# Classification of topological relations between spatial objects in two-dimensional space within the dimensionally extended 9-intersection model

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**Keywords:** topological relations; spatial object; negative conditions; dimensionally extended 9-intersection model (DE-9IM)

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## Abstract

As an important topological relation model, the dimensionally extended 9-intersection model (DE-9IM) has been widely used as a basis for standards of queries in spatial databases. However, the negative conditions for the specification of the topological relations within the DE-9IM has not been studied. The specification of the topological relations is closely related to the definition of the spatial objects and the topological relation models. The interior, boundary and exterior of the spatial objects, including the point, line and region, are defined. Within the framework of the DE-9IM, 43 negative conditions are proposed to eliminate impossible topological relations. Configurations of region/region, region/line, line/line, region/point, line/point, and point/point relations are drawn. The mutual exclusion of the negative conditions is discussed, and the topological relations within the framework of 9IM and DE-9IM are compared. The results show that (1) impossible topological relations between spatial objects can be eliminated by the application of 43 negative conditions and that (2) 12 relations between two regions, 31 relations between a region and a line, 47 relations between two lines, 3 relations between a region and a point, 3 relations between a line and a point, and 2 relations between two points can be distinguished by the DE-9IM.

## 1 INTRODUCTION

As important spatial relations, topological relations remain constant when the coordinate space is deformed, such as by twisting or stretching. Topological relations between spatial objects have always been a main research area in spatial data handling, reasoning, and querying (Clementini, Sharma, & Egenhofer, 1994; Du, Qin, Wang, & Ma, 2008; Borrmann & Rank, 2009; Praing & Schneider, 2009). For research on topological relations, an important issue is to identify the topological relations that exist in reality. Clarification of the relations between spatial objects has been widely carried out (Egenhofer & Franzosa, 1991; Egenhofer, Sharma, & Mark, 1993; Schneider & Behr, 2006). Within a given framework, the dimension of the objects, the dimension of the space and the type of boundary play important roles in the specification of the topological relations (Zlatanova, 2000a). Therefore, the related studies categorizing the topological relations that can occur in reality within a given framework, the definition of the spatial object, the topological relation models, and the methods to determine the number and specification of the topological relations between spatial objects are reviewed as follows.

Numerous spatial data models have been proposed to describe spatial objects, such as the simplicial complex (Carlson, 1987), three-dimensional formal data structure (Molenaar, 1990), tetrahedron network (Pilouk, Tempfli, & Molenaar, 1994; Pilouk, 1996), cell complex (Pigot, 1991; Brisson, 1993), geometry object model (Open GIS Consortium, 1999), simplified spatial model (Zlatanova, 2000b), multi-level model (Ramos, 2002), combinatorial data model (Lee & Kwan, 2005), common information model (Open Geospatial Consortium, 2012) and geometric algebra model (Yuan, Yu, Luo, Yi, & Lv, 2014; Yu, Luo, Yuan, Hu, Zhu, & Lv, 2016). Although there are differences in the mode of organization, spatial data objects such as points, lines, and regions, are the central concept of these spatial data models.

Before categorizing the topological relations, the formalism of the topological relations between objects must be defined. The formalization of topological relations among spatial objects (such as interval-based temporal logic (Allen, 1983; Masunaga, 1998), point-set topology (Chen, Li, Li, & Gold, 2001; Egenhofer & Franzosa, 1991; Egenhofer, Sharma, & Mark, 1993) and region connection calculus (RCC) (Randell & Cohn, 1989; Randell, Cui, & Cohn, 1992; Cohn, Randell, & Cui, 1995; Gotts, Gooday, & Cohn, 1996; Cohn, Bennett, Gooday, & Gotts, 1997; Jonsson & Drakengren, 1997)) has gained increasing attention in the last few decades. Research on formalizing topological relations based on point-set topology has been conducted, and the 4-intersection model (4IM) (Egenhofer & Franzosa, 1991), 9-intersection model (9IM) (Egenhofer & Herring, 1991), Voronoi-based 9-intersection model (Chen, Li, Li, & Gold, 2001), dimensionally extended 9-intersection model (DE-9IM) (Open GIS Consortium, 1999; Clementini, Felice, & Oosterom, 1993), intersection and difference model (Deng, Cheng, Chen, & Li, 2007), extended model expressed as 4 × 4 matrices (Liu & Shi, 2007), 9+-intersection model (Kurata, 2008), uncertain intersection and difference model (Alboody, Sedes, & Inglada, 2010), double straight line 4-intersection model (Leng, Yang, & Chen, 2017), and 27-intersection model (Shen, Zhou, & Chen, 2017) have been proposed. The intersection models and the extended models based on these have been widely studied.

Related to topological relations, infinite topological relations may exist in reality when considering some topological invariants, but the number of topological relations within some frameworks is finite. It is inevitable that many invalid spatial topological relations will appear in the above intersection models. To obtain the possible relations, some negative conditions have been proposed within the framework of the 9IM. A total of 23 conditions have been proposed to exclude the impossible relations for the spatial objects embedded in the two-dimensional space (Egenhofer & Herring, 1991). The set of 25 negative conditions were presented to derive all the possible relations for the 3D topological relations (Zlatanova, 2000a). For two-dimensional spatial objects, Egenhofer and Herring (1991) concluded that 33 relations between two simple lines, 19 relations between a simple line and a region, and 8 relations between two simple regions can be distinguished by the 9-intersection model. Schneider M and Behr (2006) concluded that 5 relations between two complex points, 14 relations between a complex point and a complex line, 7 relations between a complex point and a complex region, 82 relations between two complex lines, 43 relations between a complex line and a complex region, and 33 relations between two complex regions can be distinguished by the 9IM. For three-dimensional spatial objects, Egenhofer (1995), Zlatanova (2000a), and Kurata (2008, 2010) investigated the number of topological relations between spatial objects (points, lines, surfaces and bodies). In summary, these works have categorized the topological relations within the framework of the 9IM. Clemintini et al. (1993) described impossible cases for topological relations based on the dimension extended method. However, many details, such as the topological relations between two lines or two regions need further study. Although five topological relations (including touch, in, cross, overlap and disjoints) were taken into account, some topological relations were usually regarded as different. For example, two regions sharing a common point and two regions sharing a common line are usually identified as two different topological relations. Since detailed topological relation queries are usually needed in spatial database and spatial cognition, it is necessary to study the complete classification of the topological relations.

Compared to the 9IM that considers the content invariant (i.e., the emptiness or non-emptiness) of the intersections, the DE-9IM considers the dimension of the intersections as the refinement of the 9IM. As an important topological relation model, the DE-9IM has been used as a basis for standards of queries in spatial databases (e.g., PostGIS and SQL Server). Although much research has been performed to categorize topological relations based on the 9IM, further work is required to consider how many and what kinds of topological relations between spatial objects can be realized in reality within the framework of DE-9IM embedded in the two-dimensional Euclidean space. Therefore, the aim of this paper is to categorize the topological relations with the DE-9IM in the two-dimensional Euclidean space. To categorize the topological relations, two aspects are considered: (1) negative conditions and (2) sketches of possible configurations. The novelty of this contribution is the complete classification of the topological relations within the framework of the DE-9IM. The utility of this contribution lies in the detailed queries that can be used in spatial databases.

The remainder of this paper is outlined as follows. Section 2 introduces the conditions for non-existing topological relations. Section 3 describes the geometric interpretation of the topological relations between spatial objects. Section 4 discusses the mutual exclusion of the negative conditions and a comparison of the topological relations between 9-intersections and dimensionally extended 9-intersections. Section 5 provides the conclusions and discusses future work.

## 2 METHODS

Based on the DE-9IM, not all the topological relations between spatial objects can be realized in a particular space. The aim of this section is to introduce the processes and methods to identify the topological relations that may be realized.

### 2.1 The processes of identifying existing topological relations

Determining topological relations and their numbers may be possible in reality if the following steps are taken:

(1) All the topological relations between spatial objects based on the DE-9IM are listed and assumed to be possible. This is an effective way to find a finite number of topological relations within a given topological relation model.

(2) The negative conditions for the impossible topological relations are given by the pattern of the DE-9IM. The non-existent topological relations are excluded according to these negative conditions. Only the possible topological relations are reserved for further analysis.

(3) The value of the DE-9IM and the corresponding geometric interpretations are illustrated to verify the existence of these topological relations. Sketches of possible configurations can be drawn under the condition that the topological relations can be realized.

## 2.2 Definition of spatial objects in two-dimensional space

Spatial objects are the basis of topological relations. Topological relations may have significant differences based on different spatial objects. Therefore, spatial objects must be defined first to study the topological relations. Points, lines, and regions are defined as follows.

A point is a 0-dimensional geometry and represents a single location in the coordinate space (Figure 1(a)). The interior of a point is a point, the boundary of a point is an empty set, and the exterior of a point includes everything that is not the point.

A line is a 1-dimensional geometric object usually stored as a sequence of points with linear interpolation between the points (Figure 1(b)). A line never passes through the same point twice. The interior of a line is a line without its endpoints, the boundary of a line is its two endpoints, and the exterior of a line includes everything that does not contain the line.

A region is a 2-dimensional geometric object defined by one exterior boundary (Figure 1(c)). The interior of a region is the region without its boundary, the boundary of a region is the lines forming the boundary, and the exterior of a region includes everything that does not belong to the region.

According to the definition of spatial objects, there are six groups of topological relations. These groups are region/region, region/line, line/line, region/point, line/point, and point/point relations, and they will be described next.

## 2.3 DE-9IM for topological relations

Let spatial object *A* and spatial object *B* be the subsets of space *X*. The DE-9IM between *A* and *B* describes a topological relation with a $3 \times 3$ matrix and is represented by the following formula (Clementini, Felice, & Oosterom, 1993):

$$R_{DE-9IM}(A,B)=\begin{bmatrix} \dim(A^o \cap B^o) & \dim(A^o \cap \partial B) & \dim(A^o \cap B^-) \\ \dim(\partial A \cap B^o) & \dim(\partial A \cap \partial B) & \dim(\partial A \cap B^-) \\ \dim(A^- \cap B^o) & \dim(A^- \cap \partial B) & \dim(A^- \cap B^-) \end{bmatrix}.$$

where *A*'s interior, boundary, and exterior are respectively denoted by $A^o$, $\partial A$ and $A^-$, while *B*'s interior, boundary, and exterior are respectively denoted by $B^o$, $\partial B$ and $B^-$. The DE-9IM can also be represented as a nine-tuple (i.e., $\dim(A^o \cap B^o)$, $\dim(A^o \cap \partial B)$, $\dim(A^o \cap B^-)$, $\dim(\partial A \cap B^o)$, $\dim(\partial A \cap \partial B)$, $\dim(\partial A \cap B^-)$, $\dim(A^- \cap B^o)$, $\dim(A^- \cap \partial B)$ and $\dim(A^- \cap B^-)$). The dimension invariant is used to distinguish the dimension of the intersection and is not higher than the lowest dimension of the subset of the region. Therefore, the largest value of the dimension of the intersection between regions is 2. The value of the intersection in the DE-9IM can be drawn from {-1, 0, 1, 2}, where -1 indicates that the intersection is null, and 0, 1, and 2 represent that the dimension of intersections is 0-dimensional, 1-dimensional, and 2-dimensional, respectively. The dimension of a non-empty intersection is taken to be the maximum of the dimension of the intersections. For example, if the intersections contain both points and lines, the dimension of the intersection is set to 1. Considering the dimensions of the intersections, 262,144 ($262{,}144=4^9$) topological relations can be distinguished by the DE-9IM in theory. However, not all the 262,144 topological relations can be implemented, and some topological relations have no practical significance. Some methods should be presented to exclude the impossible topological relations.

## 2.4 Negative conditions for eliminating impossible topological relations

A total of 43 negative conditions are proposed in this research and are applied to eliminate the impossible topological relations. One negative condition may contain a variety of situations, each of which is referenced by a number and a character. Let *A* and *B* be arbitrary non-empty spatial objects. Let *a* and *b* be arbitrary parts of *A* and *B*, respectively. The value of the intersection that can take an arbitrary value will be marked by a "*".

### 2.4.1 Negative conditions for two regions

Condition 1: For region *A* and region *B*, the dimension of the intersection between *A*'s interior, boundary or exterior and *B*'s boundary is no greater than 1, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 2 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 2 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 2 & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 2 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 2 \\ * & * & * \end{bmatrix} \quad \text{(1a-e)}$$

Proof 1: The intersection between a set *a* and a set *b* is both a subset of *a* and also a subset of *b*. Therefore, their intersection is no greater than the smallest dimension of the sets.

Condition 2: For two regions, the only possible value for *A*'s interior and *B*'s interior is drawn from {-1, 2}.

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} 1 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \quad \text{(2a-b)}$$

Proof 2: If *A*'s interior intersects with *B*'s interior, the dimension of the intersection is drawn from {0, 1, 2} according to Condition 1. If a point $x$ belongs to the interior of the two regions, the neighbourhood of $x$ must be in the interior of both regions. Therefore, the intersection of the two regions can only be a region when the intersection is not null. Likewise, Condition 3 can be proven.

Condition 3: For two regions, the only possible value for *A*'s interior and *B*'s exterior is drawn from {-1, 2}, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 1 & * & * \end{bmatrix} \quad \text{(3a-d)}$$

Condition 4: For two regions, it is impossible for the dimension of the intersection between *A*'s interior and *B*'s boundary to equal 0, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{bmatrix} \quad \text{(4a-b)}$$

Proof 3: If *A*'s interior intersects with *B*'s boundary, the dimension of the intersection can only be drawn from {0, 1} according to Condition 1. If a point $x$ belongs to *A*'s interior and *B*'s boundary, then the neighbourhood of $x$ must be in the interior of a region. Moreover, a subset of *B*'s boundary must be in the neighbourhood of $x$. The dimension of the subset of *B*'s boundary is 1. Likewise, Condition 5 can be proven.

Condition 5: For two regions, it is impossible for the dimension of the intersection between *A*'s exterior and *B*'s boundary to equal 0, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & * \\ * & * & * \\ * & 0 & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & * \end{bmatrix} \qquad \text{(5a-b)}$$

Condition 6: For two regions, the only possible value for *A*'s exterior and *B*'s exterior is drawn from {2}, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & -1 \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 1 \end{bmatrix} \qquad \text{(6a-c)}$$

Likewise, Condition 6 is also valid for the topological relations between a region and a line, two lines, a region and a point, a line and a point, and two points.

Proof 4: Because $(A^o \cup \partial A) \cup A^- = X$, $(B^o \cup \partial B) \cup B^- = X$, $A \subset X$, $B \subset X$, and $(A \cup B) \subset X$, *A*'s exterior intersects with *B*'s exterior. The dimension of the intersection between *A*'s exterior and *B*'s exterior is 2.

Condition 7: For two regions, if the interiors of two regions coincide, then the only possible value for *A*'s boundary and *B*'s boundary is drawn from {1}, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 2 & -1 & -1 \\ -1 & -1 & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} 2 & -1 & -1 \\ * & 2 & * \\ * & * & * \end{bmatrix} \qquad \text{(7a-c)}$$

Proof 5: If *A*'s interior and *B*'s interior coincide, then *A* must be equal to *B*. Since the dimension of *A*'s boundary or *B*'s boundary is 1, the only possible value for *A*'s boundary and *B*'s boundary is drawn from {1}.

Condition 8: For two regions, if the interiors of two regions are disjoint, the only possible value for *A*'s interior and *B*'s exterior is drawn from {2}, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & * & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & 1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & * \\ 1 & * & * \end{bmatrix} \quad (8a\text{-}f)$$

Proof 6: If *A*'s interior and *B*'s interior are disjoint, then *A*'s interior must be the subset of *B*'s boundary and exterior. Because the dimension of *A*'s interior is 2 and the dimension of *B*'s boundary is 1, it is impossible that *A*'s interior is the subset of *B*'s boundary. Therefore, *A*'s interior intersects with *B*'s exterior.

Condition 9: For two regions, if *A*'s interior and *B*'s exterior are disjoint, then *A*'s boundary and *B*'s exterior must be disjoint, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & -1 \\ * & * & 1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ -1 & 1 & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & 0 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ -1 & 0 & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & 2 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ -1 & 2 & * \end{bmatrix} \quad (9a\text{-}f)$$

Likewise, Condition 9 is also valid for the topological relations between two lines.

Proof 7: If *A*'s interior and *B*'s exterior are disjoint, then *A*'s closure must be the subset of *B*'s closure. Since *A*'s boundary must be the subset of *B*'s closure, *A*'s boundary does not intersect with *B*'s exterior.

Condition 10: For two regions, if *A*'s interior intersects with *B*'s boundary and the only possible value of the intersection is 1, then *A*'s interior must intersect with *B*'s exterior, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 0 & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 0 & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & 1 & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 1 & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & 2 & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 2 & * & * \\ -1 & * & * \end{bmatrix} \quad (10a\text{-}f)$$

Proof 8: If a point $x$ belongs to *A*'s interior and *B*'s boundary, then the neighbourhood of $x$ must be in *A*'s interior. Moreover, the neighbourhood of $x$ must intersect with *B*'s exterior. Therefore, *A*'s interior intersects with *B*'s exterior in the condition of *A*'s interior intersecting with *B*'s boundary. Likewise, Condition 11 can be proven.

Condition 11: For two regions, if *A*'s interior does not intersect with *B*'s interior, then *A*'s interior and *B*'s boundary are disjoint, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ 0 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ 1 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & 2 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ 2 & * & * \\ * & * & * \end{bmatrix} \quad \text{(11a-f)}$$

Condition 12: For two regions, the interior, boundary or exterior of *B* must intersect with at least one of the three parts of *A*, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & * & * \\ -1 & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & -1 & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & -1 & * \\ * & -1 & * \\ * & -1 & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ -1 & -1 & -1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & -1 \\ * & * & -1 \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ -1 & -1 & -1 \end{bmatrix} \quad \text{(12a-f)}$$

Likewise, Condition 12 is also valid for the topological relations between a region and a line and between two lines.

Proof 9: Because $(A^o \cup \partial A) \cup A^- = X$ and $B^o \subset X, \partial B \subset X, B^- \subset X$, *B*'s interior, boundary or exterior intersects with at least one part of *A*.

Condition 13: For two regions, if *A*'s interior intersects with *B*'s interior and exterior, then *A*'s interior must intersect with *B*'s boundary, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 2 & * & * \\ -1 & * & * \\ 2 & * & * \end{bmatrix} \vee \begin{bmatrix} 2 & -1 & 2 \\ * & * & * \\ * & * & * \end{bmatrix} \quad \text{(13a-b)}$$

Proof 10: The only possible value for the interiors of two regions or the interior of one region and the exterior of another region is equal to 2 when the intersection is not null according to Conditions 2 and 3. For the two regions, since every path from the interior to the exterior crosses the boundary, *A*'s interior must intersect with *B*'s boundary.

Condition 14: For two regions, if *A*'s boundary and *B*'s boundary are disjoint, it is impossible that *A*'s boundary and *B*'s exterior are disjoint and *A*'s exterior and *B*'s boundary are disjoint at the same time, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & * \\ * & -1 & -1 \\ * & -1 & * \end{bmatrix} \tag{14a-a}$$

Proof 11: If *A*'s boundary and *B*'s boundary are disjoint, it implies that the boundaries of the two regions do not coincide. Therefore, at least one boundary intersects with the other region's exterior.

Condition 15: For two regions, if *A*'s interior intersects with *B*'s boundary and *A*'s boundary intersects with *B*'s interior, then *A*'s boundary must intersect with *B*'s boundary, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 1 & * \\ 1 & -1 & * \\ * & * & * \end{bmatrix} \tag{15a-a}$$

Proof 12: The only possible value for the intersection of *A*'s interior and *B*'s boundary and that of *A*'s boundary and *B*'s interior is 1 when the intersections are not null according to Condition 1 and Condition 4. If *A*'s interior intersects with *B*'s boundary and *A*'s boundary intersects with *B*'s interior, then *A*'s boundary must intersect with *B*'s interior and exterior. Since every path from the interior to the exterior crosses the boundary, *A*'s boundary must intersect with *B*'s boundary. Likewise, Conditions 16 and 17 can be proven.

Condition 16: For two regions, if *A*'s interior intersects with *B*'s exterior, then *A*'s boundary must also intersect with *B*'s exterior, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & 2 \\ * & * & -1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 2 & -1 & * \end{bmatrix} \tag{16a-b}$$

Condition 17: For two regions, if *A*'s interior and *B*'s interior are disjoint, then *A*'s exterior must intersect with *B*'s boundary, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & * & * \\ * & * & * \\ * & -1 & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & -1 \\ * & * & * \end{bmatrix} \quad (17a\text{-}b)$$

### 2.4.2 Negative conditions for a region and a line

Condition 18: For region *A* and line *B*, the dimension of the intersection between *A*'s interior, boundary, or exterior and *B*'s interior is no greater than 1, and the dimension of the intersection between *A*'s interior, boundary, or exterior and *B*'s boundary is no greater than 0, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 2 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 2 & * & * \end{bmatrix} \vee \begin{bmatrix} * & 2 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 2 & * \\ * & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 2 & * \end{bmatrix} \vee \begin{bmatrix} * & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 1 & * \end{bmatrix} \quad (18a\text{-}i)$$

Likewise, Condition 18 can be proven according to Proof 1.

Condition 19: For region *A* and line *B*, the only possible value for *A*'s interior and *B*'s interior is drawn from {-1, 1}, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} 2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \quad (19a\text{-}b)$$

Likewise, Condition 19 can be proven according to Proof 2.

Condition 20: For region *A* and line *B*, if *A*'s interior does not intersect with *B*'s interior, then *A*'s interior and *B*'s boundary are disjoint, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & 2 & * \\ * & * & * \\ * & * & * \end{bmatrix} \quad (20a\text{-}c)$$

Likewise, Condition 20 can be proven according to Proof 8.

Condition 21: For region *A* and line *B*, region *A* and point *B*, and line *A* and point *B*, the interior of *A* always intersects with the exterior of *B*, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \quad (21a\text{-}a)$$

Proof 13: If the dimension of *B* is lower than the dimension of *A*, then it is impossible that *B* completely contains *A*. Therefore, the interior of *A* always intersects with the exterior of *B* when the dimension of *A* is higher than the dimension of *B*.

Condition 22: For region *A* and line *B*, the only possible value of *A*'s boundary and *B*'s exterior is drawn from {1}, i.e.,

$$R_{9IM}(A,B) \neq \begin{bmatrix} * & * & * \\ * & * & -1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 2 \\ * & * & * \end{bmatrix} \qquad \text{(22a-c)}$$

Proof 14: For region *A* and line *B*, the boundary of *A* and the exterior of *B* are disjoint when *A*'s boundary is inside of *B*. Since the boundary of *A* is closed and the self-intersection is not allowed for line *B*, *A*'s boundary cannot be completely inside of *B*. Therefore, *B*'s exterior intersects with at least one part of *A*'s boundary. The dimension of one part of *A*'s boundary is 1.

Condition 23: For region *A* and line *B*, if *B*'s interior does not intersect with *A*'s interior and exterior, then the dimension of the intersection for *A*'s interior and *B*'s boundary is drawn from {1}, i.e.,

$$R_{9IM}(A,B) \neq \begin{bmatrix} -1 & * & * \\ -1 & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ 0 & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ 2 & * & * \\ -1 & * & * \end{bmatrix} \qquad \text{(23a-c)}$$

Proof 15: For region *A* and line *B*, if *B*'s interior does not intersect with *A*'s interior and exterior, the intersection for *B*'s interior and *A*'s boundary is *B*'s interior. Since the dimension of *B*'s interior is 1, the dimension of the intersection for *A*'s interior and *B*'s boundary is drawn from {1}.

Condition 24: For region *A* and line *B*, the only possible value for *A*'s interior and *B*'s exterior is drawn from {-1, 2}, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 1 \\ * & * & * \\ * & * & * \end{bmatrix} \qquad \text{(24a-b)}$$

Likewise, Condition 24 can be proven according to Proof 2.

Condition 25: For region *A* and line *B*, the only possible value for *A*'s exterior and *B*'s interior is drawn from {-1, 1}, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 2 & * & * \end{bmatrix} \quad \text{(25a-b)}$$

Likewise, Condition 25 can be proven according to Proof 2.

Condition 26: For region *A* and line *B*, *B*'s boundary can intersect with at most two parts of *A*'s interior, boundary and exterior, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 0 & * \\ * & 0 & * \\ * & 0 & * \end{bmatrix} \quad \text{(26a-a)}$$

Proof 16: Since *B*'s boundary is two disjoint points, *B*'s boundary intersects with at most two parts of *A*. Because the dimension of *B*'s boundary is 0, the only intersection between *B*'s boundary and one part of *A* is a point according to Condition 18.

Condition 27: For region *A* and line *B*, if *A*'s exterior and *B*'s interior are disjoint, then *A*'s exterior and *B*'s boundary must be disjoint, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & * \\ * & * & * \\ -1 & 0 & * \end{bmatrix} \quad \text{(27a-a)}$$

Proof 17: If *A*'s exterior and *B*'s interior are disjoint, then *B*'s interior must be inside *A*'s closure. Since *B*'s interior belongs to *A*'s closure, *B*'s boundary must belong to *A*'s closure.

Condition 28: For region *A* and line *B*, if *B*'s interior intersects with *A*'s interior and exterior, then *B*'s interior must intersect with *A*'s boundary, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 1 & * & * \\ -1 & * & * \\ 1 & * & * \end{bmatrix} \quad \text{(28a-a)}$$

Likewise, Condition 28 can be proven according to Proof 10.

### 2.4.3 Negative conditions for two lines

Condition 29: For line *A* and line *B*, the dimension of the intersection between *A*'s interior or exterior and *B*'s interior is no greater than 1, and the dimension of the intersection between *A*'s interior, boundary, or exterior and *B*'s boundary is no greater than 0, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 2 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 2 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & 2 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 2 & * \\ * & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 2 & * \end{bmatrix} \vee \begin{bmatrix} * & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 1 & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 2 & * & * \\ * & * & * \end{bmatrix} \quad \text{(29a-m)}$$
$$\vee \begin{bmatrix} * & * & * \\ * & * & 2 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 1 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 1 \\ * & * & * \end{bmatrix}$$

Likewise, Condition 29 can be proven according to Proof 1.

Condition 30: For line *A* and line *B*, if the two lines' interiors are disjoint, the only possible value for *A*'s interior and *B*'s exterior is drawn from {1}, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & * & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} -1 & * & 2 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & * \\ 2 & * & * \end{bmatrix} \quad \text{(30a-f)}$$

Likewise, Condition 30 can be proven according to Proof 6.

Condition 31: For two lines, if *A*'s interior intersects with *B*'s boundary, then *A*'s interior must intersect with *B*'s exterior, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 0 & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 0 & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & 1 & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 1 & * & * \\ -1 & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & 2 & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 2 & * & * \\ -1 & * & * \end{bmatrix} \quad \text{(31a-f)}$$

Proof 18: If *A*'s interior is within *B*'s interior, *A*'s interior and *B*'s exterior are disjoint. Since *A*'s interior intersects with *B*'s boundary, *A*'s interior must intersect with *B*'s exterior.

Condition 32: For two lines, if *A*'s interior and *B*'s exterior are disjoint, then *A*'s exterior intersects with both *B*'s boundary and *B*'s interior, or not at all, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & -1 \\ * & * & * \\ -1 & 1 & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & * \\ 2 & -1 & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & 1 \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 2 \\ * & * & -1 \\ -1 & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & * & -1 \\ * & * & * \\ -1 & 0 & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & * \\ -1 & 2 & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & * \\ 0 & -1 & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & * \\ 1 & -1 & * \end{bmatrix} \quad \text{(32a-l)}$$
$$\vee \begin{bmatrix} * & * & -1 \\ * & * & 0 \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & -1 \\ * & * & 2 \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 0 \\ * & * & -1 \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 1 \\ * & * & -1 \\ -1 & * & * \end{bmatrix}$$

Proof 19: If *A*'s interior and *B*'s exterior are disjoint, then *A* is equal to *B* or *A* is within *B*. If *A* is equal to *B*, *A*'s exterior intersects with both *B*'s boundary and *B*'s interior. If *A* is within *B*, *A*'s exterior and *B*'s boundary are disjoint, and *A*'s exterior and *B*'s interior are disjoint.

Condition 33: For two lines, *B*'s boundary can intersect with at most two parts of *A*'s interior, boundary and exterior, and vice versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 0 & * \\ * & 0 & * \\ * & 0 & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} \quad \text{(33a-b)}$$

Proof 20: Since one line has only two endpoints, a line's boundary can intersect with at most two parts of the other line.

Condition 34: For two lines, if *A*'s boundary is a subset of *B*'s boundary, then *A*'s interior or exterior and *B*'s boundary are disjoint, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & 0 & * \\ -1 & 0 & -1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ -1 & 0 & -1 \\ * & 0 & * \end{bmatrix} \vee \begin{bmatrix} * & -1 & * \\ 0 & 0 & * \\ * & -1 & * \end{bmatrix} \vee \begin{bmatrix} * & -1 & * \\ * & 0 & 0 \\ * & -1 & * \end{bmatrix} \quad \text{(34a-d)}$$

Proof 21: If *A*'s boundary is a subset of *B*'s boundary, then *A*'s boundary and *B*'s boundary coincide. Therefore, *A*'s interior or exterior and *B*'s boundary are disjoint.

Condition 35: For two lines, if *A*'s interior intersects with *B*'s exterior, the dimension of the intersection is drawn from {1}, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \quad \text{(35a-d)}$$

Proof 22: Since the dimension of *A*'s interior and that of *B*'s exterior are separately 1 and 2, the dimension of the intersection can only be {0, 1}. If a point belongs to *A*'s interior and *B*'s exterior, then a neighbourhood of this point must belong to *B*'s exterior and contain one part of *A*'s interior. Therefore, the dimension of the intersection is only drawn from {1}.

Condition 36: For two lines, if the dimension of the intersection between *A*'s interior and *B*'s interior is 0, then the dimension of the intersection between *A*'s interior and *B*'s exterior is 1, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 0 & * & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} 0 & * & * \\ * & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} 0 & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} 0 & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \qquad \text{(36a-d)}$$

Proof 23: If the dimension of the intersection between *A*'s interior and *B*'s interior is 0, then *A*'s interior is not the subset of *B*'s closure and *A*'s interior must intersect with *B*'s exterior. The dimension of intersection between *A*'s interior and *B*'s exterior must be 1 according to Condition 35.

### 2.4.4 Negative conditions for a region and a point

Condition 37: For any non-point object *A* and point *B*, the dimension of the intersection between *B*'s interior and any part of *A* is no greater than 0, and the dimension of the intersection between *B*'s boundary and any part of *A* is always equal to -1, i.e.,

$$\begin{aligned} R_{DE-9IM}(A,B) \neq & \begin{bmatrix} 2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 2 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 2 & * & * \end{bmatrix} \vee \begin{bmatrix} 1 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 1 & * & * \\ * & * & * \end{bmatrix} \\ & \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & 2 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 2 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 2 & * \end{bmatrix} \vee \begin{bmatrix} * & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix} \qquad \text{(37a-o)} \\ & \vee \begin{bmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 1 & * \end{bmatrix} \vee \begin{bmatrix} * & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 0 & * \end{bmatrix} \end{aligned}$$

Likewise, Condition 37 is also valid for the topological relations between a line and a point.

Likewise, Condition 37 can be proven according to Proof 1.

Condition 38: For any non-point object *A* and point *B*, *B*'s interior can only intersect with a single part of *A*, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \vee \begin{bmatrix} 0 & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \quad (38a\text{-}c)$$

Likewise, Condition 38 is also valid for the topological relations between a line and a point.

Proof 24: Since the interior of a point is a point, a point can only intersect with a single part of B.

Condition 39: For any non-point object *A* and point *B*, *B*'s interior must be a subset of one of the three parts of *A*, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & * & * \\ -1 & * & * \\ -1 & * & * \end{bmatrix} \quad (39a\text{-}a)$$

Likewise, Condition 39 is also valid for the topological relations between a line and a point.

Likewise, Condition 39 can be proven according to Proof 9.

Condition 40: For region *A* and point *B*, the dimension of the intersection between *A*'s interior and *B*'s exterior is 2, and the dimension of the intersection between *A*'s boundary and *B*'s exterior is 1, i.e.,

$$\begin{aligned} R_{DE-9IM}(A,B) \neq & \begin{bmatrix} * & * & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 1 \\ * & * & * \\ * & * & * \end{bmatrix} \\ & \vee \begin{bmatrix} * & * & * \\ * & * & -1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 2 \\ * & * & * \end{bmatrix} \end{aligned} \quad (40a\text{-}f)$$

Proof 25: Since the dimension of a region and a point are separately 2 and 0, respectively, the interior of a point cannot contain the region's interior or exterior. Therefore, the dimension of the intersection between a region's interior and a point's exterior is 2, and the dimension of the intersection between a region's boundary and a point's exterior is 1.

**2.4.5 Negative conditions for a line and a point**

Condition 41: For line *A* and point *B*, the dimension of the intersection between *A*'s interior and *B*'s exterior is 1, and the dimension of the intersection between *A*'s boundary and *B*'s exterior is 0, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} * & * & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 2 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & -1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 1 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 2 \\ * & * & * \end{bmatrix} \quad (41a\text{-}f)$$

Likewise, Condition 41 can be proven according to Proof 25.

### 2.4.6 Negative conditions for two points

Condition 42: For two points, the dimension of the intersection between *A*'s interior or exterior and *B*'s interior is no greater than 0, the dimension of the intersection between *A*'s interior, boundary, or exterior and *B*'s boundary is always equal to -1, and the dimension of the intersection between *A*'s boundary and *B*'s interior or exterior is always equal to -1, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} 2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} 1 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & 2 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & 1 & * \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & * & 2 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & 1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 2 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 1 & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & * & * \\ * & 2 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 1 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 2 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & 1 \\ * & * & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 2 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ 1 & * & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 2 & * \end{bmatrix} \vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 1 & * \end{bmatrix}$$
$$\vee \begin{bmatrix} * & * & * \\ * & * & * \\ * & 0 & * \end{bmatrix} \quad (42a\text{-}u)$$

Likewise, Condition 42 can be proven according to Proof 1.

Condition 43: For two points, *A*'s interior only intersects with *B*'s interior or exterior, and vice-versa, i.e.,

$$R_{DE-9IM}(A,B) \neq \begin{bmatrix} -1 & * & -1 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} -1 & * & * \\ * & * & * \\ -1 & * & * \end{bmatrix} \vee \begin{bmatrix} 0 & * & 0 \\ * & * & * \\ * & * & * \end{bmatrix} \vee \begin{bmatrix} 0 & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} \quad (43a\text{-}d)$$

Proof 26: Since the interior of a point is a point and the boundary of a point is null, *A*'s interior can only intersect with *B*'s interior or exterior.

## 3 RESULTS

To verify the existence of the topological relations, configurations of region/region, region/line, line/line, region/point, line/point, and point/point relations are drawn.

### 3.1 Relations between two regions

Conditions 1-17 hold for two regions. These conditions are applied to exclude the impossible topological relations between two regions. Finally, 12 relations between two regions can be realized. The dimensionally extended 9-intersections of the 12 relations are listed to verify the existence of the 12 relations. Figure 2 shows the corresponding geometric interpretation of the 12 topological relations between two regions.

### 3.2 Relations between a region and a line

Conditions 6, 12, and 18-28 hold for a region and a line. These conditions are used to exclude the impossible topological relations between a region and a line. Finally, 31 relations between a region and a line can be realized. The dimensionally extended 9-intersections of the 31 relations are listed to verify their existence and are shown in Figure 3.

### 3.3 Relations between two lines

Conditions 6, 9, 12, and 29-36 hold for two lines. These conditions are used to exclude the impossible topological relations between two lines. Finally, 47 relations between two lines can be realized. The dimensionally extended 9-intersections of the 47 relations are listed to verify their existence and are shown in Figure 4.

### 3.4 Relations between a region and a point

Conditions 6 and 37-40 hold for a region and a point. These conditions are used to exclude the impossible topological relations between a region and a point. Finally, 3 relations between a region and a point can be realized. The dimensionally extended 9-intersections of the 3 relations are listed to verify their existence and are shown in Figure 5.

### 3.5 Relations between a line and a point

Conditions 6, 37-39, and 41 hold for a line and a point. These conditions are used to exclude the impossible topological relations between a line and a point. Finally, 3 relations between a line and a point can be realized. The dimensionally extended 9-intersections of the 3 relations are listed to verify their existence and are shown in Figure 6.

### 3.6 Relations between two points

Conditions 6 and 42-43 hold for two points. These conditions are used to exclude the impossible topological relations between two points. Finally, 2 relations between two points can be realized. The dimensionally extended 9-intersections of the 2 relations are listed to verify their existence and are shown in Figure 7.

## 4 DISCUSSION

In this section, the mutual exclusion of the negative conditions is discussed, and the topological relations between spatial objects are compared with other research. The purpose of discussing the mutual exclusion is to use as few conditions as possible. The aim of comparing the topological relations within DE-9IM and those within 9IM is to discuss the refinement of DE-9IM compared with 9IM. Nine-tuples are used in Tables 1-9 for the representation of the 9-intersections and the dimensionally extended 9-intersections. To use each character to express a value, “-1” is replaced by “#” in the dimensionally extended 9-intersections of Tables 1-9.

### 4.1 Mutual exclusion of the negative conditions

To eliminate the impossible topological relations, negative conditions are defined. To minimize the number of conditions, the negative conditions should be mutually exclusive for one group of the topological relation. If a negative condition is a subset of another condition, these conditions are not mutually exclusive. For example, condition "0*****0**" is a subset of condition "******0**", and there is no need to use condition "0*****0**" to exclude the impossible topological relations. Therefore, all the subset relations for the conditions should be acquired. The following steps are taken:

(1) All the negative conditions for one group of topological relations are given by dimensionally extended 9-intersections in the form of a 9-tuple.

(2) Test whether one condition is a subset of other conditions one by one.

After executing the above two steps, Tables 1-3 were developed. Table 1, Table 2, and Table 3 are respectively the subset relations for conditions between two regions, between a region and a line, and between two lines. The subset relations for the conditions between a region and a point, between a line and a point, and between two points do not exist. After excluding the sub-conditions, the negative conditions for each group of topological relations are mutually exclusive.

For different groups of topological relations, a condition for one group may be the subset of a condition of another group. For example, Condition 33a ("*0**0**0*") is a subset of Condition 4a ("*0******* "). Because Condition 4a has a total of 128 sub-conditions, one feasible method is to define the other 127 subset conditions of 4a to eliminate the sub-relation of Conditions 33a and 4a. However, the above method is too complicated to be practical. For different groups of topological relations, subset relations between negative conditions may exist for different groups to avoid defining too many conditions.

## 4.2 Comparison of the topological relations

Egenhofer (1991) investigated the number of topological relations between spatial objects (point, line, and region) in two-dimensional space within the framework of the 9IM. Since the DE-9IM is the refinement of the 9IM, it is of interest to uncover the refinement relation of the possible topological relations of region/region, region/line, line/line, region/point, line/point, and point/point based on the DE-9IM and 9IM. Tables 4-9 are respectively the refinement relation for two regions, a region and a line, two lines, a region and a point, a line and a point, and two points.

From Tables 4 to 9, the following conclusions can be drawn. First, compared to the 9IM that considers the content invariant of the intersections, the DE-9IM considers the dimension of the intersections in a more detailed model. Moreover, the DE-9IM can distinguish more topological relations than the 9IM. Furthermore, the relationship between the 9-intersections and the dimensionally extended 9-intersections is one-to-many. Each 9-intersection corresponds to one or more dimensionally extended 9-intersections. Each dimensionally extended 9-intersection can be generalized to one type of 9-intersection.

# 5 CONCLUSION

The major novel contribution of this research is its presentation of a total of 43 negative conditions to eliminate the impossible topological relations between spatial objects embedded in the two-dimensional space. Six groups of topological relations (region/region, region/line, line/line, region/point, line/point, and point/point) are described. The DE-9IM can distinguish 12 relations between two regions, 31 relations between a region and a line, 47 relations between two lines, 3 relations between a region and a point, 3 relations between a line and a point, and 2 relations between two points. The mutual exclusion of the negative conditions is also discussed, and all the subset relations for the negative conditions are acquired. The subset conditions are not applied to eliminate the impossible topological relations. The phenomenon of a subset relation between negative conditions may exist between the conditions for different groups of topological relations. Compared with the topological relations within the framework of the 9IM, the refinement relations between the 9-intersections and the dimensionally extended 9-intersections for two regions, a region and a line, two lines, a region and a point, a line and a point, and two points have been listed.

Although all 43 negative conditions are reasonable and can exclude all the impossible topological relations, it is difficult to prove whether these negative conditions are the smallest possible set. Further analysing the relations of the negative conditions is required to determine the smallest set of negative conditions. This study addresses only the topological relations between simple objects. Complex objects, such as multi-point, multi-line, multi-region, and multi-body objects, are not considered. Topological relations between complex objects are far more complicated than topological relations between simple objects. More negative conditions may be needed for topological relations between complex objects.

The detailed categorization of topological relations within the DE-9IM may contribute to the enhancement of people's spatial cognition and the improvement of spatial query ability in spatial databases. Therefore, more experiments and algorithms need to be developed in order to further apply the research results. The implementation of topological relations is influenced by several factors, such as the definition of the spatial objects, the dimension of the space, and the topological relation model. Therefore, the topological relations of complex objects will be discussed in future work. The topological relations of spatial objects embedded in other spaces will also be discussed in future work. Furthermore, the topological relations within other frameworks will be the subject of further study.

## Acknowledgments

This research was jointly supported by the National Natural Science Foundation of China (No. 41301417), the Chongqing Natural Science Foundation (No. cstc2014jcyjA20017), and the

Fundamental Research Funds for the Central Universities (No. XDJK2015B022). We would like to thank the editors and anonymous referees for their constructive comments.

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**Table 1.** The subset relation for the conditions between two regions.

| Sub-condition | Dimensionally extended 9-intersections | Condition | Dimensionally extended 9-intersections |
|---|---|---|---|
| Condition 7c | 2##*2**** | Condition 1c | ****2**** |
| Condition 8c | #*0****** | Condition 3a | **0****** |
| Condition 8d | #*****0** | Condition 3c | ******0** |
| Condition 8e | #*1****** | Condition 3b | **1****** |
| Condition 8f | #*****1** | Condition 3d | ******1** |
| Condition 9a | **#**0*** | Condition 5b | *****0*** |
| Condition 9c | ******#0* | Condition 5a | *******0* |
| Condition 9e | **#**2*** | Condition 1e | *****2*** |
| Condition 9f | ******#2* | Condition 1c | *******2* |
| Condition 10c | *0#****** | Condition 4a | *0******* |
| Condition 10d | ***0**#** | Condition 4b | ***0***** |
| Condition 10e | *2#****** | Condition 1a | *2******* |
| Condition 10f | ***2**#** | Condition 1d | ***2***** |
| Condition 11c | #0******* | Condition 4a | *0******* |
| Condition 11d | #**0***** | Condition 4b | ***0***** |
| Condition 11e | #2******* | Condition 1a | *2******* |
| Condition 11f | #**2***** | Condition 1d | ***2***** |
| Condition 12a | #**#**#** | Condition 8b | #*****#** |
| Condition 12b | ###****** | Condition 8a | #*#****** |
| Condition 12e | **#**#**# | Condition 6a | ********# |
| Condition 12f | ******### | Condition 6a | ********# |

**Table 2.** The subset relations for the conditions between a region and a line.

| Sub-condition | Dimensionally extended 9-intersections | Condition | Dimensionally extended 9-intersections |
|---|---|---|---|
| Condition 12b | ###****** | Condition 21a | **#****** |
| Condition 12d | ***###*** | Condition 22a | *****#*** |
| Condition 12e | **#**#**# | Condition 6a | ********# |
| Condition 12e | **#**#**# | Condition 21a | **#****** |
| Condition 12e | **#**#**# | Condition 22a | *****#*** |
| Condition 12f | ******### | Condition 6a | ********# |
| Condition 18a | 2******** | Condition 19b | 2******** |
| Condition 18c | ******2** | Condition 26b | ******2** |
| Condition 24a | #**1**#0* | Condition 28a | ******#0* |
| Condition 24b | #0*1**#** | Condition 20a | #0******* |

**Table 3.** The subset relations for the conditions between two lines.

| Sub-condition | Dimensionally extended 9-intersections | Condition | Dimensionally extended 9-intersections |
|---|---|---|---|
| Condition 9b | **#**1*** | Condition 29m | *****1*** |
| Condition 9d | ******#1* | Condition 29i | *******1* |
| Condition 12a | #**#**#** | Condition 30b | #*****#** |
| Condition 12c | **#**#**# | Condition 6a | ********# |
| Condition 12d | ###****** | Condition 30a | #*#****** |
| Condition 12f | ******### | Condition 6a | ********# |
| Condition 30c | #*0****** | Condition 35a | **0****** |
| Condition 30d | #*****0** | Condition 35b | ******0** |
| Condition 30e | #*2****** | Condition 29b | **2****** |
| Condition 30f | #*****2** | Condition 29c | ******2** |
| Condition 31c | *1#****** | Condition 29g | *1******* |
| Condition 31d | ***1**#** | Condition 29l | ***1***** |
| Condition 31e | *2#****** | Condition 29d | *2******* |
| Condition 31f | ***2**#** | Condition 29j | ***2***** |
| Condition 32a | **#***#1* | Condition 9d | ******#1* |
| Condition 32a | **#***#1* | Condition 29i | *******1* |
| Condition 32b | **#***2#* | Condition 29c | ******2** |
| Condition 32c | **#**1#** | Condition 9b | **#**1*** |
| Condition 32c | **#**1#** | Condition 29m | *****1*** |
| Condition 32d | **2**##** | Condition 29b | **2****** |
| Condition 32e | **#***#0* | Condition 9c | ******#0* |
| Condition 32f | **#***#2* | Condition 29f | *******2* |
| Condition 32g | **#***0#* | Condition 35b | ******0** |
| Condition 32i | **#**0#** | Condition 9a | **#**0*** |
| Condition 32j | **#**2#** | Condition 29k | *****2*** |

| | | | |
|---|---|---|---|
| Condition 32k | **0**##** | Condition 35a | **0****** |
| Condition 36c | 0*0****** | Condition 35a | **0****** |
| Condition 36d | 0*****0** | Condition 35b | ******0** |

**Table 4.** The refinement relation for two regions between the 9-intersections and the dimensionally extended 9-intersections.

| 9-intersections (Egenhofer, 1991) | Dimensionally extended 9-intersections |
|---|---|
| 001001111 | (1)##2##1212 |
| 111001001 | (8)212##1##2 |
| 100100111 | (5)2##1##212 |
| 100010001 | (4)2###1###2 |
| 001011111 | (2)##2#01212; (3)##2#11212 |
| 111011001 | (9)212#01##2; (10)212#11##2 |
| 100110111 | (6)2##10#212; (7)2##11#212 |
| 111111111 | (11)212101212; (12)212111212 |

**Table 5.** The refinement relation for a region and a line between the 9-intersections and the dimensionally extended 9-intersections.

| 9-intersections (Egenhofer, 1991) | Dimensionally extended 9-intersections |
|---|---|
| 001001111 | (1)##2##1102 |
| 111001001 | (20)102##1##2 |
| 001101111 | (4)##20#1102; (7)##21#1102 |
| 101101111 | (12)1#20#1102; (16)1#21#1102 |
| 111101001 | (22)1020#1##2; (27)1021#1##2 |
| 111101101 | (23)1020#11#2; (28)1021#11#2 |
| 111101111 | (24)1020#1102; (29)1021#1102 |
| 001011101 | (2)##2#011#2 |
| 001011111 | (3)##2#01102 |
| 101011001 | (11)1#2#01##2 |
| 111011001 | (21)102#01##2 |
| 001111001 | (8)##2101##2 |
| 001111101 | (5)##20011#2; (9)##21011#2 |
| 001111111 | (6)##2001102; (10)##2101102 |
| 101111001 | (13)1#2001##2; (17)1#2101##2 |
| 101111101 | (14)1#20011#2; (18)1#21011#2 |
| 101111111 | (15)1#2001102; (19)1#2101102 |
| 111111001 | (25)102001##2; (30)102101##2 |
| 111111101 | (26)1020011#2; (31)1021011#2 |

**Table 6.** The refinement relation for two lines between the 9-intersections and the dimensionally extended 9-intersections.

| 9-intersections (Egenhofer, 1991) | Dimensionally extended 9-intersections |
|---|---|
| 001001111 | (1)##1##0102 |
| 101001111 | (15)0#1##0102; (32)1#1##0102 |
| 011001101 | (7)#01##01#2 |
| 011001111 | (8)#01##0102 |
| 111001001 | (38)101##0##2 |
| 111001101 | (21)001##01#2; (39)101##01#2 |
| 111001111 | (22)001##0102; (40)101##0102 |
| 001100111 | (4)##10##102 |
| 100100111 | (30)1##0##102 |
| 101100111 | (18)0#10##102; (35)1#10##102 |
| 011100101 | (10)#010##1#2 |
| 011100111 | (11)#010##102 |
| 111100101 | (24)0010##1#2; (43)1010##1#2 |
| 111100111 | (25)0010##102; (44)1010##102 |
| 001101111 | (5)##10#0102 |
| 101101111 | (19)0#10#0102; (36)1#10#0102 |
| 011101101 | (12)#010#01#2 |
| 011101111 | (13)#010#0102 |
| 111101101 | (26)0010#01#2; (45)1010#01#2 |
| 111101111 | (27)0010#0102; (46)1010#0102 |
| 001010101 | (2)##1#0#1#2 |
| 100010001 | (29)1###0###2 |
| 101010101 | (16)0#1#0#1#2; (33)1#1#0#1#2 |
| 001011111 | (3)##1#00102 |
| 101011111 | (17)0#1#00102; (34)1#1#00102 |

| | |
|---|---|
| 011011101 | (9)#01#001#2 |
| 111011001 | (41)101#00##2 |
| 111011101 | (23)001#001#2; (42)101#001#2 |
| 001110111 | (6)##100#102 |
| 100110111 | (31)1##00#102 |
| 101110111 | (20)0#100#102; (37)1#100#102 |
| 011110101 | (14)#0100#1#2 |
| 111110101 | (28)00100#1#2; (47)10100#1#2 |

**Table 7.** The refinement relation for a region and a point between the 9-intersections and the dimensionally extended 9-intersections.

| 9-intersections | Dimensionally extended 9-intersections |
|---|---|
| 001001101 | (1)##2##10#2 |
| 001101001 | (2)##20#1##2 |
| 101001001 | (3)0#2##1##2 |

**Table 8.** The refinement relation for a line and a point between the 9-intersections and the dimensionally extended 9-intersections.

| 9-intersections | Dimensionally extended 9-intersections |
|---|---|
| 001001101 | (1)##1##00#2 |
| 001101001 | (2)##10#0##2 |
| 101001001 | (3)0#1##0##2 |

**Table 9.** The refinement relation for two points between the 9-intersections and the dimensionally extended 9-intersections.

| 9-intersections | Dimensionally extended 9-intersections |
|---|---|
| 001000101 | (1)##0###0#2 |
| 100000001 | (2)0#######2 |

## Figure Captions

**Figure 1** Spatial objects including interior, boundary and exterior. (a) A point with its interior, boundary and exterior. (b) A line with its interior, boundary and exterior. (c) A region with its interior, boundary and exterior.

**Figure 2** A geometric interpretation of the 12 topological relations between two regions.

**Figure 3** A geometric interpretation of the 31 topological relations between a region and a line.

**Figure 4** A geometric interpretation of the 47 topological relations between two lines.

**Figure 5** A geometric interpretation of the 3 topological relations between a region and a point.

**Figure 6** A geometric interpretation of the 3 topological relations between a line and a point.

**Figure 7** A geometric interpretation of the 2 topological relations between two points.

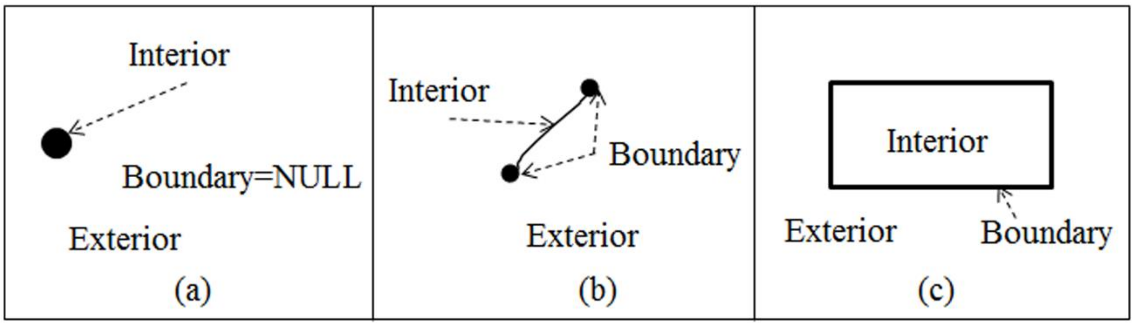


**Figure 1** Spatial objects including interior, boundary and exterior. (a) A point with its interior, boundary and exterior. (b) A line with its interior, boundary and exterior. (c) A region with its interior, boundary and exterior.

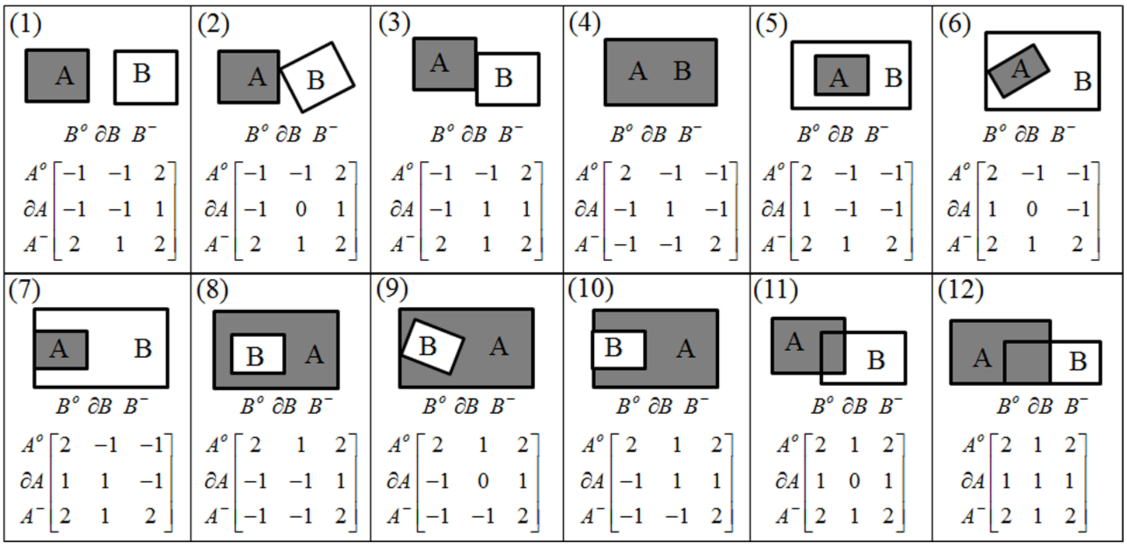

**Figure 2** A geometric interpretation of the 12 topological relations between two regions.

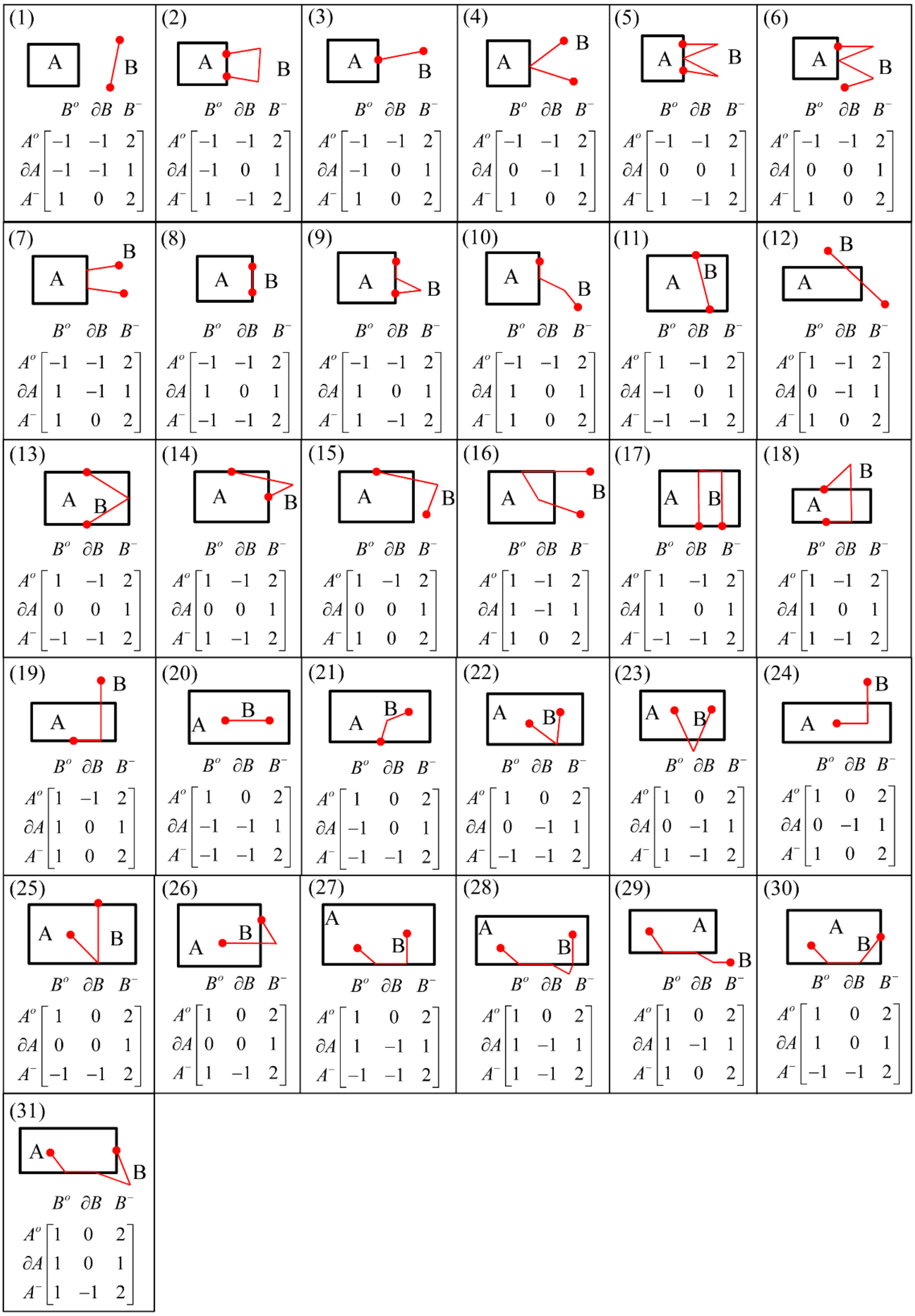

**Figure 3** A geometric interpretation of the 31 topological relations between a region and a line.

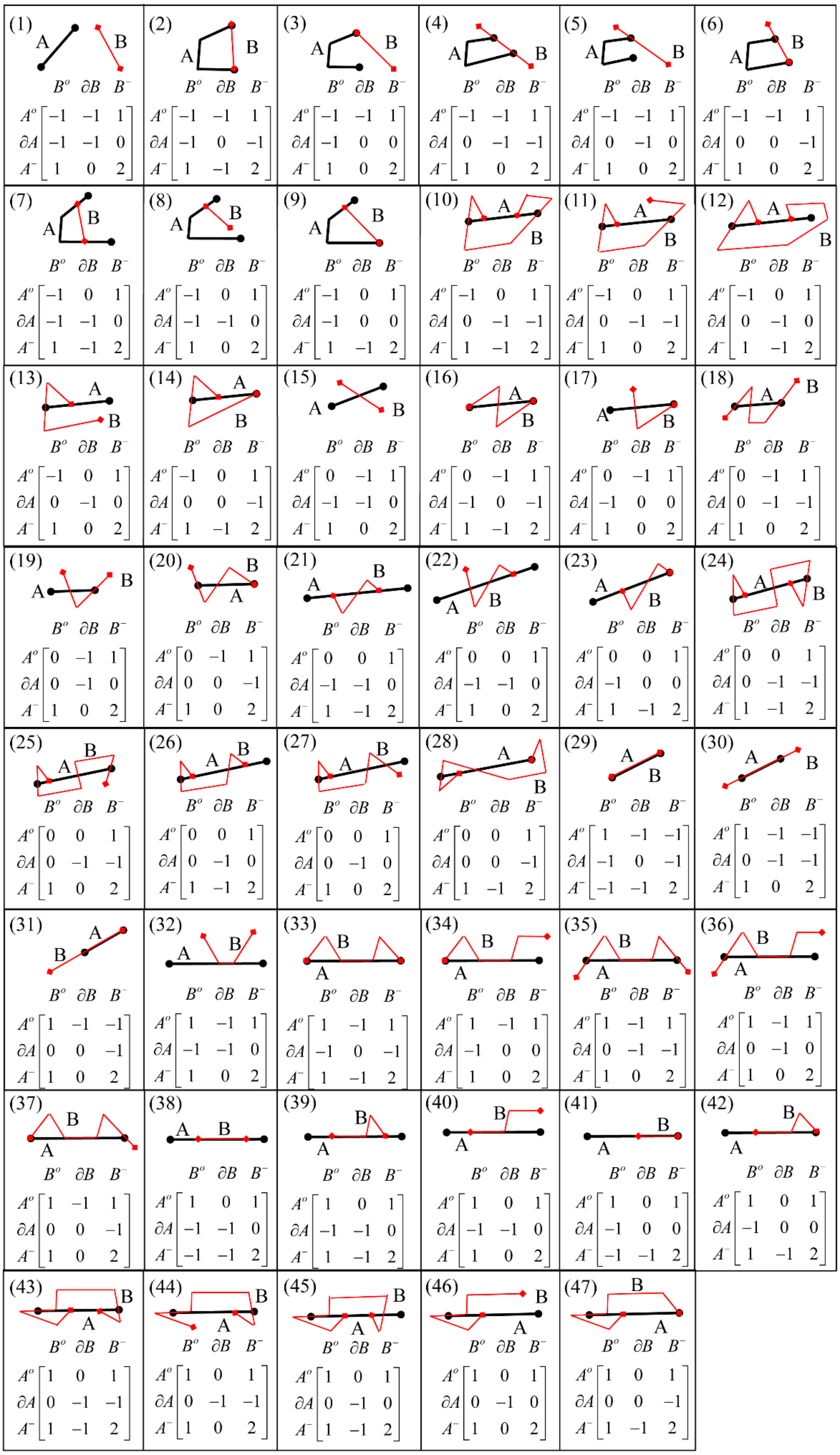


**Figure 4** A geometric interpretation of the 47 topological relations between two lines.

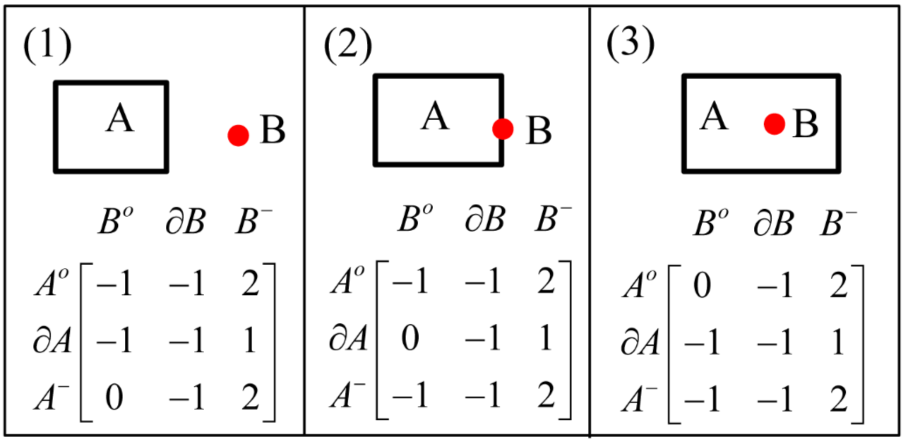


**Figure 5** A geometric interpretation of the 3 topological relations between a region and a point.

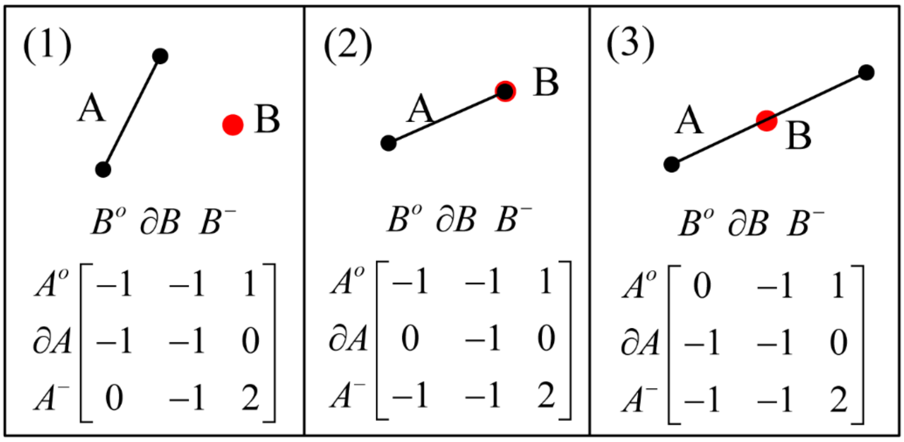


**Figure 6** A geometric interpretation of the 3 topological relations between a line and a point.

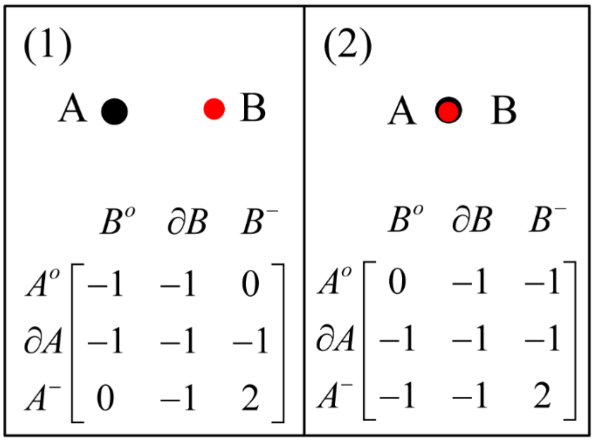


**Figure 7** A geometric interpretation of the 2 topological relations between two points.