begin{Bmatrix} q^m \\ q^s \end{Bmatrix} = \bar{\boldsymbol{\Phi}}^m q^m + \bar{\boldsymbol{\Phi}}^s q^s \tag{A.9}$$

According to Eq. (A.7), Eq. (A.9) also can be expressed as

$$\mathbf{E}^{[k-1]}\mathbf{z}_i^m = \left[\mathbf{I} - \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C} - \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}\right]\bar{\mathbf{F}}\bar{\mathbf{M}}\mathbf{E}^{[k-2]}\mathbf{z}_i^m\lambda_i = \bar{\boldsymbol{\Phi}}^m q^m + \bar{\boldsymbol{\Phi}}^s q^s \tag{A.10}$$

It is obvious that $\left[\bar{\boldsymbol{\Phi}}^m\right]^{\mathrm{T}}\bar{\mathbf{M}}\bar{\mathbf{F}} = \left[\bar{\boldsymbol{\Phi}}^m\right]^{\mathrm{T}}\bar{\mathbf{M}}\bar{\boldsymbol{\Phi}}^s\bar{\boldsymbol{\Lambda}}_s^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}} = \mathbf{0}$ and similarly $\left[\bar{\boldsymbol{\Phi}}^m\right]^{\mathrm{T}}\bar{\mathbf{M}}\mathbf{S} = \mathbf{0}$. So pre-multiplying Eq. (A.10) with $\left[\bar{\boldsymbol{\Phi}}^m\right]^{\mathrm{T}}\bar{\mathbf{M}}$ yields [22, 25]

$$q^m = \mathbf{0} \tag{A.11}$$

Consequently, Eq. (A.10) is expressed as

$$\mathbf{E}^{[k-1]}\mathbf{z}_i^m = \bar{\boldsymbol{\Phi}}^s q^s \tag{A.12}$$

Substituting Eq. (A.12) into Eq. (A.7) leads to

$$\mathbf{E}^{[k]}\mathbf{z}_i^m = \mathbf{P}\mathbf{F}\bar{\mathbf{M}}\bar{\boldsymbol{\Phi}}^s q^s\lambda_i = \mathbf{P}\bar{\boldsymbol{\Phi}}^s\left(\bar{\boldsymbol{\Lambda}}^s\right)^{-1}\left[\bar{\boldsymbol{\Phi}}^s\right]^{\mathrm{T}}\bar{\mathbf{M}}\bar{\boldsymbol{\Phi}}^s q^s\lambda_i = \mathbf{P}\bar{\boldsymbol{\Phi}}^s\left(\bar{\boldsymbol{\Lambda}}^s\right)^{-1}q^s\lambda_i \tag{A.13}$$

where $\mathbf{P} = \mathbf{I} - \bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C} - \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\boldsymbol{\Gamma}^m\left(\mathbf{C}\bar{\mathbf{F}}\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}$. It can be proved that $\mathbf{P}^2$=$\mathbf{P}$, i.e. $\mathbf{P}$ is an idempotent matrix. Therefore, there exists a specific subordinate matrix norm of $\mathbf{P}$ that

satisfies $\left\|\mathbf{P}\right\|=1$. Hence,

$$\left\|\mathbf{E}^{[k]}\mathbf{z}_i^m\right\| \le \left\|\mathbf{P}\right\|\left\|\bar{\mathbf{\Phi}}^s\left[\left(\bar{\mathbf{\Lambda}}^s\right)^{-1}\lambda_i\right]q^s\right\| = \left\|\bar{\mathbf{\Phi}}^s\left[\left(\bar{\mathbf{\Lambda}}^s\right)^{-1}\lambda_i\right]q^s\right\| \tag{A.14}$$

$\bar{\mathbf{\Lambda}}^s$ encloses the largest $s^{(j)}$ eigenvalues of all the substructures, i.e. $\lambda^{(j)}_{m^{(j)}+1}$, $\lambda^{(j)}_{m^{(j)}+2}$, …, $\lambda^{(j)}_{m^{(j)}+s^{(j)}}$ (j=1, 2, …, NS). Therefore, with the consideration of Eq. (A.12), Eq. (A.14) is rewritten as

$$\left\|\mathbf{E}^{[k]}\mathbf{z}_i^m\right\| \le \frac{\lambda_i}{\min\limits_{j=1,2,..,NS}\left\{\lambda^{(j)}_{m^{(j)}+1}\right\}}\left\|\mathbf{E}^{[k-1]}\mathbf{z}_i^m\right\| \tag{A.15}$$

Therefore, if $\dfrac{\lambda_i}{\min\limits_{j=1,2,..,NS}\left\{\lambda^{(j)}_{m^{(j)}+1}\right\}} \le 1$ is satisfied for all the interested modes, $\left\|\mathbf{E}^{[k]}\mathbf{z}_i^m\right\| \le \left\|\mathbf{E}^{[k-1]}\mathbf{z}_i^m\right\|$.

That is, when the eigenvalues of all interested modes are smaller than the $m^{(j)}$+1 eigenvalue of any substructure, $\mathbf{E}^{[k]}\mathbf{z}_i^m$ is smaller than $\mathbf{E}^{[k-1]}\mathbf{z}_i^m$ in norm, leading to the convergence of the proposed method for eigensolution.

Similarly, another error matrix is defined to estimate the error of $\left(\dfrac{\partial\mathbf{T}}{\partial r}\right)^{[k]}$ as

$$\mathbf{\varepsilon}^{[k]} = \left(\frac{\partial\mathbf{T}}{\partial r}\right)^{[k]} - \frac{\partial\mathbf{T}}{\partial r} \tag{A.16}$$

As $\mathbf{T}^{[k]}$ and $\mathbf{M}_D^{[k]}$ converge to $\mathbf{T}$ and $\mathbf{M}_D$ respectively after several iterations, $\mathbf{T}^{[k]}\approx\mathbf{T}$ and $\mathbf{M}_D^{[k]}\approx\mathbf{M}_D$. Accordingly, substitution of Eqs. (43) and (55) into Eq. (A.16) gives

$$\mathbf{\varepsilon}^{[k]} \approx \mathbf{S}\bar{\mathbf{M}}\left[\left(\frac{\partial\mathbf{T}}{\partial r}\right)^{[k-1]} - \frac{\partial\mathbf{T}}{\partial r}\right]\mathbf{M}_D^{-1}\mathbf{K}_C - \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\left[\left(\frac{\partial\mathbf{M}_D}{\partial r}\right)^{[k]} - \frac{\partial\mathbf{M}_D}{\partial r}\right]\mathbf{M}_D^{-1}\mathbf{K}_C \tag{A.17}$$

Substitute Eqs. (42) and (54) into Eq. (A.17), one can obtain

$$\boldsymbol{\varepsilon}^{[k]} = \mathbf{S}\bar{\mathbf{M}}\left[\left(\frac{\partial \mathbf{T}}{\partial r}\right)^{[k-1]} - \frac{\partial \mathbf{T}}{\partial r}\right]\mathbf{M}_D^{-1}\mathbf{K}_C - \mathbf{S}\bar{\mathbf{M}}\mathbf{T}\mathbf{M}_D^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\left[\left(\frac{\partial \mathbf{T}}{\partial r}\right)^{[k-1]} - \frac{\partial \mathbf{T}}{\partial r}\right]\mathbf{M}_D^{-1}\mathbf{K}_C$$
$$= \mathbf{S}\bar{\mathbf{M}}\left[\mathbf{I} - \mathbf{T}\mathbf{M}_D^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\right]\boldsymbol{\varepsilon}^{[k-1]}\mathbf{M}_D^{-1}\mathbf{K}_C \tag{A.18}$$

Considering $\lambda_i \mathbf{z}_i^m = \mathbf{M}_D^{-1}\mathbf{K}_C\mathbf{z}_i^m$, post-multiplication of Eq. (A.18) with $\mathbf{z}_i^m$ leads to

$$\boldsymbol{\varepsilon}^{[k]}\mathbf{z}_i^m = \mathbf{S}\bar{\mathbf{M}}\left[\mathbf{I} - \mathbf{T}\left(\mathbf{M}_D\right)^{-1}\mathbf{T}_C^{\mathrm{T}}\bar{\mathbf{M}}\right]\boldsymbol{\varepsilon}^{[k-1]}\mathbf{z}_i^m\lambda_i \tag{A.19}$$

It is noted that Eq. (A.19) has the same form as Eq. (A.4). Based on the similar derivations in Eqs. (A.5)-(A.14), one can obtain

$$\left\|\boldsymbol{\varepsilon}^{[k]}\mathbf{z}_i^m\right\| \le \frac{\lambda_i}{\min\limits_{j=1,2,..,NS}\left\{\lambda_{m^{(j)}+1}^{(j)}\right\}}\left\|\boldsymbol{\varepsilon}^{[k-1]}\mathbf{z}_i^m\right\| \tag{A.20}$$

Therefore, $\left(\frac{\partial \mathbf{T}}{\partial r}\right)^{[k]}$ also converges to $\frac{\partial \mathbf{T}}{\partial r}$ if $\frac{\lambda_i}{\min\limits_{j=1,2,..,NS}\left\{\lambda_{m^{(j)}+1}^{(j)}\right\}} \le 1$ is satisfied for all the interested modes, leading to the convergence of the proposed method for the eigensensitivity.

Appendix B. Multiplication Count (MC)

Assume the half bandwidth of the system matrices of the jth substructure is $b^{(j)}$, calculation of the eigensolutions for the first l modes with the proposed substructuring method consumes the MC of

$$\mathrm{MC}_{sub1} = \sum_{j=1}^{NS}\left\{\left[n^{(j)}\right]^2 b^{(j)} + n^{(j)}\left[b^{(j)}\right]^2 + 5m^{(j)}n^{(j)}b^{(j)}\right\} + 2p\sum_{j=1}^{NS}m^{(j)}\left[n^{(j)}\right]^2 + \left(\frac{1}{4}n_m^3 + \frac{5}{2}ln_m^2 + \sum_{j=1}^{NS}ln_m n^{(j)}\right) \tag{B.1}$$

where p is the number of iterations required in the proposed method. The first item represents the MC consumed by the initial step to compute each substructural solutions and the intermediate variables used in later steps. The second item indicates the MC cost in the iterative process. The third item implies the MC used to solve the eigensolutions from the reduced

eigenequation.

Assume the designed parameter locates in the *R*th substructure, computation of the eigensensitivities for the first *l* modes by the proposed method consumes the MC of

$$\mathrm{MC}_{sub2} = \left\{3m^{(R)}\left[n^{(R)}\right]^2 + 8m^{(R)}n^{(R)}b + 4n^{(R)}b^2 + \left[n^{(R)}\right]^2 b + 2\bar{N}^2 b + 2n_m\bar{N}b\right\} + p\left(6n_m^2\bar{N} + 13n_m\bar{N}b\right) + 2l\bar{N}n_m \quad \text{(B.2)}$$

where the first term denotes the MC consumed in the initial step to calculate the substructural eigensensitivities and the intermediate variables to be used in later steps. The second term represent the MC consumed in the iterative process to calculate the eigenvalue derivatives. The third term indicates the MC used in the calculation of eigenvector derivatives.

In the traditional method, the eigensolutions of the global structure are calculated with the Lanczos method [33, 34] and the eigensensitivities are computed with Nelson's method [32]. Calculation of the eigensolutions for the first *l* modes with the Lanczos method consumes the MC of

$$\mathrm{MC}_{G1} = Nb^2 + 5lNb \quad \text{(B.3)}$$

Employing the Nelson's method, calculation of the eigensensitivity for the first *l* modes requires the MC of

$$\mathrm{MC}_{G2} = 2lNb + l\left(N^2 + 6Nb\right) \quad \text{(B.4)}$$

where the first item represents the MC consumed in the calculation of eigenvalue derivatives and the second item indicates the MC consumed in the calculation of eigenvector derivatives.

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