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Normalization of correlated random variables in structural reliability analysis using fourth-moment transformation

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Abstract: In this paper, a fourth-moment transformation technique is proposed to transform correlated nonnormal random variables into independent standard normal ones. The procedure mainly includes two steps: First, the correlated nonnormal random variables are transformed into correlated standard normal ones using the fourth-moment transformation, where the complete mathematical formula of the correlation coefficient in standard normal space, i.e., equivalent correlation coefficient, is developed and the upper and lower bounds of original correlation coefficient are identified to ensure the transformation executable; Second, the correlated standard normal random variables are transformed into independent standard normal ones using Cholesky decomposition. For the cases of original correlation matrix with very small eigenvalues, the equivalent correlation matrix might become a nonpositive semidefinite matrix. A recently developed method for solving the problem is adopted to make Cholesky decomposition ready. A first-order reliability method (FORM) for structural reliability analysis involving correlated random variables is developed using the proposed transformation technique. Several numerical examples are presented to demonstrate the efficiency and

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accuracy of the proposed method for structural reliability assessment considering correlated random variables.

Keywords: Correlated random variables; Structural reliability; Normal transformation; Fourth-moment transformation; Equivalent correlation coefficient.

1. Introduction

A fundamental problem in structural reliability theory is the computation of the following multi-fold probability integral [1, 2]:

$$P_f = \int_{G(\mathbf{X})\leq 0} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where P_f is probability of structural failure; $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$ (where T denotes matrix transpose) is an n-dimensional vector of random variables representing uncertain quantities such as loads, material properties, geometric dimensions, and boundary conditions; $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of $\mathbf{X}$; $G(\mathbf{X})$ is the performance function; and $G(\mathbf{X}) \leq 0$ is the domain of integration, which denotes the failure region of the structure.

The difficulty in computing the probability integral of Eq. (1) has led to the development of various reliability approximation techniques, such as the first- and second-order reliability method, moment methods and simulation methods for estimating the failure probability [3-6]. The first-order reliability method (FORM) [1, 7, 8] is considered to be one of the most reliable computational methods [5, 9] and has become a basic method for structural reliability analysis. As the random variables encountered in engineering practice are often nonnormally distributed and correlated [10-17], a tractable procedure for efficient transformation from correlated nonnormal random variables into independent standard normal ones (referred to as normal transformation hereafter) is necessary for structural reliability analysis.

If the joint PDF of basic random variables, i.e., $f_{\mathbf{X}}(\mathbf{x})$ in Eq. (1), is known, Rosenblatt transformation [10] is available to realize the normal transformation. While this method leads to n! different transformations considering the ordering of the inputs, and not all of them give rise to the same favourable numerical properties [2, 18]. Moreover, the joint PDF is seldom available in many practical applications. Thus, the Nataf transformation [19] is a useful alternative to normalize the inputs, which requires the marginal PDFs and correlation matrix of the input random variables. It has been pointed out [18, 19] that the major obstacle for Nataf transformation is to evaluate the equivalent correlation matrix in standard normal space that should be determined by solving two-dimensional nonlinear integral equations. To avoid the complexity, Der Kiureghian and Liu [20] developed 54 empirical formulas for 10 kinds of distributions through numerical experiments. It was demonstrated that the formulas can produce sufficiently accurate results and Nataf transformation has no shortcoming of varying with the transformation order of random variables. In spite of many cases included, the empirical formulas cannot cover arbitrary cases (e.g., the truncated distributions) because there are various probability distributions in engineering practice [18]. More recently, using the first three moments (mean, standard deviation, and skewness) and correlation matrix of basic random variables, a third-moment transformation technique is proposed [17] for transforming the correlated variables into independent standard normal ones, which complements normal transformation when the joint PDF and marginal PDFs of the basic random variables are unknown. However, the third-moment transformation is not flexible enough to reflect the fourth moment, i.e., kurtosis, of a basic random variable or statistical data, which may lead to inappropriate results when the kurtosis is important and should be accounted for. To efficiently utilize the information of kurtosis as well as the mean, standard deviation, and skewness of the basic random variables, a fourth-moment transformation has been proposed [21-25] to transforming the independent random variables into independent standard normal ones. However, the application and efficiency of the fourth-moment transformation technique for normal transformation involving correlated random variables has not been investigated adequately yet.

The objective of the paper is to extend the fourth-moment transformation technique to transform correlated nonnormal random variables into independent standard normal ones. The paper is organized as follows: Section 2 presents a fourth-moment transformation for normalization of correlated nonnormal random variables. The complete mathematical formula of equivalent correlation coefficient is proposed, and the application regions of the proposed fourth-moment transformation is also investigated; Section 3 presents a computational procedure of FORM based on the proposed fourth-moment transformation for performance function involving correlated random variables. In Section 4, the simplicity, efficiency and accuracy of the proposed method are demonstrated through several numerical examples. Finally, the findings of the paper are summarized in Section 5.

2. Fourth-moment transformation for correlated random variables

2.1. Inverse normal transformation based on fourth-moment transformation

Without loss of generality, a random variable X_i can be standardized as:

$$X_{is} = \left(X_i - \mu_{X_i} \right) \big/ \sigma_{X_i} \tag{2}$$

where μ_{X_i} and σ_{X_i} are the mean and standard deviation of X_i, respectively.

According to the fourth-moment transformation technique suggested by Fleishman [21], the standardized variable X_{is} can be approximated by:

$$X_{is} = S_z(Z_i) = a_i + b_i Z_i + c_i Z_i^2 + d_i Z_i^3 \tag{3}$$

where Z_i is the ith correlated standard normal random variable; $S_z(Z_i)$ is the third-order polynomial of Z_i; and a_i, b_i, c_i, and d_i are the polynomial coefficients, which can be determined by making the first four central moments (i.e., mean, standard deviation, skewness, and kurtosis) of $S_z(Z_i)$ equal to those of X_{is} [21] as shown in Appendix A.

Combining Eqs. (2) and (3), the relation between X_i and Z_i can be written as:

$$X_i = \mu_{X_i} + \sigma_{X_i}(a_i + b_i Z_i + c_i Z_i^2 + d_i Z_i^3), \quad (i = 1, \ 2, \ \ldots, \ n) \tag{4}$$

Assume that the correlation coefficient between X_i and X_j is ρ_{ij}, and the correlation coefficient between Z_i and Z_j is ρ_{0ij} referred as equivalent correlation coefficient. According to the definition of correlation coefficient and the relation between X_i and Z_i given in Eq. (4), the relationship between ρ_{ij} and ρ_{0ij} can be formulated as [26]:

$$\rho_{ij} = (b_i b_j + 3d_i b_j + 3b_i d_j + 9d_i d_j)\cdot \rho_{0ij} + 2c_i c_j \rho_{0ij}^2 + 6d_i d_j \rho_{0ij}^3 \tag{5a}$$

It is worth noting that: (i) ρ_{0ij} should be an increasing function of ρ_{ij}; (ii) $\rho_{0ij} = 0$ for $\rho_{ij} = 0$; and (iii) the valid solution of ρ_{0ij} should be restricted by the following conditions to satisfy the definition of the correlation coefficient:

$$-1 \le \rho_{0ij} \le 1 \ \text{ and } \ \rho_{ij}\cdot\rho_{0ij} \ge 0 \tag{5b}$$

According to Eq. (5*a*) and the properties of equation between ρ_{0ij} and ρ_{ij}, the equivalent correlation coefficient ρ_{0ij} can be determined and summarized at Eqs. (6*a*)-(6*c*). The detailed derivations and the application regions of each formula are shown in Section 2.3.

(1) When $d_i d_j \neq 0$, ρ_{0ij} is formulated as:

$$\rho_{0ij} = \begin{cases} -2r\cos\left[(\theta+\pi)/3\right] - t_2/3, & d_i d_j < 0 \\ \sqrt[3]{A} + \sqrt[3]{B} - t_2/3, & d_i d_j > 0 \ \text{and} \ \Delta \ge 0 \\ 2r\cos(\theta/3) - t_2/3, & d_i d_j > 0 \ \text{and} \ \Delta < 0 \ \text{and} \ t_2 > 0 \\ -2r\cos\left[(\theta-\pi)/3\right] - t_2/3, & d_i d_j > 0 \ \text{and} \ \Delta < 0 \ \text{and} \ t_2 < 0 \end{cases} \tag{6a}$$

(2) When $d_i d_j = 0$ and $c_i c_j \neq 0$, ρ_{0ij} is:

$$\rho_{0ij} = \frac{1}{4c_i c_j}\left[-(b_i b_j + 3b_i d_j + 3d_i b_j) + \sqrt{(b_i b_j + 3b_i d_j + 3d_i b_j)^2 + 8c_i c_j \rho_{ij}}\right] \tag{6b}$$

(3) When $d_i d_j = 0$ and $c_i c_j = 0$, ρ_{0ij} is formulated as:

$$\rho_{0ij} = \frac{\rho_{ij}}{b_i b_j + 3b_i d_j + 3d_i b_j} \tag{6c}$$

The parameters of r, θ, t_2, A, B, and Δ in Eq. (6*a*) are given by Eqs. (7*a*)-(7*c*).

$$r=\sqrt{-\frac{p}{3}},\ \ \theta=\arccos\left(-\frac{q}{2r^{3}}\right),\ \ t_{2}=\frac{c_{i}c_{j}}{3d_{i}d_{j}} \tag{7a}$$

$$A=-\frac{q}{2}+\sqrt{\Delta},\ \ B=-\frac{q}{2}-\sqrt{\Delta},\ \ \Delta=\left(\frac{p}{3}\right)^{3}+\left(\frac{q}{2}\right)^{2} \tag{7b}$$

$$p=t_{1}-\frac{t_{2}^{2}}{3},\ \ q=\frac{2t_{2}^{3}}{27}-\frac{t_{1}t_{2}}{3}-\frac{\rho_{ij}}{6d_{i}d_{j}},\ \ t_{1}=\frac{b_{i}b_{j}+3b_{i}d_{j}+3d_{i}b_{j}+9d_{i}d_{j}}{6d_{i}d_{j}} \tag{7c}$$

From the preceding discussion, any two arbitrary correlated random variables with known the first four moments and correlation coefficients can be approximated by two correlated standard normal variables, in which the correlation coefficients of standard normal random variables can be readily determined by Eqs. (6*a*)-(6*c*). Then, the correlation matrix in standard normal variables can be summarized as $\mathbf{C_Z}$. The relationship between the correlated standard normal random vector **Z** and the independent standard normal vector **U** can be expressed as:

$$\mathbf{Z}=\mathbf{L}_{0}\mathbf{U} \tag{8}$$

where $\mathbf{L}_0$ is a lower triangular matrix obtained from Cholesky decomposition of $\mathbf{C_Z}$.

Theoretically, $\mathbf{C_Z}$ is positive semidefinite once the fully correlated variables are extended. However, because the probability information used in the fourth-moment transformation is incomplete, small negative eigenvalues of $\mathbf{C_Z}$ might appear during the transformation from nonnormal variables to normal variables, especially in the cases of original correlation matrix with very small eigenvalues. Under this circumstance, a method introduced by Ji et al. [27] is adopted. $\mathbf{C_Z}$ is rewritten as:

$$\mathbf{C_Z}=\mathbf{V\Lambda V}^{T} \tag{9}$$

where **V** and **Λ** are the eigenvector and diagonal eigenvalue matrices of $\mathbf{C_Z}$, respectively. The small negative eigenvalues in **Λ** are substituted by small positive values, e.g., 0.001, to make Cholesky decomposition ready.

According to Eq. (8), Z_i is expressed as:

$$Z_i = \sum_{k=1}^{i} l_{ik} U_k, \quad (i = 1,\ 2,\ \cdots,\ n) \tag{10}$$

where Z_i is the ith correlated standard normal random variable; U_k is the kth independent standard normal random one; and l_{ik} is the ith row kth column element of matrix $\mathbf{L}_0$.

Substituting Eq. (10) into Eq. (4), the inverse normal transformation based on the fourth-moment transformation is expressed as:

$$X_i = \mu_{X_i} + \sigma_{X_i}\left[a_i + b_i \sum_{k=1}^{i} l_{ik} U_k + c_i \left(\sum_{k=1}^{i} l_{ik} U_k\right)^2 + d_i \left(\sum_{k=1}^{i} l_{ik} U_k\right)^3\right], \quad (i = 1,\ 2,\ \cdots,\ n) \tag{11}$$

2.2. Normal transformation based on fourth-moment transformation

According to Eq. (3) and (4), the **X** to **Z** transformation is to find the proper roots of the third-order polynomial functions, i.e.,

$$Z_i = S_z^{-1}(X_{is}) \quad (i = 1, 2, \cdots, n) \tag{12}$$

where $S_z^{-1}(\cdot)$ is the inverse function of $S_z(\cdot)$. According to Zhao et al. [24], the completed expression of $S_z^{-1}(\cdot)$ is summarized in Table 1.

Table 1. Complete expression of $S_z^{-1}(\cdot)$

Parameter			Range of X_i	Expression of $S_z^{-1}(x_{is})$
$d_i < 0$			$J_2 < x_i < J_1$	$-2r_i \cos[(\theta_i + \pi)/3] - c_i/(3d_i)$
$d_i > 0$	$p_i < 0$	$\alpha_{3X_i} \ge 0$	$J_1 < x_i < J_2$	$2r_i \cos(\theta_i/3) - c_i/(3d_i)$
			$x_i \ge J_2$	$\sqrt[3]{A_i} + \sqrt[3]{B_i} - c_i/(3d_i)$
		$\alpha_{3X_i} < 0$	$J_1 < x_i < J_2$	$-2r_i \cos[(\theta_i - \pi)/3] - c_i/(3d_i)$
			$x_i \le J_1$	$\sqrt[3]{A_i} + \sqrt[3]{B_i} - c_i/(3d_i)$
	$p_i \ge 0$		$(-\infty, +\infty)$	$\sqrt[3]{A_i} + \sqrt[3]{B_i} - c_i/(3d_i)$
$d_i = 0$		$\alpha_{3X_i} \ne 0$	$b_i^2 + 4c_i(c_i + x_{is}) \ge 0$	$[-b_i + \sqrt{b_i^2 + 4c_i(c_i + x_{is})}]/(2c_i)$
		$\alpha_{3X_i} = 0$	$(-\infty, +\infty)$	x_{is}

In Table 1, the parameters p_i, r_i, θ_i, A_i, B_i, J_1, and J_2 can be obtained using Eq. (13a)-(13c).

$$p_i = \frac{3b_i d_i - c_i^2}{3d_i^2},\;\; r_i = \sqrt{-\frac{p_i}{3}},\;\; \theta_i = \arccos\left(-\frac{q_i}{2r_i^3}\right),\;\; q_i = \frac{2c_i^3}{27d_i^3} - \frac{b_i c_i}{3d_i^2} - \frac{c_i}{d_i} - \frac{x_{is}}{d_i} \tag{13a}$$

$$A_i = -\frac{q_i}{2} + \sqrt{\Delta_i}\,,\;\; B_i = -\frac{q_i}{2} - \sqrt{\Delta_i}\,,\;\; \Delta_i = \sqrt{p_i^3 + q_i^2}\,, \tag{13b}$$

$$J_1 = \sigma_{X_i} d_i \left(-2r_i^3 + \frac{2c_i^3}{27d_i^3} - \frac{b_i c_i}{3d_i^2} - \frac{c_i}{d_i}\right) + \mu_{X_i}\,,\;\; J_2 = \sigma_{X_i} d_i \left(2r_i^3 + \frac{2c_i^3}{27d_i^3} - \frac{b_i c_i}{3d_i^2} - \frac{c_i}{d_i}\right) + \mu_{X_i} \tag{13c}$$

Then, according to Eq. (8), the correlated standard normal random vector **Z** can be transformed into independent standard normal vector **U**,

$$\mathbf{U} = \mathbf{L}_0^{-1}\mathbf{Z} \tag{14}$$

where $\mathbf{L}_0^{-1}$ is the inverse matrix of $\mathbf{L}_0$, which is also a lower triangular matrix.

According to Eq. (14), U_i is computed as:

$$U_i = \sum_{k=1}^{i} h_{ik} Z_k,\quad (i = 1,\ 2,\ \cdots,\ n) \tag{15}$$

where h_{ik} is the ith row kth column element of matrix $\mathbf{L}_0^{-1}$.

Substituting Eq. (12) into Eq. (15) and using the relation between X_i and X_{is} given in Eq. (2), the normal transformation based on the fourth-moment transformation is:

$$U_i = \sum_{k=1}^{i} h_{ik} S_z^{-1}(X_{ks}) = \sum_{k=1}^{i} h_{ik} S_z^{-1}\left[(X_k - \mu_{X_k})/\sigma_{X_k}\right],\quad (i = 1,\ 2,\ \cdots,\ n) \tag{16}$$

2.3. Application regions of the proposed fourth-moment transformation

The suitable regions of the fourth-moment transformation technique for the translation of independent nonnormal random variables into independent standard normal ones, i.e., the compatibility and limitations of the pairs of skewness and kurtosis, have been discussed by Zhao et al. [25]. In this section, the selection of the formula for equivalent correlation coefficient, i.e., Eqs. (6*a*)-(6*c*), with detailed derivations and the applicable upper and lower bounds of original correlation coefficient ρ_{ij} are investigated.

If $d_i d_j \neq 0$, Eq. (5*a*) is a cubic equation about ρ_{0ij} and can be equivalently expressed as:

$$\frac{\rho_{ij}}{6d_id_j} = \frac{b_ib_j + 3b_id_j + 3d_ib_j + 9d_id_j}{6d_id_j}\rho_{0ij} + \frac{c_ic_j}{3d_id_j}\rho_{0ij}^2 + \rho_{0ij}^3 \tag{17}$$

For convenience of exposition, the right side of Eq. (17) is expressed as $h(\rho_{0ij})$, i.e.,

$$\frac{\rho_{ij}}{6d_id_j} = h(\rho_{0ij}) = \rho_{0ij}^3 + t_2\rho_{0ij}^2 + t_1\rho_{0ij} \tag{18a}$$

$$t_1 = \frac{b_ib_j + 3b_id_j + 3d_ib_j + 9d_id_j}{6d_id_j}, \quad t_2 = \frac{c_ic_j}{3d_id_j} \tag{18b}$$

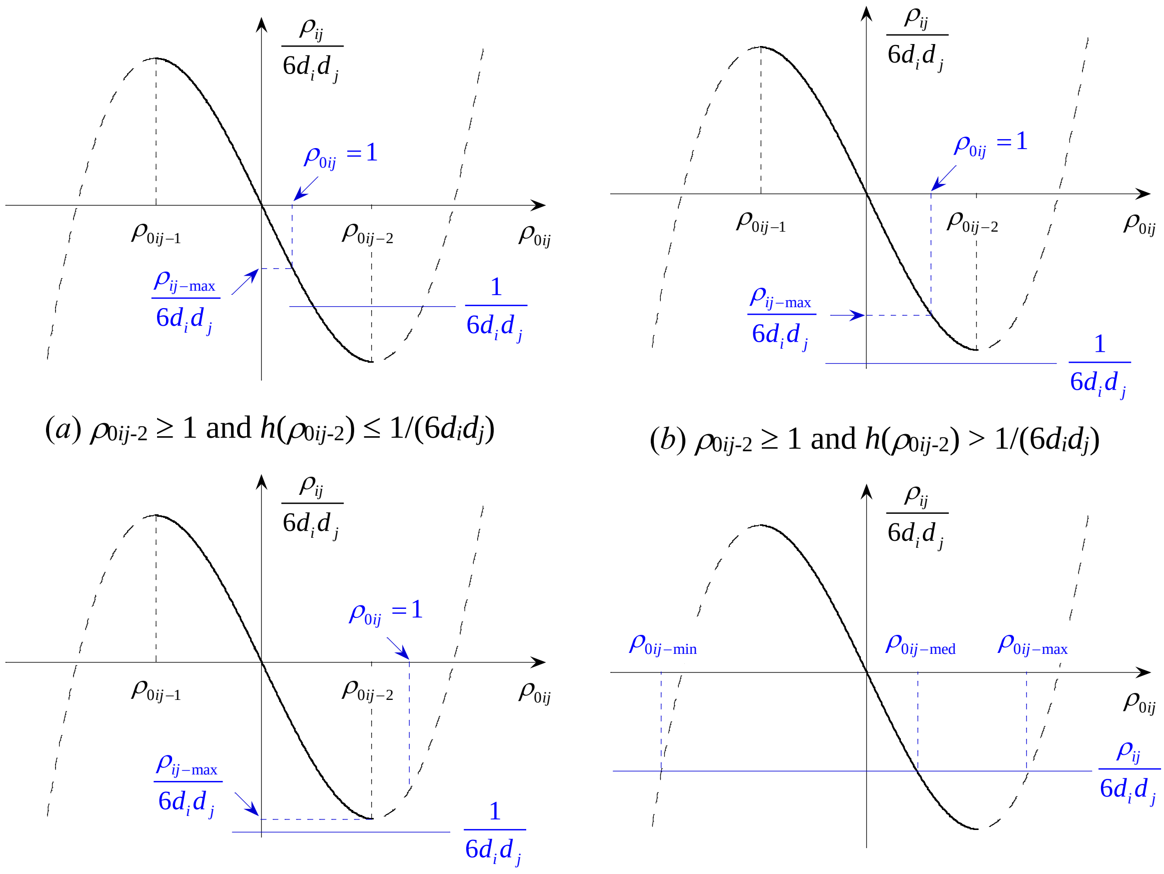
The first- and second-order derivative of $h(\rho_{0ij})$ with respect to ρ_{0ij} are formulated as:

$$h'(\rho_{0ij}) = dh(\rho_{0ij})/d\rho_{0ij} = 3\rho_{0ij}^2 + 2t_2\rho_{0ij} + t_1 \tag{19a}$$

$$h''(\rho_{0ij}) = d^2h(\rho_{0ij})/d\rho_{0ij}^2 = 6\rho_{0ij} + 2t_2 \tag{19b}$$

According to the compatibility and limitations of the pairs of skewness and kurtosis in the fourth-moment transformation technique for independent non-normal random variables given by Zhao et al. [25], the minimum of $b_ib_j+3b_id_j+3d_ib_j+9d_id_j$ (i.e., the numerator of t_1) is determined as a positive value. Accordingly, t_1 is positive for $d_id_j >0$ and negative for $d_id_j <0$.

(1) When $d_id_j < 0$, there exist two real roots $\rho_{0ij-1} = (-t_2 - \sqrt{t_2^2 - 3t_1})/3$ and $\rho_{0ij-2} = (-t_2 + \sqrt{t_2^2 - 3t_1})/3$ for the quadratic equation of $h'(\rho_{0ij}) = 0$ due to the fact that its discriminant $D = (2t_2)^2 - 4\cdot 3\cdot t_1 = 4(t_2^2 - 3t_1)$ is positive. It should be noted that $\rho_{0ij\text{-}1}$ and $\rho_{0ij\text{-}2}$ are not correlation coefficients but just the roots of Eq. (19*a*) equal to zero. Moreover, since $t_1 < 0$ for $d_id_j < 0$, it can be obtained that $h(\rho_{0ij-1})\cdot h(\rho_{0ij-2}) = t_1^2(4t_1 - t_2^2)/27 < 0$ and $\rho_{0ij-1}\cdot\rho_{0ij-2} = t_1/3 < 0$. According to the property of cubic function, the shape of $h(\rho_{0ij})$ for $d_id_j < 0$ is depicted in Fig. 1, in which the solid line denotes the region satisfying the condition that $\rho_{ij}\cdot\rho_{0ij} \geq 0$ and ρ_{0ij} is an increasing function of ρ_{ij}. Note that the correlation coefficient ρ_{ij} is a negative value when the vertical axis, $\rho_{ij}/(6d_id_j)$, takes a positive value, as $d_id_j < 0$.



(a) $\rho_{0ij\text{-}2} \geq 1$ and $h(\rho_{0ij\text{-}2}) \leq 1/(6d_id_j)$

(b) $\rho_{0ij\text{-}2} \geq 1$ and $h(\rho_{0ij\text{-}2}) > 1/(6d_id_j)$

(c) $\rho_{0ij\text{-}2} < 1$

(d) Intersections of $h(\rho_{0ij})$ and $\rho_{ij}/(6d_id_j)$

Fig. 1. The shape of $h(\rho_{0ij})$ for $d_id_j < 0$

To ensure ρ_{0ij} satisfies the definition of correlation coefficient, i.e., $-1 \leq \rho_{0ij} \leq 1$, ρ_{ij} should be within a value interval, i.e., $\rho_{ij} \in [\rho_{ij\text{-min}}, \rho_{ij\text{-max}}]$, in which $\rho_{ij\text{-min}}$ and $\rho_{ij\text{-max}}$ are the lower and upper bounds, respectively. The upper bound, $\rho_{ij\text{-max}}$, is first discussed as follows. When $\rho_{0ij\text{-}2} \geq 1$, $h(1)$ (i.e., $\rho_{0ij} = 1$) must be larger than $1/(6d_id_j)$ (i.e., $\rho_{ij} \leq 1$) because $|\rho_{ij}| \leq |\rho_{0ij}|$ is valid for any type of random variables [20, 28], as shown in Figs. 1(a)-(b). From Figs. 1(a)-(b), the upper bound $\rho_{ij\text{-max}}$ for $\rho_{0ij\text{-}2} \geq 1$ can be determined as $6d_id_j \cdot h(1)$ to ensure $\rho_{0ij} \leq 1$; When $\rho_{0ij\text{-}2} < 1$, as shown in Fig. 1(c), the upper bound can be determined as $\rho_{ij\text{-max}} = 6d_id_j \cdot h(\rho_{0ij\text{-}2})$ to ensure ρ_{0ij} increasing with ρ_{ij}. Therefore, when $d_id_j < 0$, the suitable maximum original correlation coefficient, i.e., the upper bound of ρ_{ij}, can be summarized as:

$$\rho_{ij-\max} = \begin{cases} 6d_id_j \cdot h(\rho_{0ij-2}), & \rho_{0ij-2} < 1 \\ 6d_id_j \cdot h(1), & \text{otherwise} \end{cases} \tag{20a}$$

Similarly, the lower bound of ρ_{ij} can be determined as:

$$\rho_{ij-\min} = \begin{cases} 6d_i d_j \cdot h(\rho_{0ij-1}), & \rho_{0ij-1} > -1 \\ 6d_i d_j \cdot h(-1), & \text{otherwise} \end{cases} \tag{20b}$$

The intersections of $h(\rho_{0ij})$ and $\rho_{ij}/(6d_id_j)$ are depicted in Fig. 1(d). From Fig. 1(d), it can be observed that there are three intersections for $\rho_{ij} \in [\rho_{ij\text{-min}}, \rho_{ij\text{-max}}]$, i.e., $h(\rho_{0ij}) = \rho_{ij}/(6d_id_j)$ have three real roots. The equivalent correlation coefficient ρ_{0ij} is the medial horizontal coordinate of these intersections. According to the solution of cubic equation and its property, as shown in Appendix B, ρ_{0ij} is expressed as:

$$\rho_{0ij} = -2r\cos\left(\frac{\theta+\pi}{3}\right) - \frac{t_2}{3} \tag{21}$$

(2) When $d_id_j > 0$ and $D = (2t_2)^2 - 4 \cdot 3 \cdot t_1 = 4(t_2^2 - 3t_1) \le 0$, there always exists $h'(\rho_{0ij}) \ge 0$ for arbitrary ρ_{0ij}. Therefore, the tendency of $h(\rho_{0ij})$ is increasing for $\rho_{0ij} \in (-\infty, +\infty)$. The shape of $h(\rho_{0ij})$ for $d_id_j > 0$ and $D \le 0$ is depicted in Fig. 2.

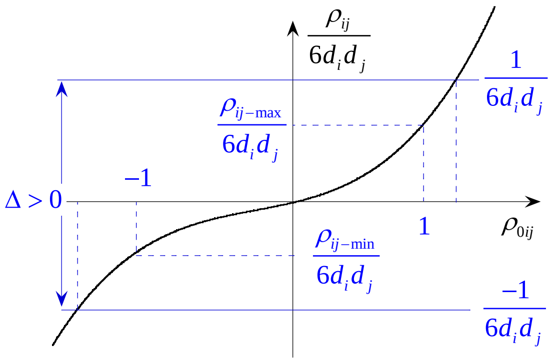


Fig. 2. The shape of $h(\rho_{0ij})$ for $d_id_j > 0$ and $D \le 0$

Because $|\rho_{ij}| \le |\rho_{0ij}|$ is valid for any type of random variables [20, 28], the horizontal coordinate of the intersection of $h(\rho_{0ij})$ and $1/6d_id_j$ (i.e., $\rho_{ij} = 1$) is larger than 1, and that of $h(\rho_{0ij})$ and $-1/6d_id_j$ (i.e., $\rho_{ij} = -1$) is lesser than -1, as indicated in Fig. 2. To ensure $-1 \le \rho_{0ij} \le 1$, the original correlation coefficient ρ_{ij} should be in between $\rho_{ij\text{-}max}$ and $\rho_{ij\text{-}min}$:

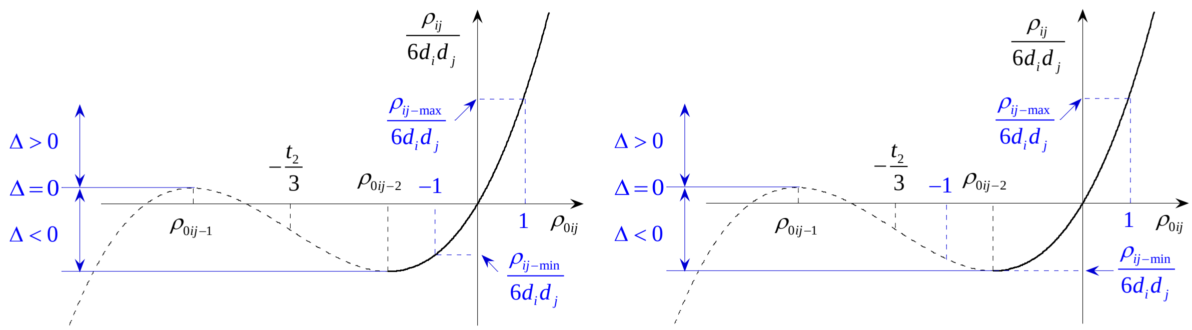
$$\rho_{ij-\max} = 6d_i d_j \cdot h(1) \tag{22a}$$

$$\rho_{ij-\min} = 6d_i d_j \cdot h(-1) \tag{22b}$$

Because $h(\rho_{0ij})$ is a monotonical increasing function for $\rho_{ij} \in [\rho_{ij\text{-min}}, \rho_{ij\text{-max}}]$, there is only one intersection of $h(\rho_{0ij})$ and $\rho_{ij}/(6d_i d_j)$. Therefore, equation $h(\rho_{0ij}) = \rho_{ij}/(6d_i d_j)$ has only one real root. According to the solution of cubic equation shown in Appendix B, the equivalent correlation coefficient ρ_{0ij} can be expressed as:

$$\rho_{0ij} = \sqrt[3]{A} + \sqrt[3]{B} - \frac{t_2}{3} \tag{23}$$

(3) When $d_i d_j > 0$ and $D = (2t_2)^2 - 4\cdot 3\cdot t_1 = 4(t_2^2 - 3t_1) > 0$, equation $h'(\rho_{0ij}) = 0$ has two real roots $\rho_{0ij\text{-}1}$ and $\rho_{0ij\text{-}2}$. Therefore, the tendency of $h(\rho_{0ij})$ is increasing for $\rho_{0ij} \in (-\infty, \rho_{0ij-1}) \cup (\rho_{0ij-2}, +\infty)$ and decreasing for $\rho_{0ij} \in [\rho_{0ij-1}, \rho_{0ij-2}]$. According to $h''(\rho_{0ij}) = 0$, the inflection point of $h(\rho_{0ij})$ is computed as $\rho_{0ij} = -t_2/3$. Moreover, $\rho_{0ij-1} \cdot \rho_{0ij-2} = t_1/3$ is positive because $t_1 > 0$. The shape of $h(\rho_{0ij})$ for this case is depicted in Fig. 3 for $t_2 > 0$ and Fig. 4 for $t_2 < 0$, in which the solid lines denote the region satisfying the condition that $\rho_{ij} \cdot \rho_{0ij} \geq 0$ and ρ_{0ij} is an increasing function of ρ_{ij}.



(a) $\rho_{0ij\text{-}2} \leq -1$ (b) $\rho_{0ij\text{-}2} > -1$

Fig. 3. The shape of $h(\rho_{0ij})$ for $d_i d_j > 0$, $D > 0$, and $t_2 > 0$

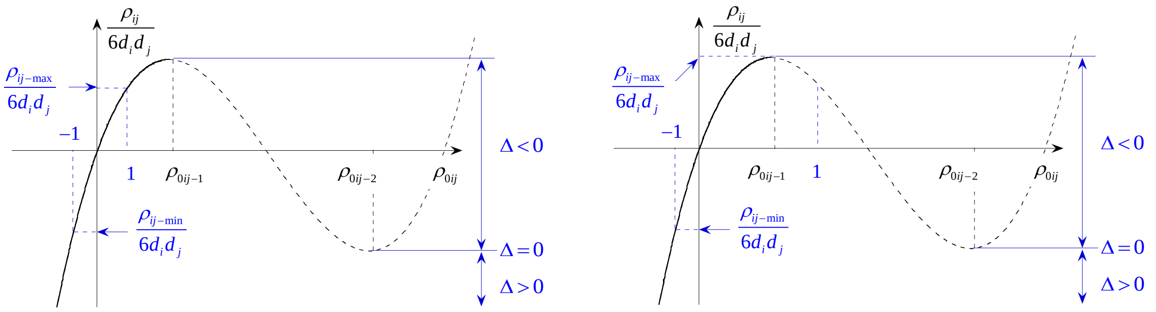
The derivation of the bounds of ρ_{ij} for $d_i d_j > 0$, $D > 0$, and $t_2 > 0$ is similar to that of the preceding cases. As shown in Fig. 3, the bounds of ρ_{ij} can be expressed as:

$$\rho_{ij-\max} = 6d_i d_j \cdot h(1) \tag{24a}$$

$$\rho_{ij-\min} = \begin{cases} 6d_i d_j \cdot h(\rho_{0ij-2}), & \rho_{0ij-2} > -1 \\ 6d_i d_j \cdot h(-1), & \text{otherwise} \end{cases} \tag{24b}$$

According to the properties of cubic equation, the discriminant Δ expressed in Eq. (7*b*) can be used to determine the number of real roots. The values of Δ for $d_i d_j > 0$, $D > 0$, and $t_2 > 0$ are also depicted in Fig. 3. It can be observed from Fig. 3 that there is only one real root for $\Delta > 0$, two roots for $\Delta = 0$, and three roots for $\Delta < 0$. According to Fig. 3 and the solution of cubic equation given in Appendix B, the equivalent correlation coefficient ρ_{0ij} can be determined: (i) when $\Delta > 0$, there is only one intersection of $h(\rho_{0ij})$ and $\rho_{ij}/(6d_i d_j)$, and the solution of ρ_{0ij} is the horizontal coordinate of this intersection, which is identical with Eq. (23); (ii) when $\Delta = 0$, there are two intersections, and the solution of ρ_{0ij} is the larger of the horizontal coordinate of these intersections, which is identical with Eq. (23); and (iii) when $\Delta < 0$, there exist three intersections, and the solution of ρ_{0ij} is the largest of the horizontal coordinate of these intersections, which is expressed as:

$$\rho_{0ij} = 2r\cos\left(\frac{\theta}{3}\right) - \frac{t_2}{3} \tag{25}$$



(*a*) $\rho_{0ij\text{-}1} \le 1$ (*b*) $\rho_{0ij\text{-}1} > 1$

Fig. 4. The shape of $h(\rho_{0ij})$ for $d_i d_j > 0$, $D > 0$, and $t_2 < 0$

Similarly, as shown in Fig. 4, the application range of ρ_{ij} for $d_i d_j > 0$, $D > 0$, and $t_2 < 0$ can be determined as:

$$\rho_{ij-\max} = \begin{cases} 6d_i d_j \cdot h(\rho_{0ij-1}), & \rho_{0ij-1} > 1 \\ 6d_i d_j \cdot h(1), & \text{otherwise} \end{cases} \tag{26a}$$

$$\rho_{ij-\min} = 6d_i d_j \cdot h(-1) \tag{26b}$$

From Fig. 4 and with the solution of cubic equation given in Appendix B, the equivalent correlation coefficient ρ_{0ij} can be determined: (i) when $\Delta > 0$, there is only one intersection of $h(\rho_{0ij})$ and $\rho_{ij}/(6d_id_j)$, and the solution of ρ_{0ij} is the horizontal coordinate of this intersection, which is identical with Eq. (23); (ii) when $\Delta = 0$, there are two intersections, and the solution of ρ_{0ij} is the smaller of the horizontal coordinate of these intersections, which is identical with Eq. (23); and (iii) when $\Delta < 0$, there exist three intersections, and the solution of ρ_{0ij} is the smallest of the horizontal coordinate of these intersections, which is expressed as:

$$\rho_{0ij} = -2r\cos\left(\frac{\theta-\pi}{3}\right) - \frac{t_2}{3} \tag{27}$$

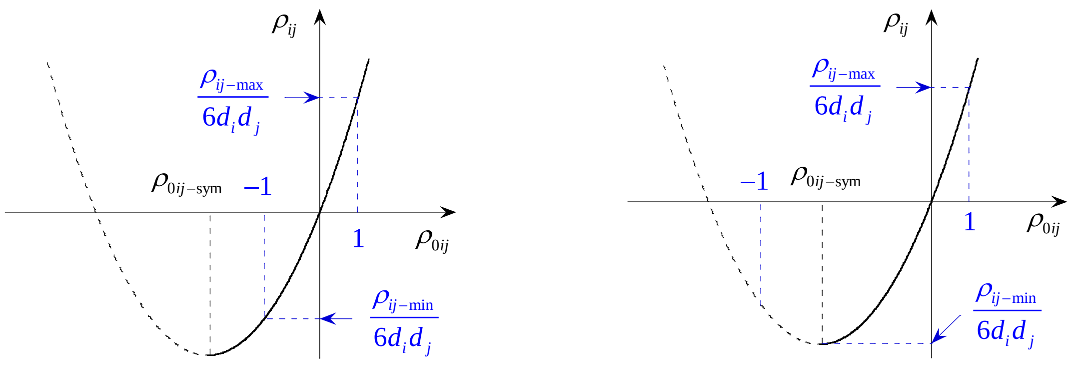
In summary, the solutions of equivalent correlation coefficient ρ_{0ij} for $d_id_j \neq 0$ (i.e., Eq. (5*a*) is a cubic equation about ρ_{0ij}) are summarized as indicated in Eq. (6*a*).

(4) If $d_id_j = 0$ and $c_ic_j \neq 0$, Eq. (5*a*) reduces to a quadratic equation. For brevity, the right side of Eq. (5*a*) is expressed as $h_2(\rho_{0ij})$, i.e.,

$$\rho_{ij} = (b_ib_j + 3d_ib_j + 3b_id_j)\cdot\rho_{0ij} + 2c_ic_j\rho_{0ij}^2 = h_2(\rho_{0ij}) \tag{28}$$

The axis of symmetry of $h_2(\rho_{0ij})$ is at:

$$\rho_{0ij} = \rho_{0ij-\mathrm{sym}} = -\frac{b_ib_j + 3b_id_j + 3d_id_j}{4c_ic_j} \tag{29}$$



(*a*) $\rho_{0ij\text{-sym}} > -1$ (*b*) $\rho_{0ij\text{-sym}} < -1$

Fig. 5. The shape of $h_2(\rho_{0ij})$ for $c_ic_j > 0$

Because $b_i b_j$+3$b_i d_j$+3$d_i b_j$ is larger than zero, $\rho_{0ij\text{-sym}}$ is positive for $c_i c_j < 0$ and negative for $c_i c_j > 0$. The shape of $h_2(\rho_{0ij})$ is depicted in Fig. 5 for $c_i c_j > 0$, in which the solid lines denote the region satisfying the condition that $\rho_{ij} \cdot \rho_{0ij} \geq 0$ and ρ_{0ij} is an increasing function of ρ_{ij}.

From Fig. 5, the application bound of ρ_{ij} for $c_i c_j > 0$ is expressed as:

$$\rho_{ij-\max} = h_2(1) \tag{30a}$$

$$\rho_{ij-\min} = \begin{cases} h_2(\rho_{0ij-\mathrm{sym}}), & \rho_{0ij-\mathrm{sym}} > -1 \\ h_2(-1), & \text{otherwise} \end{cases} \tag{30b}$$

And the equivalent correlation coefficient ρ_{0ij} is obtained as:

$$\rho_{0ij} = \frac{1}{4c_i c_j}\left[-(b_i b_j + 3b_i d_j + 3d_i b_j) + \sqrt{(b_i b_j + 3b_i d_j + 3d_i b_j)^2 + 8c_i c_j \rho_{ij}}\right] \tag{31}$$

Similarly, the solution of ρ_{0ij} for $d_i d_j = 0$ and $c_i c_j < 0$ can be determined, which is identical with Eq. (31), and the application bounds of ρ_{ij} are expressed as:

$$\rho_{ij-\max} = \begin{cases} h_2(\rho_{0ij-\mathrm{sym}}), & \rho_{0ij-\mathrm{sym}} < 1 \\ h_2(1), & \text{otherwise} \end{cases} \tag{32a}$$

$$\rho_{ij-\min} = h_2(-1) \tag{32b}$$

(5) If $d_i d_j = 0$ and $c_i c_j = 0$, Eq. (5*a*) reduces to a linear equation. The equivalent correlation coefficient ρ_{0ij} can then be determined as:

$$\rho_{0ij} = \frac{\rho_{ij}}{b_i b_j + 3b_i d_j + 3d_i b_j} \tag{33}$$

And the application bound of ρ_{ij} is expressed as:

$$\rho_{ij-\max} = b_i b_j + 3b_i d_j + 3d_i b_j \tag{34a}$$

$$\rho_{ij-\min} = -(b_i b_j + 3b_i d_j + 3d_i b_j) \tag{34b}$$

3. FORM for structural reliability analysis involving correlated random variables based on fourth-moment transformation

Using the proposed fourth-moment transformation described above, FORM for structural reliability analysis involving correlated random variables can be readily conducted. The computation procedure for FORM based on the fourth-moment normal transformation is described as follows:

(1) Obtain the first four moments of each random variable and original correlation matrix $\mathbf{C}_\mathbf{X}$ by the probability information (e.g., the joint PDF or the marginal PDFs and correlation matrix or statistical moments and correlation matrix);

(2) Determine the polynomial coefficients in Eq. (3) if the pairs of skewness and kurtosis lie in the suitable region given by Zhao et al. [25] and the equivalent correlation coefficients ρ_{0ij} using Eqs. (6*a*)-(6*c*) if ρ_{ij} lies in application region given in section 2.3. Then, determine the lower triangular matrix $\mathbf{L}_0$ and its inverse matrix $\mathbf{L}_0^{-1}$;

(3) Assume an initial checking point $\mathbf{x}_0$ (generally is the mean vector of $\mathbf{X}$);

(4) Obtain the corresponding checking point in the independent standard normal space, $\mathbf{u}_0$, using Eq. (16), and determine the initial reliability index β_0;

(5) Determine the Jacobian matrix $\mathbf{J} = \partial \mathbf{X} / \partial \mathbf{U}$ evaluated at $\mathbf{u}_0$, where the element of Jacobian matrix derived from Eq. (11) is given by

$$\frac{\partial X_i}{\partial U_j} = \sigma_{X_i} l_{ij} \left[b_i + 2c_i \sum_{k=1}^{i} l_{ik} U_k + 3d_i \left(\sum_{k=1}^{i} l_{ik} U_k \right)^2 \right] \quad (i, j = 1, 2, \ldots, n) \tag{35}$$

(6) Evaluate the performance function and gradient vector at $\mathbf{u}_0$:

$$g(\mathbf{u}_0) = G(\mathbf{x}_0), \quad \nabla g(\mathbf{u}_0) = \mathbf{J}^T \cdot \nabla G(\mathbf{x}_0) \tag{36}$$

where $\nabla g(\mathbf{u}_0)$ is the gradient vector of performance function $g(\mathbf{U})$ at $\mathbf{u}_0$; and $\nabla G(\mathbf{x}_0)$ is the gradient vector of performance function $G(\mathbf{X})$ at $\mathbf{x}_0$;

(7) Obtain new checking point $\mathbf{u}_1$ in independent standard normal space:

$$\mathbf{u}_1 = \frac{1}{\nabla^T g(\mathbf{u}_0)\cdot\nabla g(\mathbf{u}_0)}\left[\nabla^T g(\mathbf{u}_0)\cdot\mathbf{u}_0 - g(\mathbf{u}_0)\right]\nabla g(\mathbf{u}_0) \tag{37}$$

Then, the corresponding reliability index can be determined as $\beta = (\mathbf{u}_1^T \cdot \mathbf{u}_1)^{1/2}$; and

(8) Calculate the absolute difference between β and β_0, i.e., ε_r = abs (β–β_0). If $\varepsilon_r > \varepsilon$, where ε is the permissible error (generally $\varepsilon = 10^{-6}$), determine the new checking point in original space using the following equation:

$$\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{J}(\mathbf{u}_1 - \mathbf{u}_0) \tag{38}$$

Repeat the step 4 through step 8 using $\mathbf{x}_1$ as the new point until convergence is achieved.

Particularly, if the performance function $G(\mathbf{X})$ is explicit, substituting Eq. (11) into the performance function, it can be formulated as:

$$G(\mathbf{X}) = G(X_1, X_2, \cdots, X_n) = g(U_1, U_2, \cdots, U_n) = g(\mathbf{U}) \tag{39}$$

where $g(\mathbf{U})$ is an explicit function including only independent standard normal random variables. The reliability analysis can thus be readily conducted using the general FORM.

4. Numerical examples and investigations

To demonstrate the simplicity, efficiency and accuracy of the proposed method for structural reliability analysis involving correlated random variables, four numerical examples are investigated. The first example illustrates the computational procedure of the proposed method step by step, and the advantages of the proposed method over FORM based on third-moment transformation is demonstrated. In the second example, the proposed method having no shortcoming of varying with the transformation order of input random variables is numerically examined. The complexity of Nataf transformation for evaluating the equivalent correlation matrix for truncated distributions in standard normal space is illustrated in the third example. Finally, the application of the developed reliability analysis method for implicit performance function, which involves correlated random variables with known marginal PDFs and correlation matrix, is shown in the fourth example by a three-bay five-story frame structure with 21 random variables. Herein, the finite element method is utilized to conduct deterministic structural analyses and numerical differentiation method is used to determine the gradient vector of the performance function.

4.1 Example 1: Illustration of the computational procedure of the proposed method

The first example considers the following simple performance function:

$$G(\mathbf{X}) = R - S \tag{40}$$

where R is the resistance; S is the load; and R and S are log-normally distributed random variables with means and standard deviations of μ_R = 100, σ_R = 20, and μ_S = 50, σ_S = 20, respectively. The correlation coefficient between R and S is ρ = 0.8.

Because R and S are both lognormal distributed random variables, the joint PDF of R and S can be obtained from the marginal PDFs and correlation coefficient. The probability of failure for Eq. (40) can be obtained directly by numerical integral as P_f =1.5099×10^{-3} and the corresponding reliability index is 2.97, which is taken as the exact value. According to the probability distributions of R and S, their first four moments are readily obtained as μ_R = 100, σ_R = 20, α_{3R} = 0.608, α_{4R} = 3.664, μ_S = 50, σ_S = 20, α_{3S} = 1.265, and α_{4S} = 5.969, which lie in the suitable region given by Zhao et al. [25]. Using Eq. (55) presented in Appendix A, the polynomial coefficients are determined as $a_R = -c_R$ = –0.09740, b_R = 0.9706, d_R = 6.6705×10^{-3}, $a_S = -c_S$ = –0.1804, b_S = 0.8880, and d_S = 0.02681, respectively. The suitable region of original correlation coefficient is obtained as [-0.9253, 0.9956] using Eq. (22). According to Eq. (6*a*) or (23), the equivalent correlation coefficient, ρ_0, is determined as 0.8093. For the initial checking point $\mathbf{x} = (100, 50)^T$, the calculation results are summarized in Table 2, which shows that the reliability index obtained by FORM based on the proposed fourth-moment transformation is in close agreement with the exact one.

Table 2. Summary of the iteration computation of the proposed method for Example 1

Iteration	Checking point			Jacobian matrix	New checking point $\mathbf{u}_1$	Reliability index β
	$\mathbf{x}_0$	$\mathbf{z}_0$	$\mathbf{u}_0$			
1	$\begin{pmatrix}100\\50\end{pmatrix}$	$\begin{pmatrix}0.0993\\0.1952\end{pmatrix}$	$\begin{pmatrix}0.0993\\0.1955\end{pmatrix}$	$\begin{pmatrix}19.803 & 0\\15.563 & 11.285\end{pmatrix}$	$\begin{pmatrix}-1.5086\\4.0185\end{pmatrix}$	4.2923
2	$\begin{pmatrix}68.157\\68.157\end{pmatrix}$	$\begin{pmatrix}-1.8356\\0.9948\end{pmatrix}$	$\begin{pmatrix}-1.836\\4.2227\end{pmatrix}$	$\begin{pmatrix}13.610 & 0\\21.472 & 15.584\end{pmatrix}$	$\begin{pmatrix}1.3257\\2.6278\end{pmatrix}$	2.9433
3	$\begin{pmatrix}111.18\\111.18\end{pmatrix}$	$\begin{pmatrix}0.6342\\2.2608\end{pmatrix}$	$\begin{pmatrix}0.6342\\2.9751\end{pmatrix}$	$\begin{pmatrix}22.044 & 0\\34.232 & 24.845\end{pmatrix}$	$\begin{pmatrix}1.2994\\2.6488\end{pmatrix}$	2.9503
4	$\begin{pmatrix}125.84\\125.84\end{pmatrix}$	$\begin{pmatrix}1.2589\\2.5886\end{pmatrix}$	$\begin{pmatrix}1.2589\\2.6724\end{pmatrix}$	$\begin{pmatrix}24.951 & 0\\38.216 & 27.736\end{pmatrix}$	$\begin{pmatrix}1.2745\\2.6650\end{pmatrix}$	2.9541
5	$\begin{pmatrix}126.23\\126.23\end{pmatrix}$	$\begin{pmatrix}1.2745\\2.5968\end{pmatrix}$	$\begin{pmatrix}1.2745\\2.6650\end{pmatrix}$	$\begin{pmatrix}25.028 & 0\\38.320 & 27.812\end{pmatrix}$	$\begin{pmatrix}1.2738\\2.6653\end{pmatrix}$	2.9541

Since Eq. (40) is an explicit performance function, it can be transformed into a function of independent standard normal random variables by the fourth-moment transformation. According to Eq. (11), if the transformation order R to S is used, $G(\mathbf{X})$ can be formulated as:

$$\begin{aligned} G(\mathbf{X}) = g_1(\mathbf{U}) = 51.66 + 5.039U_1 - 0.4156U_1^2 - 0.1508U_1^3 - 10.43U_2 - 3.431U_1U_2 \\ - 0.6189U_1^2U_2 - 1.2450U_2^2 - 0.4492U_1U_2^2 - 0.1087U_2^3 \end{aligned} \tag{41}$$

For an explicit function with inclusion of random variables being independent standard normal random variables only, its reliability index can be obtained using the FORM. When initial checking point is selected to be $\mathbf{u} = (0, 0)^T$, the iteration procedure for Eq. (41) is summarized in Table 3.

If the transformation order S to R is used, the performance function can be formulated as:

$$\begin{aligned} G(\mathbf{X}) = g_2(\mathbf{U}) = 51.66 - 2.05U_1 - 2.33U_1^2 - 0.47U_1^3 + 11.40U_2 + 1.85U_1U_2 \\ + 0.15U_1^2U_2 + 0.67U_2^2 + 0.11U_1U_2^2 + 0.027U_2^3 \end{aligned} \tag{42}$$

Similarly, the reliability index can also be easily obtained using general FORM. The iteration procedure is also summarized in Table 3. The two limit state curves corresponding to Eqs. (41) and (42), and their corresponding design points and reliability indices are illustrated in Fig. 6.

Table 3. Iteration computation of explicit performance function for Example 1

Iteration	Eq. (41)			Eq. (42)		
	Checking point		Reliability index	Checking point		Reliability index
	u	**x**		**u**	**x**	
0	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}98.052\\46.392\end{pmatrix}$	0	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}98.052\\46.392\end{pmatrix}$	0
1	$\begin{pmatrix}-1.9397\\4.0153\end{pmatrix}$	$\begin{pmatrix}66.753\\62.905\end{pmatrix}$	4.4592	$\begin{pmatrix}0.7886\\-4.3889\end{pmatrix}$	$\begin{pmatrix}66.753\\62.905\end{pmatrix}$	4.4592
2	$\begin{pmatrix}1.2733\\2.8452\end{pmatrix}$	$\begin{pmatrix}126.204\\131.289\end{pmatrix}$	3.1172	$\begin{pmatrix}2.7018\\-1.5548\end{pmatrix}$	$\begin{pmatrix}126.204\\131.289\end{pmatrix}$	3.1172
3	$\begin{pmatrix}1.3397\\2.6348\end{pmatrix}$	$\begin{pmatrix}127.877\\127.907\end{pmatrix}$	2.9559	$\begin{pmatrix}2.6319\\-1.3455\end{pmatrix}$	$\begin{pmatrix}127.877\\127.907\end{pmatrix}$	2.9559
4	$\begin{pmatrix}1.2713\\2.6665\end{pmatrix}$	$\begin{pmatrix}126.154\\126.153\end{pmatrix}$	2.9541	$\begin{pmatrix}2.5951\\-1.4113\end{pmatrix}$	$\begin{pmatrix}126.154\\126.153\end{pmatrix}$	2.9541
5	$\begin{pmatrix}1.2740\\2.6665\end{pmatrix}$	$\begin{pmatrix}126.220\\126.220\end{pmatrix}$	2.9541	$\begin{pmatrix}2.5965\\-1.4087\end{pmatrix}$	$\begin{pmatrix}126.220\\126.220\end{pmatrix}$	2.9541

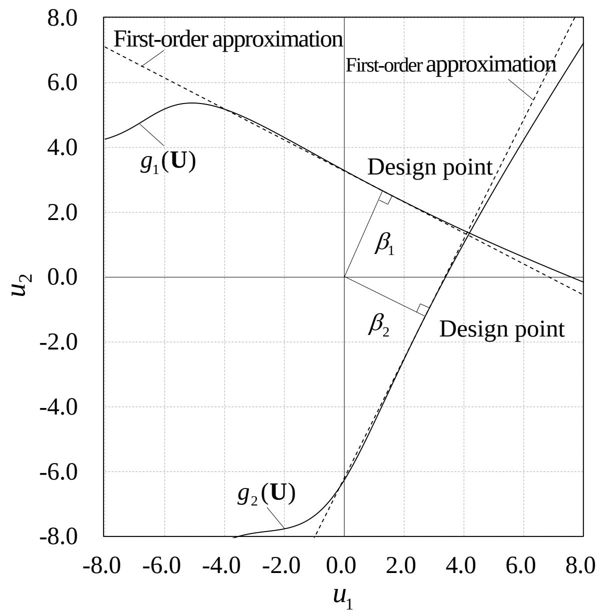


Fig. 6. Limit state curves in independent standard normal space for Example 1

Compare Table 2 with Table 3, it can be observed that the reliability index of explicit performance function obtained from the different computational procedures is identical. From Fig. 6 and Table 3, it can be observed that: (1) The first-order reliability indices obtained by FORM based on the fourth-moment transformation are the same even the transformation order is different; (2) Although the design points corresponding to the two limit state curves expressed in Eqs. (41) and (42) are different in independent standard normal space, they are identical in original space.

If R and S are assumed to be Gumbel distributed random variables, with means and coefficient of variations (COV) of μ_R, V_R = 0.2, and μ_S = 50, V_S = 0.4, respectively, and the correlation coefficient between R and S is ρ = 0.5. Reliability analysis based on Rosenblatt transformation cannot be used because the joint PDF of R and S are unavailable. While the reliability analysis can be realized using FORM based on the Nataf, fourth-moment, and third-moment transformations since the marginal PDFs and correlation matrix of the basic random variables are given. The corresponding first-order reliability indices varying with respect to the mean of R, μ_R, can be obtained and are illustrated in Fig. 7.

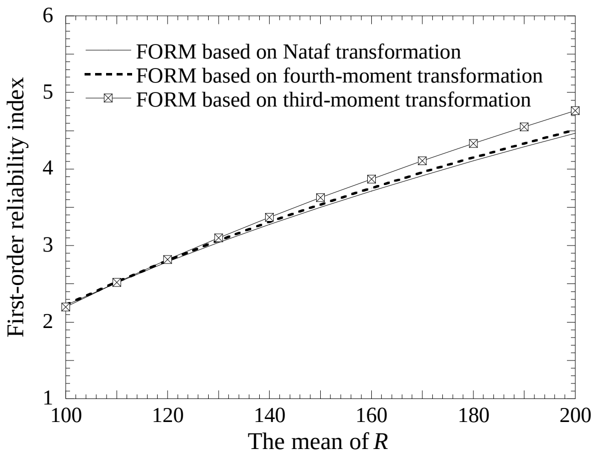


Fig. 7. Comparison of the first-order reliability indices by different methods for Example 1

From Fig. 7, it can be observed that: (1) FORM based on the fourth-moment transformation provides almost the same results with FORM based on Nataf transformation for the entire investigation range; (2) The differences between the reliability indices obtained from FORM based on third-moment transformation and those of FORM based on Nataf transformations become larger as the mean of R (μ_R) increases. The reason lies in that as the mean of R increases, the corresponding absolute value of U_R in the standard normal space becomes larger. Previous study [23] has shown that the third-moment transformation yields certain errors in the X-U and U-X transformations when the absolute value of X_s or U is relatively large. Since only the information of the first three moments is used, the underlying shortcoming of the third-moment transformation is that it cannot be flexible enough to reflect the kurtosis of a basic random variable or statistical data, which limits its applicable range and may lead to inappropriate results when the kurtosis is important and should be accounted for.

4.2 Example 2: Investigation of the influence of transformation order

The second example considers the following performance function:

$$G(\mathbf{X}) = X_1^2 - 2X_2 - X_3 \tag{43}$$

where X_1 is independent of X_2 and X_3; X_2 and X_3 are correlated random variables. X_1 is a lognormally distributed random variable with parameters λ = 1.590 and ζ = 0.198. X_2 and X_3 are bivariate exponentially distributed random variables, and their joint PDF is expressed as:

$$f_{X_2X_3}(x_2, x_3) = (x_2 + x_3 + x_2x_3)\exp[-(x_2 + x_3 + x_2x_3)]; \quad x_2, x_3 \geq 0 \tag{44}$$

Therefore, the joint PDF of **X** is given as:

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1) \cdot f_{X_2X_3}(x_2, x_3) \tag{45}$$

where $f_{X_1}(x_1)$ is the PDF of X_1.

Because the joint PDF of **X** and performance function are available, the probability of failure can be obtained directly by numerical integral as 4.928×10^{-4}, and the corresponding reliability index is 3.295, which is taken as the exact value. With the aid of the joint PDF of **X**, the reliability indices with different transformation orders can be obtained by FORM based on Rosenblatt transformation, which are listed in Table 4, together with the design points in original space and in independent standard normal space. Because the marginal PDFs, the first four moments, and correlation matrix of the basic random variables can be determined from the joint PDF, the reliability indices can also be obtained using FORM based on the Nataf and fourth-moment transformations. The corresponding design points and reliability indices are also listed in Table 4 for comparison.

Table 4. Comparison of results by FORM based on different transformations for Example 2

Transformation used in FORM	Transformation order	Design point		Reliability index
		$\mathbf{u}^T = \{u_1, u_2, u_3\}$	$\mathbf{x}^T = \{x_1, x_2, x_3\}$	
Rosenblatt transformation	$X_1 \to X_2 \to X_3$	{-1.973, 2.623, 0.069}	{3.319, 5.438, 0.136}	3.283
	$X_1 \to X_3 \to X_2$	{-1.920, -1.103, 2.277}	{3.353, 5.549, 0.145}	3.176
	$X_2 \to X_1 \to X_3$	{2.623, -1.973, 0.069}	{3.319, 5.438, 0.136}	3.283
	$X_2 \to X_3 \to X_1$	{2.623, 0.069, -1.973}	{3.319, 5.438, 0.136}	3.283
	$X_3 \to X_1 \to X_2$	{-1.103, -1.920, 2.277}	{3.353, 5.549, 0.145}	3.176
	$X_3 \to X_2 \to X_1$	{-1.103, 2.277, -1.920}	{3.353, 5.549, 0.145}	3.176
Nataf transformation	$X_1 \to X_2 \to X_3$	{-1.975, 2.631, 0.0571}	{3.316, 5.460, 0.0776}	3.291
	$X_1 \to X_3 \to X_2$	{-1.975, -1.442, 2.202}	{3.316, 5.460, 0.0776}	3.291
	$X_2 \to X_1 \to X_3$	{2.631, -1.975, 0.0571}	{3.316, 5.460, 0.0776}	3.291
	$X_2 \to X_3 \to X_1$	{-1.442, -1.975, 2.202}	{3.316, 5.460, 0.0776}	3.291
	$X_3 \to X_1 \to X_2$	{2.631, 0.0571, -1.975}	{3.316, 5.460, 0.0776}	3.291
	$X_3 \to X_2 \to X_1$	{-1.442, 2.202, -1.975}	{3.316, 5.460, 0.0776}	3.291
Fourth-moment transformation	$X_1 \to X_2 \to X_3$	{-1.970, 2.640, 0.006}	{3.319, 5.472, 0.0746}	3.293
	$X_1 \to X_3 \to X_2$	{-1.970, -1.550, 2.137}	{3.319, 5.472, 0.0746}	3.293
	$X_2 \to X_1 \to X_3$	{2.640, -1.970, 0.006}	{3.319, 5.472, 0.0746}	3.293
	$X_2 \to X_3 \to X_1$	{-1.550, -1.970, 2.137}	{3.319, 5.472, 0.0746}	3.293
	$X_3 \to X_1 \to X_2$	{2.640, 0.0006, -1.970}	{3.319, 5.472, 0.0746}	3.293
	$X_3 \to X_2 \to X_1$	{-1.550, 2.137, -1.970}	{3.319, 5.472, 0.0746}	3.293

Table 4 shows that: (1) The results of FORM based on Rosenblatt transformation are different due to the variation of the transformation order of input random variables, and not all of them give a good estimation to the exact value; (2) FORM based on the fourth-moment and Nataf transformations have no shortcoming of varying with the transformation order of random variables; and (3) The reliability indices obtained from FORM based on the fourth-moment and Nataf transformation are in close agreement with the exact value.

4.3 Example 3: A performance function with truncated probability distributions

The third example considers the following flexural limit state function in the midspan cross-section of a simple steel beam subjected to a uniformly distributed load:

$$G(\mathbf{X}) = W \cdot F_y - q \cdot L^2 / 8 \tag{46}$$

where W is the plastic modulus; F_y is the yield stress; q is the uniformly distributed load; and L is the beam length. The probabilistic information of random variables is listed in Table 5. With the exception of W and q is correlated, the others are mutually independent. The correlation coefficient between W and q is $\rho = 0.5$.

Table 5. Probabilistic information of the basic random variables for Example 3

Variable	Distribution	Mean	Standard deviation	Skewness	Kurtosis
W	Truncated normal [90, 110]	100.0 cm^3	4.3981 cm^3	0.0	2.3655
F_y	Truncated normal [350, 450]	400.0 Mpa	19.0919 Mpa	0.0	2.6242
q	Lognormal	5.0 kN/m	4.0 kN/m	2.912	21.125
L	Truncated normal [3.5, 4.5]	4.0 m	0.1909 m	0.0	2.6242

Because the marginal PDFs and correlation matrix are known, FORM based on Nataf transformation is theoretically available. However, the truncated normal distributions are not included in the empirical formulas of evaluating the equivalent correlation coefficient in correlated standard normal space developed by Der Kiureghian and Liu [20]. The determination of the equivalent correlation coefficient ρ_0 involves solving the following two-dimensional nonlinear integral equation:

$$\rho = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{F_W^{-1}\left[\Phi(z_1)\right] - \mu_W}{\sigma_W} \cdot \frac{F_q^{-1}\left[\Phi(z_2)\right] - \mu_q}{\sigma_q} \cdot \phi_2\left(z_1, z_2, \rho_0\right) dz_1 dz_2 \tag{47}$$

where μ_W, σ_W, and μ_q, σ_q are the mean and standard deviation of W and q, respectively; $F_W^{-1}(\cdot)$ and $F_q^{-1}(\cdot)$ are the inverse CDF of W and q, respectively; $\phi_2(z_1, z_2, \rho_0)$ is a bivariate normal PDF of Z_1 and Z_2; and $F_W(\cdot)$ is expressed as:

$$F_W(w)=\frac{1}{2\Phi(2)-1}\Phi\left(\frac{w-\mu_W}{\sigma_W}\right)-\frac{1-\Phi(2)}{2\Phi(2)-1} \tag{48}$$

Obviously, it is not easy to obtain the explicit analytical expression between ρ_0 and ρ because the solution of Eq. (47) is quite complex. On the other hand, if the proposed method used, the explicit analytical expression between ρ_0 and ρ can be expressed by Eqs. (6*a*)-(6*c*), and the reliability index can be easily obtained as 2.109. According to Eq. (11), the random samples of correlated random variables can be generated by the independent standard normal random variables, based on which the probability of failure obtained by Monte Carlo simulation (MCS) is 0.0173 (the coefficient of variation (COV) of the MCS result is 0.75%) with corresponding reliability index of β_{MCS} = 2.113. It can be concluded that the proposed method is simpler than FORM based on Nataf transformation for the cases of empirical formulas for equivalent correlation coefficients being not included.

4.4 Example 4: Applications to reliability problems with implicit performance functions

The fourth example considers a three-bay five-story frame structure subjected to lateral loads as shown in Fig. 8. The total height of the frame structural is H = 21.0 m. In this example, 21 random variables are considered. The random variables associated with the frame elements are shown in Table 6, and their probabilistic information is listed in Table 7. The implicit performance function is formulated as [29]:

$$G(\mathbf{X})=u_{\mathrm{lim}}-u(\mathbf{X}) \tag{49}$$

where u_{lim} = H/350 = 0.06 m; and $u(\mathbf{X})$ is the top-floor displacement, which is determined by finite element analysis. The correlation matrix of basic random variables is given as:

$$\mathbf{C}_{\mathbf{X}} = \begin{matrix} P_1 \\ P_2 \\ P_3 \\ E_1 \\ E_2 \\ I_1 \\ \vdots \\ A_8 \end{matrix} \begin{pmatrix} 1 & & & & & & & \\ 0.5 & 1 & & & & & & \\ 0.5 & 0.5 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0.9 & 1 & & & \\ 0 & 0 & 0 & 0 & 0.13 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0.13 & \cdots & 0.13 & 1 \end{pmatrix} \tag{50}$$

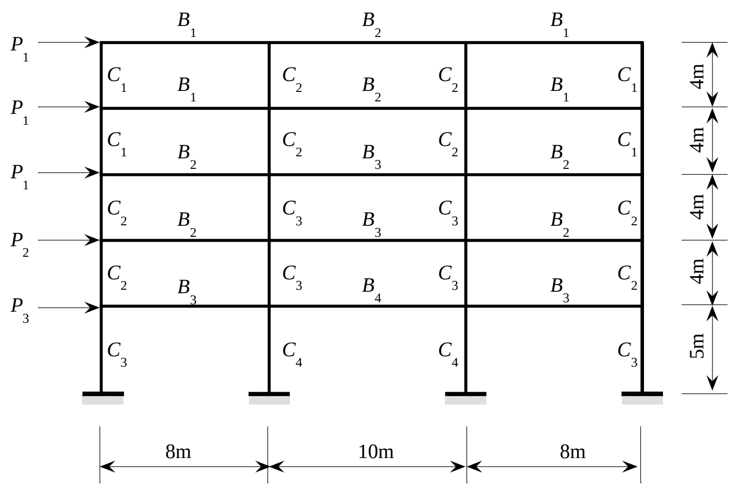


Fig. 8. Three-bay five-story frame structure considered in Example 4

Table 6. Characteristics of the frame elements in Example 4

Element	Young's modulus	Moment of inertia	Cross-sectional area
B_1	E_1	I_1	A_1
B_2	E_1	I_2	A_2
B_3	E_1	I_3	A_3
B_4	E_1	I_4	A_4
C_1	E_2	I_5	A_5
C_2	E_2	I_6	A_6
C_3	E_2	I_7	A_7
C_4	E_2	I_8	A_8

Because the joint PDF of random variables is not available, the reliability analysis based on Rosenblatt transformation can no longer be used. While the reliability analysis can be realized using FORM based on the proposed fourth-moment and Nataf transformations since the marginal PDFs, the first four moments, and correlation matrix of basic random variables are given. The gradient vector of the implicit performance function is determined by numerical differentiation method. The reliability indices obtained from FORM based on the proposed fourth-moment and Nataf transformations with respect to the COV of lateral loads V_P ranging from 0.1 to 1.5 are depicted in Fig. 9, together with results obtained from MCS. It can be observed from Fig. 9 that the results obtained by reliability analysis methods based on the proposed fourth-moment transformation are in close agreement with those based on Nataf transformation for the entire investigation range.

Table 7. Probabilistic information of the basic random variables for Example 4

Variable	Distribution	Mean	Standard deviation	Skewness	Kurtosis
P_1	Gumbel	135.0 kN	V_P×135.0 kN	1.134	5.40
P_2	Gumbel	90.0 kN	V_P×90.0 kN	1.134	5.40
P_3	Gumbel	70.0 kN	V_P×70.0 kN	1.134	5.40
E_1	Lognormal	2.1×10^{7} kN/m^2	1.05×10^{6} kN/m^2	0.150	3.040
E_2	Lognormal	2.4×10^{7} kN/m^2	2.1×10^{6} kN/m^2	0.150	3.040
I_1	Lognormal	8.11×10^{-3} m^4	8.11×10^{-4} m^4	0.301	3.162
I_2	Lognormal	0.011 m^4	1.1×10^{-3} m^4	0.301	3.162
I_3	Lognormal	0.0213 m^4	2.13×10^{-3} m^4	0.301	3.162
I_4	Lognormal	0.0295 m^4	2.95×10^{-3} m^4	0.301	3.162
I_5	Lognormal	0.0108 m^4	1.08×10^{-3} m^4	0.301	3.162
I_6	Lognormal	0.0141 m^4	1.41×10^{-3} m^4	0.301	3.162
I_7	Lognormal	0.0232 m^4	2.32×10^{-3} m^4	0.301	3.162
I_8	Lognormal	0.0259 m^4	2.59×10^{-3} m^4	0.301	3.162
A_1	Lognormal	0.312 m^2	0.0312 m^2	0.301	3.162
A_2	Lognormal	0.372 m^2	0.0372 m^2	0.301	3.162
A_3	Lognormal	0.505 m^2	0.0505 m^2	0.301	3.162
A_4	Lognormal	0.557 m^2	0.0557 m^2	0.301	3.162
A_5	Lognormal	0.253 m^2	0.0253 m^2	0.301	3.162
A_6	Lognormal	0.291 m^2	0.0291 m^2	0.301	3.162
A_7	Lognormal	0.372 m^2	0.0372 m^2	0.301	3.162
A_8	Lognormal	0.418 m^2	0.0418 m^2	0.301	3.162

Note: V_p denotes the COV of lateral loads, varying from 0.1 to 1.5.

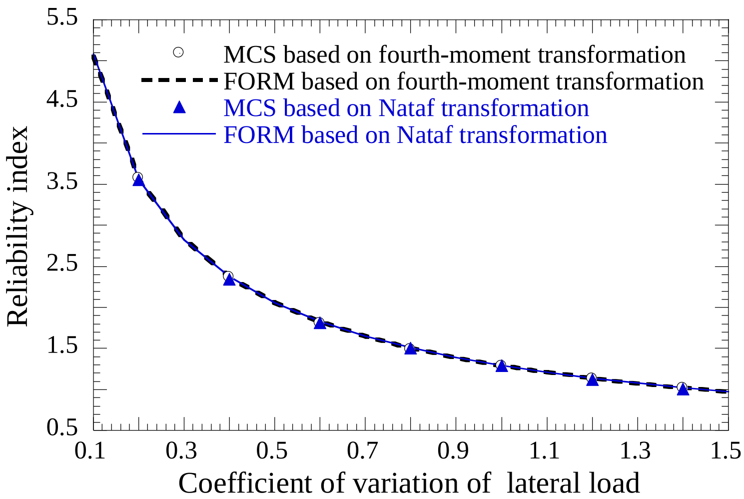


Fig. 9. Comparison of results by different methods for Example 4

5. Concluding remarks

The present paper propose a fourth-moment transformation technique for transforming the correlated nonnormal random variables into independent standard normal ones, in which the correlated nonnormal random variables are firstly transformed into correlated standard normal random ones using the fourth-moment transformation and the complete explicit expressions of equivalent correlation coefficients are proposed; and the correlated standard normal random variables are then transformed into independent standard normal random ones using Cholesky decomposition. The application scope of the proposed fourth-moment transformation is investigated, in which the upper and lower bounds of original correlation coefficient are given.

Based on the proposed fourth-moment transformation, a FORM for structural reliability analysis involving correlated random variables is developed. Four numerical examples including explicit and implicit performance functions are studied, from which it can be concluded that: (1) Since the proposed fourth-moment transformation efficiently utilizes the kurtosis as well as the mean, standard deviation, and skewness of the basic random variable, it generally provides more accurate results than the third-moment transformation; (2) The proposed fourth-moment transformation has no shortcoming of varying with the transformation order of random variables, which is considered to be the main weakness of Rosenblatt transformation; (3) In general, the proposed method provides almost the same results compared with the FORM based on Nataf transformation, and the proposed method is simpler for the cases of empirical formulas for equivalent correlation coefficients being not included in Nataf transformation; and (4) Since only the first four moments and correlation matrix of input random variables are used in the proposed method, it can be applied for structural reliability assessment even if probability distributions of the basic random variables are unknown.

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Appendix A: Moment-matching method for determining the polynomial coefficients

The polynomial coefficient a_i, b_i, c_i, and d_i in Eq. (3) can be obtained by making the first four moments of left side of Eq. (3) equal to those of the right side, i.e.,

$$a_i + c_i = 0 \tag{51a}$$

$$b_i^2 + 2c_i^2 + 6b_i d_i + 15d_i^2 = 1 \tag{51b}$$

$$6b_i^2 c_i + 8c_i^3 + 72b_i c_i d_i + 270c_i d_i^2 = \alpha_{3X_i} \tag{51c}$$

$$3(b_i^4 + 20b_i^3 d_i + 210b_i^2 d_i^2 + 1260b_i d_i^3 + 3465d_i^4) + 12c_i^2(5b_i^2 + 5c_i^2 + 78b_i d_i + 375d_i^2) = \alpha_{4X_i} \tag{51d}$$

Simplifying Eqs. (51*a*)-(51*d*), the equations about b_i and d_i can be obtained as following:

$$2A_1 A_2^2 = \alpha_{3X_i}^2 \tag{52a}$$

$$3A_1 A_3 + 3A_4 = \alpha_{4X_i} \tag{52b}$$

in which,

$$A_1 = 1 - b_i^2 - 6b_i d_i - 15d_i^2, \quad A_2 = 2 + b_i^2 + 24b_i d_i + 105d_i^2 \tag{53a}$$

$$A_3 = 5 + 5b_i^2 + 126b_i d_i + 675d_i^2, \quad A_4 = b_i^4 + 20b_i^3 d_i + 210b_i^2 d_i^2 + 1260b_i d_i^3 + 3465d_i^4 \tag{53b}$$

After the coefficients b_i and d_i have been obtained, a_i and c_i can be readily given as:

$$c_i = -a_i = \frac{\alpha_{3X_i}}{2A_2} \tag{54}$$

When $|\alpha_{3X_i}| \le 2$ and $(4\alpha_{3X_i}^2 + 7)/3 \le \alpha_{4X_i} \le 12$, the explicit expressions of the four coefficients suggested by Zhao and Lu [23] can be used, which are expressed as:

$$a_i = -c_i = -\frac{\alpha_{3X_i}}{6(1+6k_i)}, \quad b_i = \frac{1-3k_i}{1+a_i^2-k_i^2}, \quad d_i = \frac{k_i}{1+a_i^2+12k_i^2} \tag{55a}$$

$$k_i = \frac{1}{36}\left(\sqrt{6\alpha_{4X_i} - 8\alpha_{3X_i}^2 - 14} - 2\right) \tag{55b}$$

Appendix B: The solution of cubic equation and its property

According to Cardano formula [30], for a cubic equation $x^3+t_2x^2+t_1x+t_0 = 0$ with one variable x, its number of real roots can be discriminate using Δ, which is expressed as:

$$\Delta = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \quad p = t_1 - \frac{t_2^2}{3}, \quad q = \frac{2t_2^3}{27} - \frac{t_1 t_2}{3} + t_0 \tag{56}$$

When $\Delta > 0$, there is only one real root, which is expressed as:

$$x = \sqrt[3]{A} + \sqrt[3]{B} - \frac{t_2}{3}, \quad A = -\frac{q}{2} + \sqrt{\Delta}, \quad B = -\frac{q}{2} - \sqrt{\Delta} \tag{57}$$

When $\Delta = 0$, there are two real roots, one of which is horizontal coordinate of the stationary point of cubic function $f(x) = x^3+t_2x^2+t_1x$. The real toots for $\Delta = 0$ are expressed as:

$$x_1 = \sqrt[3]{A} + \sqrt[3]{B} - \frac{t_2}{3} = 2\left(-\frac{q}{2}\right)^{1/3} - \frac{t_2}{3}, \quad x_2 = \left(\frac{q}{2}\right)^{1/3} - \frac{t_2}{3} \tag{58}$$

In which x_2 is the horizontal coordinate of stationary point; and $x_1 \ge x_2$ for $q < 0$ and $x_1 < x_2$ for $q \ge 0$.

When $\Delta < 0$, there exist three real roots, which are respectively expressed as:

$$x_1 = 2r\cos\left(\frac{\theta}{3}\right) - \frac{t_2}{3},\quad x_2 = -2r\cos\left(\frac{\theta+\pi}{3}\right) - \frac{t_2}{3},\quad x_3 = -2r\cos\left(\frac{\theta-\pi}{3}\right) - \frac{t_2}{3} \qquad (59)$$

where $r = \sqrt{-p/3}$, $\theta = \arccos[-q/(2r^3)]$, According to the property of cosine function, there exists $x_1 \geq x_2 \geq x_3$ because $0 \leq \theta = \arccos[-q/(2r^3)] \leq \pi$.

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