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Automated On-Ramp Merging and Gap Development with Speed Constraints – A State-Constrained Optimal Control Approach

Yue Zhou, Michael E. Cholette, Ashish Bhaskar, and Edward Chung

Abstract— **This paper presents an automated trajectory planning method for motorway on-ramp merging and gap development based on connected and automated vehicles. The method is composed of two relevant optimal control problems – one for the mainline facilitating vehicle and one for the on-ramp merging vehicle. Previous studies have shown that the speed of facilitating maneuver of a mainline vehicle to develop a suitable gap for accommodating an on-ramp merging vehicle could drop to an undesired low level under some conditions. To address this issue, this paper formulates the trajectory planning task of the facilitating vehicle as a state-constrained optimal control problem. The optimal control problem is constrained in the control variable as well. Moreover, the optimal control problem of the facilitating vehicle features flexible terminal time (merge location) which is automatically determined in the optimization process. The Pontryagin Maximum Principle is applied to solve the optimal control problems. The solutions are then implemented in a recursive fashion so as to accommodate constantly changing external environment. A numerical example is conducted to demonstrate the effectiveness of the proposed methodology.**

I. INTRODUCTION

Motorway on-ramp merge sections can contribute to travel delay [1], cause capacity drop [2], cause traffic instabilities [3], and excessive vehicular emissions [4]. Conventionally, ramp metering is used to regulate macroscopic traffic state variables (e.g. density, occupancy, flow rate) by controlling on-ramp inflow rate. However, ramp metering is unable to regulate movements of individual vehicles. In particular, ramp metering cannot interfere with movements of mainline vehicles which are critical in creating suitable gaps for merging vehicles. Connected and automated vehicle (CAV) technologies are providing new opportunities to manage individual vehicle trajectories and have initiated new perspectives for improving traffic operations at on-ramp merge sections.

Automated on-ramp merging and gap development strategies can be broadly classified into two categories: feedback control and optimization approaches. Feedback control strategies e.g. [5-7], are simple but are in general not optimal in any sense. Moreover, they are unable to simultaneously satisfy terminal speed and gap requirements. Optimization approaches have been explored in a couple of studies, e.g. [8-11]. Among these, [9, 10] derived analytical solutions for the optimal control problems of vehicle trajectory planning. Compared to numerical solutions, the main advantage of analytical solutions is that they have higher computational efficiencies and thus are more suitable for online applications such as model predictive control (MPC). Control bounds, i.e. the limits of vehicle acceleration, were not taken into account by [9, 10]. Zhou et al. [12] explicitly considered the limits of vehicle acceleration and derived analytical solutions for the control-constrained optimal control. No existing studies, however, have considered the speed constraint in formulating optimal control problems. However, previous studies [10, 11] have indicated that the speed of facilitating maneuver of a mainline vehicle to develop a suitable gap for accommodating an on-ramp merging vehicle could drop to an undesired low level under some conditions. Therefore, it is desirable to impose a constraint on the speed of the facilitating manoeuvre so as to mitigate its negative impact on mainline traffic.

In this paper, we formulate the trajectory planning task of the mainline facilitating vehicle as an optimal control problem with both control and speed constraints. With regard to the speed constraint, only a low speed constraint is considered, because the automated on-ramp merging strategy is mostly needed when the mainline traffic is at high density, under which situation a high speed constraint is not a limiting factor. The trajectory planning task of the on-ramp merging vehicle is formulated as an optimal control problem with only control bounds. The Pontryagin Maximum Principle (PMP) is applied to solve the optimal control problems. The solutions are then implemented in a recursive framework so as to accommodate the constantly changing external disturbances.

Clearly, a complete design of the automated on-ramp merging methodology includes additional considerations, but the scope of this paper is limited to the formulation of the optimal control problems, solution using the PMP, and implementation in a recursive fashion to characterize the optimal gap development and merging. It is envisioned that the outcomes of this study will form the basis for a complete design of on-ramp merging methodology in the future. The remainder of this paper is organized as follows. Section II formulates the optimal control problems. Section III applies

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Yue Zhou (e-mail: y59.zhou@hdr.qut.edu.au) is with the Smart Transport Research Center, School of Civil Engineering and Built Environment, Science and Engineering Faculty, Australia.

Michael E. Cholette (e-mail: michael.cholette@qut.edu.au) is with the School of Chemistry, Physics and Mechanical Engineering, Science and Engineering Faculty, Queensland University of Technology, Australia.

Ashish Bhaskar (e-mail: ashish.bhaskar@qut.edu.au) is with the Smart Transport Research Center, School of Civil Engineering and Built Environment, Science and Engineering Faculty, Australia.

Edward Chung (edward.cs.chung@polyu.edu.hk) is with the Department of Electrical Engineering, Hong Kong Polytechnic University, Hong Kong, PR China.

the PMP to solve the optimal control problems, and then presents a recursive implementation framework. Section IV conducts numerical analysis and present a numerical example. Section V concludes the paper.

II. Formulation of Optimal Control Problems

A. Overview of On-Ramp Merging Process

TABLE I
NOTATION LIST

Notation	Description
$u(t)$	Acceleration trajectory (i.e. control history)
$v(t)$	Position trajectory
$x(t)$	Speed trajectory
x_0^{fac}	Initial position of the facilitating vehicle
v_0^{fac}	Initial speed of the facilitating vehicle
x_0^{mer}	Initial position of the merging vehicle
v_0^{mer}	Initial speed of the merging vehicle
$v_{t_{\mathrm{f}}}^{\mathrm{mer}}$	Desired merge-in speed of the merging vehicle
x_M	Desired merge location
$v_l(t)$	Speed trajectory of the leading vehicle
$x_l(t)$	Position trajectory of the leading vehicle
x_0^{lead}	Initial position of the leading vehicle
v_0^{lead}	Initial speed of the leading vehicle
t_{f}	Control terminal time
v_{min}	Speed constraint
a_{min}	Maximum deceleration
a_{max}	Maximum acceleration
λ	Penalty weight on control terminal time
λ_1	Penalty weight on deviation from desired merge location
λ_2	Penalty weight on deviation from desired merge speed
L	Vehicle length
s_0	Standstill clearance
τ	Constant time gap as in the car-following rule

The on-ramp merging and gap development process considered in this paper can be illustrated by Fig. 1. The driving direction is from right to left. Fig. 1 (a) shows the condition at the time when the merging process begins. The shaded blue vehicle (with no flag) is the on-ramp merging vehicle whose front bumper is currently at the predefined call-for-assistance (CFA) point and who sends out a request for assistance to mainline vehicles. It is assumed for simplicity that communications are instantaneous. The flagged vehicle is the mainline vehicle who agrees to facilitate and who initiates the process of creating a suitable gap between itself and the leading vehicle, i.e. the grey shaded vehicle. Fig. 1 (b) is a snapshot at some time in the middle of the merging process. Note that at this time the gap is not yet suitable for merging. Fig. 1 (c) shows the condition when the merging process terminates: The gap has satisfied certain conditions and the on-ramp vehicle is ready to merge into the mainline. As soon as the merging vehicle merges into the gap, it switches to car-following operation, so does the facilitating vehicle. Obviously, the time and location where the above process terminates depends on the trajectory of the leading vehicle and the conditions specifying the successful development of a suitable gap, which are defined by terminal conditions of the optimal control problems presented later. In this paper, we do not consider lateral movements of vehicles, as in [9, 10].

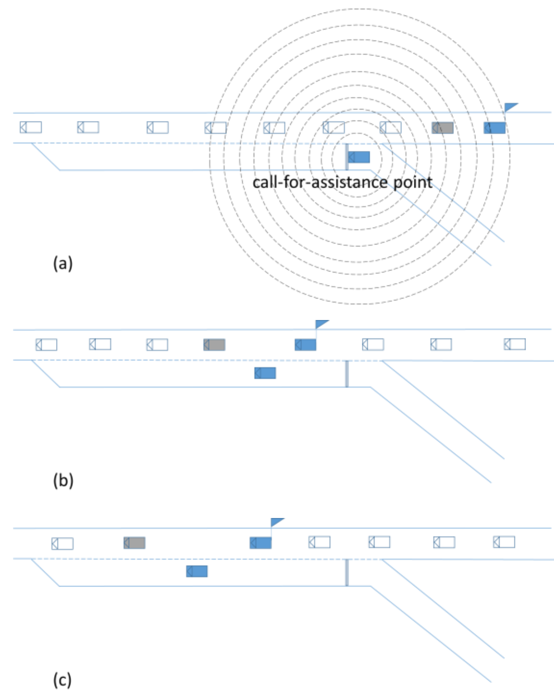

Fig. 1. The on-ramp merging process: (a) the time when the process is initiated; (b) a time in the middle of the process; (c) the time when the process is finished.

The leading vehicle is an external disturbance and its trajectory need to be predicted. In practice, this prediction is obviously not perfect. A common way of compensating for such uncertainty is to update the prediction and correspondingly re-solve the problem in recursive fashion.

B. Formulation of Optimal Control Problems

The notation used in this paper are summarized in Table I. The optimal control problem of the mainline facilitating vehicle is given by (1) through (8).

$$\min_{u(t)} \int_0^{t_{\mathrm{f}}} \frac{1}{2}[u(t)^2 + \lambda]\, dt \tag{1}$$

subject to:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ u(t) \end{bmatrix} \tag{2}$$

$$v(t) \geq v_{\mathrm{min}} \tag{3}$$

$$a_{\mathrm{min}} \leq u(t) \leq a_{\mathrm{max}} \tag{4}$$

with initial conditions:

$$x(0) = x_0^{\mathrm{fac}} \tag{5}$$

$$v(0) = v_0^{\mathrm{fac}} \tag{6}$$

with terminal state conditions:

$$x(t_{\mathrm{f}}) = x_l(t_{\mathrm{f}}) - 2(L + s_0) - 2\tau v_l(t_{\mathrm{f}}) \tag{7}$$

$$v(t_{\mathrm{f}}) = v_l(t_{\mathrm{f}}) \tag{8}$$

In the above, (1) through (8) describe a linear-quadratic optimal control problem with a flexible terminal time and terminal states which move along a given curve as a known function of the final time. The physical implication of the terminal conditions (7) and (8) is to render zero feedback errors in a constant time-gap car-following strategy [13-15] when the merging process is completed and the facilitating and merging vehicles switch to car-following operations.

The optimal control problem that governs the maneuver of the on-ramp merging vehicle is given as (9) through (15). Note that here the terminal time t_f is fixed, as determined by the optimal control problem of the facilitating vehicle and transmitted to the merging vehicle via V2V communication.

$$\min_{u(t)} \int_0^{t_\mathrm{f}} \frac{1}{2}u(t)^2\,dt + \frac{1}{2}\lambda_1[x(t_\mathrm{f}) - x_M]^2 + \frac{1}{2}\lambda_2\left[v(t_\mathrm{f}) - v_{t_\mathrm{f}}^{\mathrm{mer}}\right]^2 \tag{9}$$

subject to:

$$\mathbf{x}(t) = \begin{bmatrix}\dot{x}(t)\\ \dot{v}(t)\end{bmatrix} = \begin{bmatrix}v(t)\\ u(t)\end{bmatrix} \tag{10}$$

$$a_{\min} \le u(t) \le a_{\max} \tag{11}$$

with initial conditions:

$$x(0) = x_0^{\mathrm{mer}} \tag{12}$$

$$v(0) = v_0^{\mathrm{mer}} \tag{13}$$

and desired terminal state:

$$x_M = x_l(t_\mathrm{f}) - (L + s_0) - \tau v_l(t_\mathrm{f}) \tag{14}$$

$$v_{t_\mathrm{f}}^{\mathrm{mer}} = v_l(t_\mathrm{f}) \tag{15}$$

III. Analytical Solution And A Recursive Implementation Framework

A. Solution Using Pontryagin Maximum Principle with State Constraints

In this section we apply the PMP to solve the optimal control problem of the facilitating vehicle. Solution of the optimal control problem of the merging vehicle is comparatively straightforward, and thus is not presented due to space limitations. Because of the existence of the state constraint (3), a maximum principle with state constraints must be used. As is well-known, when there are neither control constraints nor state constraints, the maximum principle reduces to $\mathcal{H}_{\boldsymbol{u}}^* = \mathbf{0}$, i.e. the first-order partial derivatives of the Hamiltonian with respect to controls vanish. When there exist control constraints but no state constraints, the maximum principle requires maximization of the Hamiltonian over the admissible control domain. When there are both control constraints and state constraints, the Hamiltonian has to be maximized over the admissible control domain and meanwhile not to violate the state constraints. The method of Lagrange multipliers can be applied, by adjoining the control and state constraints to the Hamiltonian. Solution to this case can be much more complicated than the first two cases, especially when pure state constraints are present, as in our problem. Difficulties often lie in the so-called jump conditions [16] that must be satisfied for points where the extremal state trajectories make a contact with the state constraining arcs. In this paper we use a maximum principle with state constraints proposed by [16].

1. Hamiltonian and Lagrangian

Consider the optimal control problem of the facilitating vehicle as defined by (1) to (8). To apply the PMP, first covert (1) to be maximization of the cost function.

$$\max_{u(t)} \int_0^{t_\mathrm{f}} -\frac{1}{2}[u(t)^2 + \lambda]\,dt \tag{16}$$

Form the Hamiltonian:

$$\mathcal{H} := -\frac{1}{2}u(t)^2 - \frac{1}{2}\lambda + p_1(t)v(t) + p_2(t)u(t) \tag{17}$$

In the above, $p_1(t)$ and $p_2(t)$ are the co-state variables associated with the state variables $x(t)$ and $v(t)$, respectively. To maximize the Hamiltonian over the constrained control domain and constrained state space, we adjoin the control constraints (4) and the state constraint (3) to the Hamiltonian (17) to form the Lagrangian:

$$\begin{aligned}\mathcal{L} := -\frac{1}{2}u(t)^2 - \frac{1}{2}\lambda + p_1(t)v(t) + p_2(t)u(t)\\ + \rho_1(t)[u(t) - a_{\min}] + \rho_2(t)[-u(t) + a_{\max}]\\ + \mu(t)[v(t) - v_{\min}]\end{aligned} \tag{18}$$

$$\rho_1(t)[u(t) - a_{\min}] = 0 \tag{19}$$

$$\rho_2(t)[-u(t) + a_{\max}] = 0 \tag{20}$$

$$\mu(t)[v(t) - v_{\min}] = 0 \tag{21}$$

$$\rho_1(t), \rho_2(t), \mu(t) \ \ge 0 \tag{22}$$

In the above, $\rho_1(t)$ and $\rho_2(t)$ are the Lagrangian multiplier associated with the control constraints; $\mu(t)$ is the Lagrangian multiplier associated with the state (speed) constraint.

2. Maximization of the Lagrangian

This yields, from (18) through (22), $\forall t \in [0, t_\mathrm{f}]$:

$$u^*(t) = \begin{cases} p_2^*(t), & \text{for } a_{\min} < p_2^*(t) < a_{\max}\\ a_{\max}, & \text{for } p_2^*(t) \ge a_{\max}\\ a_{\min}, & \text{for } p_2^*(t) \le a_{\min}\end{cases} \tag{23}$$

3. State equations

$$\dot{x}^*(t) = v^*(t) \tag{24}$$

$$\dot{v}^*(t) = u^*(t) \tag{25}$$

4. Co-state equations

$$\dot{p}_1^*(t) = -\mathcal{L}_x^* = 0 \tag{26}$$

$$\dot{p}_2^*(t) = -\mathcal{L}_v^* = -p_1^*(t) - \mu^*(t) \tag{27}$$

5. Initial conditions

$$x^*(0) = x_0^{\text{fac}} \tag{28}$$

$$v^*(0) = v_0^{\text{fac}} \tag{29}$$

6. Transversality conditions

Let

$$h(t) := v(t) - v_{min} \tag{30}$$

$$b_1(t_\text{f}) := x(t_\text{f}) - x_l(t_\text{f}) + 2(L + s_0) + 2\tau v_l(t_\text{f}) \tag{31}$$

$$b_2(t_\text{f}) := v(t_\text{f}) - v_l(t_\text{f}) \tag{32}$$

The transversality conditions are:

$$p_1^*(t_\text{f}^{*-}) = \beta_1 \frac{\partial b_1^*}{\partial x}(t_\text{f}^*) + \beta_2 \frac{\partial b_2^*}{\partial x}(t_\text{f}^*) + \omega \frac{\partial h^*}{\partial x}(t_\text{f}^*) = \beta_1 \tag{33}$$

$$p_2^*(t_\text{f}^{*-}) = \beta_1 \frac{\partial b_1^*}{\partial v}(t_\text{f}^*) + \beta_2 \frac{\partial b_2^*}{\partial v}(t_\text{f}^*) + \omega \frac{\partial h^*}{\partial v}(t_\text{f}^*) = \beta_2 + \omega \tag{34}$$

$$\omega h^*(t_\text{f}^*) = \omega\, [v^*(t_\text{f}^*) - v_\text{min}] = 0 \tag{35}$$

$$\omega \geq 0 \tag{36}$$

In our problem, since the terminal time t_f is free, we need one more transversality condition:

$$\mathcal{H}^*(t_\text{f}^{*-}) + \beta_1 \frac{\partial b_1^*}{\partial t_\text{f}}(t_\text{f}^*) + \beta_2 \frac{\partial b_2^*}{\partial t_\text{f}}(t_\text{f}^*) + \omega \frac{\partial h^*}{\partial t_\text{f}}(t_\text{f}^*) = 0 \tag{37}$$

which, combined with (33) and (8), yields

$$-\frac{1}{2}u^*(t_\text{f}^{*-})^2 - \frac{1}{2}\lambda + p_2^*(t_\text{f}^{*-})u^*(t_\text{f}^{*-}) + (2\tau\beta_1 - \beta_2)\dot{v}_l(t_\text{f}^*) = 0 \tag{38}$$

From (26) we obtain

$$p_1^*(t) \equiv c_1 \tag{39}$$

Combining (39) and (33) gives

$$\beta_1 = c_1 \tag{40}$$

Plugging (40) into (38) yields

$$-\frac{1}{2}u^*(t_\text{f}^{*-})^2 - \frac{1}{2}\lambda + p_2^*(t_\text{f}^{*-})u^*(t_\text{f}^{*-}) + (2\tau c_1 - \beta_2)\dot{v}_l(t_\text{f}^*) = 0 \tag{41}$$

7. $\frac{d\mathcal{H}^*}{dt} = \frac{d\mathcal{L}^*}{dt} = \frac{\partial \mathcal{L}^*}{\partial t}$

This yields

$$\dot{\mu}^*(t)\, [v^*(t_\text{f}) - v_\text{min}] + \mu^*(t)u^*(t) = 0 \tag{42}$$

$$-u^*(t)\dot{u}^*(t) + \dot{p}_1^*(t)v^*(t) + p_1^*(t)u^*(t) + \dot{p}_2^*(t)u^*(t) + p_2^*(t)\dot{u}^*(t) = 0 \tag{43}$$

It can be verified from previous information that (42) and (43) always hold.

8. Jump conditions analyses

Jump conditions must be satisfied at points where an extremal state trajectory contacts state constraining arcs. The purpose of jump conditions analysis is to help to identify admissible extremal control sequences and to obtain the jump conditions of co-state variables which will be used in the construction of algebraic equations to solve for control constants and switching times. In our problem, since the only constrained state variable is the speed, thus we only need analyse jump conditions associated with the extremal speed trajectories. To start our analysis, we first consider such a general case of extremal speed trajectory, namely Case (a), which is graphically illustrated by Fig. 2 (a).

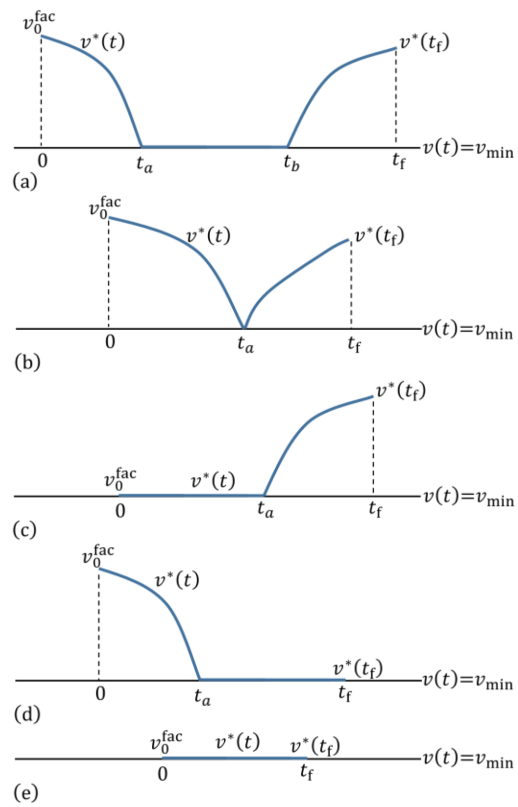

Fig. 2. The five cases of admissible extremal speed trajectories when the speed constraint can be temporarily active.

Case (a): The extremal speed trajectory starts at a regular point (i.e. not on the constraining arc $v(t) = v_\text{min}$) and reaches the constraining arc, remains at it for a finite time interval, and then leaves it.

According to [16], the jump conditions at the points $t = t_a$ and $t = t_b$ can be deduced to be as (44) through (49).

$$p_1^*(t_a^-) - p_1^*(t_a^+) = 0 \tag{44}$$

$$p_2^*(t_a^-) - p_2^*(t_a^+) \geq 0 \tag{45}$$

$$\mathcal{H}^*(t_a^-) - \mathcal{H}^*(t_a^+) = 0 \tag{46}$$

and

$$p_1^*(t_b^-) - p_1^*(t_b^+) = 0 \tag{47}$$

$$p_2^*(t_b^-) - p_2^*(t_b^+) \geq 0 \tag{48}$$

$$\mathcal{H}^*(t_b^-) - \mathcal{H}^*(t_b^+) = 0 \tag{49}$$

Now we use the above conditions to infer admissible extremal control sequences that can render the concerned extremal speed trajectory possible. First, note that (44) and (47) always hold because of (39). Therefore we only need to check (45), (46), (48) and (49).

Case (a1). $u^*(t)_{t\le t_a} = p_2^*(t)$; $u^*(t)_{t\ge t_b} = p_2^*(t)$

We use the notation $u^*(t)_{t\le t_a}$ to denote the extremal control history over a finite time interval that ends at time t_a. The notation $u^*(t)_{t\ge t_b}$ is defined similarly. Since

$$\begin{aligned}\mathcal{H}^*(t)_{t\le t_a} &= -\frac{1}{2}u^*(t)^2 - \frac{1}{2}\lambda + p_1^*(t)v^*(t)\\ &+ p_2^*(t)u^*(t) = \frac{1}{2}p_2^*(t)^2 - \frac{1}{2}\lambda + c_1 v^*(t)\end{aligned} \tag{50}$$

thus

$$\mathcal{H}^*(t_a^-) = \frac{1}{2}p_2^*(t_a^-)^2 - \frac{1}{2}\lambda + c_1 v_{\min} \tag{51}$$

On the other hand, since

$$\begin{aligned}\mathcal{H}^*(t)_{t\ge t_a} &= -\frac{1}{2}u^*(t)^2 - \frac{1}{2}\lambda + p_1^*(t)v^*(t)\\ &+ p_2^*(t)u^*(t) = -\frac{1}{2}\lambda + c_1 v^*(t)\end{aligned} \tag{52}$$

thus

$$\mathcal{H}^*(t_a^+) = -\frac{1}{2}\lambda + c_1 v_{\min} \tag{53}$$

Plugging (53) and (51) into (46) yields

$$p_2^*(t_a^-) = 0 \tag{54}$$

This implies that the co-state trajectory $p_2^*(t)$ is continuous at the point $t = t_a$. Finally, notice that such a result satisfies (45) as well. Similarly, we can show that the co-state trajectory $p_2^*(t)$ is continuous at the point $t = t_b$.

The functional form of $p_2^*(t)$ when the extremal speed trajectory is not on the constraining arc can be deduced from (21), (27), and (39) as:

$$p_2^*(t)_{t\le t_a} = -c_1 t + c_2 \tag{55}$$

$$p_2^*(t)_{t\ge t_b} = -c_1 t + c_3 \tag{56}$$

where c_2 and c_3 are constants.

The fact that $p_2^*(t)$ is continuous throughout the whole control process, combined with (55) and (56), leads to an important conclusion: Once an extremal speed trajectory leaves the constraining arc $v(t) = v_{\min}$, it never comes back.

Now we show that the following three sub-cases are all impossible.

Case (a2). $u^*(t)_{t\le t_a} = a_{\min}$; $u^*(t)_{t\ge t_b} = p_2^*(t)$

Case (a3). $u^*(t)_{t\le t_a} = p_2^*(t)$; $u^*(t)_{t\ge t_b} = a_{\max}$

Case (a4). $u^*(t)_{t\le t_a} = a_{\min}$; $u^*(t)_{t\ge t_b} = a_{\max}$

Theorem 1. $u^*(t)_{t\le t_a} = a_{\min}$ is not admissible.
Proof:
Assume that $u^*(t)_{t\le t_a} = a_{\min}$ is admissible. Then,

$$\begin{aligned}&\mathcal{H}^*(t_a^-)\\ &= -\frac{1}{2}{a_{\min}}^2 - \frac{1}{2}\lambda + c_1 v_{\min} + p_2^*(t_a^-)a_{\min}\end{aligned} \tag{57}$$

Combing (57), (53) and (46) gives

$$-\frac{1}{2}{a_{\min}}^2 + p_2^*(t_a^-)a_{\min} = 0 \tag{58}$$

which yields

$$p_2^*(t_a^-) = \frac{1}{2}a_{\min} \tag{59}$$

On the other hand, we have assumed that $u^*(t)_{t\le t_a} = a_{\min}$ is admissible, which implies

$$p_2^*(t_a^-) \le a_{\min} \tag{60}$$

It is obvious that (59) and (60) contradict.

■

Theorem 2. $u^*(t)_{t\ge t_b} = a_{\max}$ is not admissible.
Proof:
Similar to the proof of Theorem 1.

■

Consequently, we can conclude that Cases (a2), (a3) and (a4) are all impossible.

Based on the above analysis, we can conclude that admissible extremal control sequences associated with Case (a) can only include:

$\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$,

$\{a_{\min}, p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$,

$\{a_{\min}, p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}, a_{\max}\}$,

$\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}, a_{\max}\}$.

Now we consider the follow cases of extremal speed trajectories, corresponding to Fig. (2b) through Fig. (2e), respectively.

Case (b): The extremal speed trajectory starts at a regular point, reaches the constraining arc, and immediately leaves it;

Case (c): The extremal speed trajectory starts from the constraining arc, remains at it for a finite time interval, and then leaves;

Case (d): The extremal speed trajectory starts at a regular initial point, then reaches the constraining arc, remains on it until the final time; and a trivial case:

Case (e): The extremal speed trajectory remains at the constraining arc for the entire control process.

Remark 1. It can be shown that the admissible extremal control sequences associated with the extremal speed trajectory Case (b) are the same as when the speed constraining arc is never touched, i.e. $\{p_2^*(t)\}$, $\{a_{\min}, p_2^*(t)\}$, $\{a_{\min}, p_2^*(t), a_{\max}\}$, $\{p_2^*(t), a_{\max}\}$, $\{a_{\max}, p_2^*(t), a_{\min}\}$, $\{a_{\max}, p_2^*(t)\}$, $\{p_2^*(t), a_{\min}\}$, $\{a_{\min}\}$, and $\{a_{\max}\}$. For a detailed treatment of these options, please refer to [12]. Moreover, from (23) and the monotony of $p_2^*(t)$, it is obvious that for Case (b), it is also true that once the extremal speed trajectory leaves the speed constraining arc, it never returns.

Remark 2. It can be shown that the admissible extremal control sequences associated with the extremal speed trajectory Case (c) can only include $\{0, p_2^*(t)\}$ and $\{0, p_2^*(t), a_{\max}\}$. From the analysis for Case (a), we can conclude that for Case (c), it is also true that once the extremal speed trajectory leaves the speed constraining arc, it never returns.

Remark 3. It can be shown that the admissible extremal control sequences associated with the extremal speed trajectory Case (d) can only include $\{p_2^*(t), 0\}$ and $\{a_{\min}, p_2^*(t), 0\}$. But it can be further shown that, when $v_l(t) \equiv v_{\min}$, the extremal speed trajectory Case (d) and Case (e) cannot be possible.

9. Construction of algebraic equations

From the above analysis, we have that there are a total of 18 admissible control sequences, summarized in Table II.

TABLE II
ADMISSIBLE CONTROL SEQUENCES

$\{p_2^*(t)\}$	$\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$
$\{a_{\min}, p_2^*(t)\}$	$\{a_{\min}, p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$
$\{a_{\min}, p_2^*(t), a_{\max}\}$	$\{a_{\min}, p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}, a_{\max}\}$
$\{p_2^*(t), a_{\max}\}$	$\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}, a_{\max}\}$
$\{a_{\max}, p_2^*(t), a_{\min}\}$	$\{0, p_2^*(t)\}$
$\{a_{\max}, p_2^*(t)\}$	$\{0, p_2^*(t), a_{\max}\}$
$\{p_2^*(t), a_{\min}\}$	$\{p_2^*(t), 0\}$
$\{a_{\min}\}$	$\{a_{\min}, p_2^*(t), 0\}$
$\{a_{\max}\}$	$\{0\}$

For each admissible extremal control sequence, algebraic equations need be constructed at control switching times (if any) and the terminal time. Due to space limitations, this paper only constructs equations for one of the admissible extremal control sequences under which the speed constraint is temporarily active, i.e. $\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$. Algebraic equations for other admissible extremal control sequences can be constructed in a similar manner.

If the optimal control sequence is $\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$, this implies that there exists two times t_a and t_b, $0 < t_a < t_b < t_f$, such that

$$u^*(t) = \begin{cases} p_2^*(t)_{t\le t_a} & \text{for } t \in [0, t_a) \\ 0 & \text{for } t \in [t_a, t_b) \\ p_2^*(t)_{t\ge t_b} & \text{for } t \in [t_b, t_f] \end{cases} \quad (61)$$

At $t = t_a$, using the continuities of state variables as well as the proved continuity of the control variable, we can construct the following equations:

$$-c_1 t_a + c_2 = 0 \quad (62)$$

$$-\frac{1}{2}c_1 t_a^2 + c_2 t_a + v_0^{\text{fac}} = v_{\min} \quad (63)$$

$$-\frac{1}{6}c_1 t_a^3 + \frac{1}{2}c_2 t_a^2 + v_0^{\text{fac}} t_a + x_0^{\text{fac}} = v_{\min} t_a + c_4 \quad (64)$$

At $t = t_b$ we have:

$$0 = -c_1 t_b + c_3 \quad (65)$$

$$v_{\min} = -\frac{1}{2}c_1 t_b^2 + c_3 t_b + c_5 \quad (66)$$

$$v_{\min} t_b + c_4 = -\frac{1}{6}c_1 t_b^3 + \frac{1}{2}c_3 t_b^2 + c_5 t_b + c_6 \quad (67)$$

At $t = t_f$ we have:

$$\frac{1}{2}(-c_1 t_f + c_3)^2 - \frac{1}{2}\lambda + (2\tau c_1 + c_1 t_f - c_3)\dot{v}_l(t_f) = 0 \quad (68)$$

$$-\frac{1}{2}c_1 t_f^2 + c_3 t_f + c_5 = v_l(t_f) \quad (69)$$

$$-\frac{1}{6}c_1 t_f^3 + \frac{1}{2}c_3 t_f^2 + c_5 t_f + c_6 = x_l(t_f) - 2(L + s_0) - 2v_l(t_f) \quad (70)$$

In the above, (68) is obtained from combining (34), (41) and (56). Note that for $\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$ to be feasible, the extremal speed trajectory must not terminate at the speed constraint, thus ω in (34) must be zero, per (35). Therefore, we have nine unknowns, c_1, c_2, c_3, c_4, c_5, c_6, t_a, t_b, t_f, and nine algebraic equations (62) through (70), which can be solved using MATLAB's "vpasolve" function. Furthermore, once these unknown constants are solved, they need to be checked for feasibility. The following conditions must be satisfied for the control sequence $\{p_2^*(t)_{t\le t_a}, 0, p_2^*(t)_{t\ge t_b}\}$ to be feasible:

$$0 < t_a < t_b < t_f \quad (71)$$

$$a_{\min} < c_2 < 0 \quad (72)$$

$$0 < -c_1 t_f + c_3 < a_{\max} \quad (73)$$

$$-c_1 > 0 \quad (74)$$

Using the algebraic conditions on the corners and end time, as well as the feasibility conditions on the solved parameters, for each of these control sequences, the solutions are examined for feasibility. Since the Hamiltonian is concave in control and the control and state constraints are all linear, once an admissible control sequence is found to be feasible, it is optimal. Note that it can be possible that no feasible solution exists for an initial condition. If this happens, another facilitating vehicle needs to be selected and the on-ramp vehicle needs to wait. We leave the issue of vehicle selection to future work. We have verified the above analytical solution method by discretizing the optimal control problem and then solving using numerical method (the "fmincon" function of the MATLAB®).

B. A Recursive Implementation Framework

In practice, the leading vehicle's trajectories, i.e. $x_l(\cdot)$ and $v_l(\cdot)$, cannot be perfectly known in advance. Therefore, to compensate for the errors it is necessary to recursively update the prediction of $x_l(\cdot)$ and $v_l(\cdot)$ and re-solve the optimal control problems. In this paper, the future speed of the leading vehicle is assumed to be the same as the instant when the planning is updated, as in [9, 17].

At each sampling time $t_0 = 0$ (at each sampling time, the optimal controller resets the initial time as zero, being consistent with the analytical solutions above), the speed and position of the leading vehicle are obtained, and the optimal control problems for the facilitating and merging vehicle are solved, yielding the optimal control histories $u^{\text{fac}^*}(t)$ and $u^{\text{mer}^*}(t)$ for $0 \leq t \leq t_\text{f}$. The first portion of the controls are applied over a pre-specified updating interval Δt_up, e.g. 0.1 sec, 0.5 sec, etc., and then the leading vehicle trajectory is resampled and the optimal control problems are re-solve. The process is repeated until the termination condition $t_\text{f} < \overline{t_\text{f}}$ is met, after which the entire optimal control histories are applied. Note that $\overline{t_\text{f}}$ is needed because here the length of planning horizon is shrinking with the increase of the recursive step index. In this paper, the control implementation interval is 0.1 seconds, i.e. $\Delta t_\text{im} = 0.1$ sec. Numerical simulation studies revealed that, when$\Delta t_\text{im} = 0.1$ sec, then the threshold $\overline{t_\text{f}} = \max\{0.5 \text{ sec}, 2\Delta t_\text{up}\}$ is sufficient to generate a well-behaved solution. If the length of control implementation interval can be shorter, e.g. $\Delta t_\text{im} = 0.01$ sec, then the threshold value can be accordingly made shorter.

IV. Numerical Analysis and Example

A. Effects of λ

One feature of our methodology as compared to previous studies is the flexible merge time t_f, which enables more flexibility in integrating the methodology with an upper-level traffic management scheme. The parameter to alter the merge time is λ in the optimal control problem of the facilitating vehicle. Fig. 3 illustrates the effect of the value of λ on the merge time. Table III lists the optimal control sequences. The calculations were based on the following assumptions: $v_l(t) \equiv 23$ m/sec, $x_0^\text{lead} = -13$ m, $x_0^\text{fac} = -48$ m, $v_0^\text{fac} = 25$ m/sec, $v_\text{min} = 20$ m/sec, $a_\text{min} = -3$ m/sec^2, $a_\text{max} = 3$ m/sec^2.

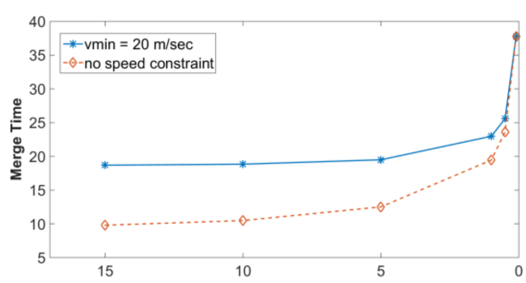

Fig. 3. The effect of the value of λ on the merge time t_f.

Two observations can be made from the above results: 1) For a given value of λ, the merge time of the case when the speed constraint is imposed is longer than that of the case when there is no speed constraint; 2) With the decrease of the value of λ, the optimal control solutions for the two cases eventually converge to a same solution under which neither the control bounds nor the speed constraint ever gets active. In applications, the value of λ can be determined through an upper-level controller which supervises the overall traffic flow efficiency.

TABLE III
Optimal Control Sequences For Different Values of λ

Value of λ	**Control sequence** (v_min= 20 m/sec)	**Control sequence** (no v_min)
15	$\{a_\text{min}, p_2^*(t)_{t\leq t_a}, 0, p_2^*(t)_{t\geq t_b}, a_\text{max}\}$	$\{a_\text{min}, p_2^*(t), a_\text{max}\}$
10	$\{a_\text{min}, p_2^*(t)_{t\leq t_a}, 0, p_2^*(t)_{t\geq t_b}, a_\text{max}\}$	$\{a_\text{min}, p_2^*(t), a_\text{max}\}$
5	$\{p_2^*(t)_{t\leq t_a}, 0, p_2^*(t)_{t\geq t_b}\}$	$\{p_2^*(t)\}$
1	$\{p_2^*(t)_{t\leq t_a}, 0, p_2^*(t)_{t\geq t_b}\}$	$\{p_2^*(t)\}$
0.5	$\{p_2^*(t)_{t\leq t_a}, 0, p_2^*(t)_{t\geq t_b}\}$	$\{p_2^*(t)\}$
0.1	$\{p_2^*(t)\}$	$\{p_2^*(t)\}$

B. Recursive Implementation Example

We study the effectiveness of the proposed strategy under the influence of a leading vehicle whose actual speed is varying sinusoidally:$v_l(t) = 23\left[1 - \frac{1}{5}\sin\left(\frac{\pi}{15}t\right)\right]$, $v_0^\text{lead} = 22.04$ m/sec, $x_0^\text{lead} = -7.43$ m. The initial states of the facilitating vehicle and the merging vehicle are: $v_0^\text{fac} = 22.87$ m/sec, $x_0^\text{fac} = -49.53$ m; $v_0^\text{mer} = 7$ m/sec, $x_0^\text{mer} = 0$ m. Set $\lambda = 15$, $\lambda_1 = \lambda_2 = 25$, $v_\text{min} = 16.67$ m/sec, $a_\text{min} = -3$ m/sec^2, $a_\text{max} = 3$ m/sec^2.

Fig. 4 and Fig. 5 show the acceleration, speed, and position trajectories of the facilitating and merging vehicles, when the optimal control problem of the facilitating vehicle is unconstrained and constrained, respectively. From Fig. 4 (b) it can be seen that when the speed constraint is not imposed, the facilitating vehicle's speed can drop to a very low level. On the other hand, from Fig. 5 (b) it can be seen that when the speed constraint is imposed, the facilitating vehicle's speed never drops below the specified constraining speed. Fig. 4 and Fig. 5 also demonstrate that under the proposed optimal control model and the recursive implementation framework, although the leading vehicle is constantly changing its speed, the facilitating and merging vehicles successfully maneuver to proper final positions with proper final speeds with which the merging vehicle merges into the developed gap and then both vehicles switch to car-following operations.

V. Conclusions

This paper develops an optimal control based strategy for planning facilitating and merging maneuvers of automated vehicles at motorway on-ramp merges. Particularly, the optimal control problem of the facilitating vehicle considers a speed constraint in order to prevent an excessive drop of the mainline speed. An analytical solution using the Pontryagin Maximum Principle is derived and the solutions are implemented in a recursive framework so as to accommodate constantly changing external environment. A numerical example is presented which demonstrates the effectiveness of the proposed strategy. The outcomes of this study will form the basis for a complete design of on-ramp merging and gap development methodology in the future.

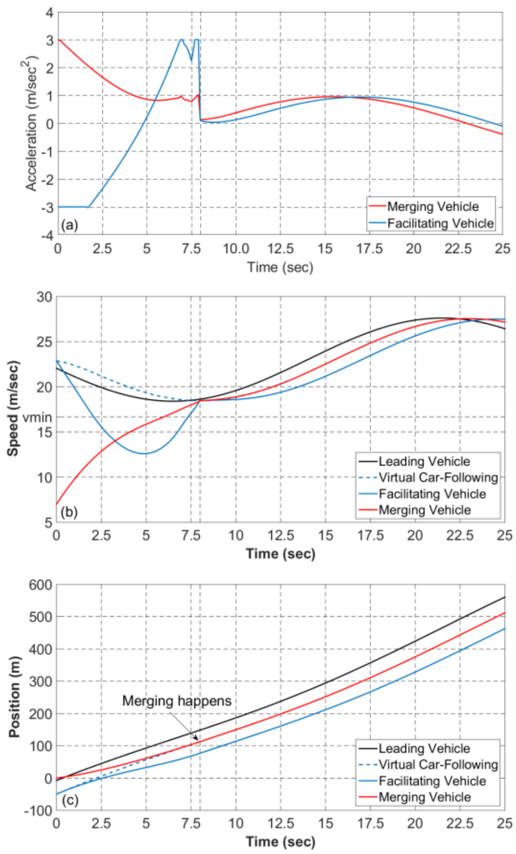

Fig. 4. Trajectories when the speed constraint is not imposed: (a) accelerations; (b) speeds; (c) positions.

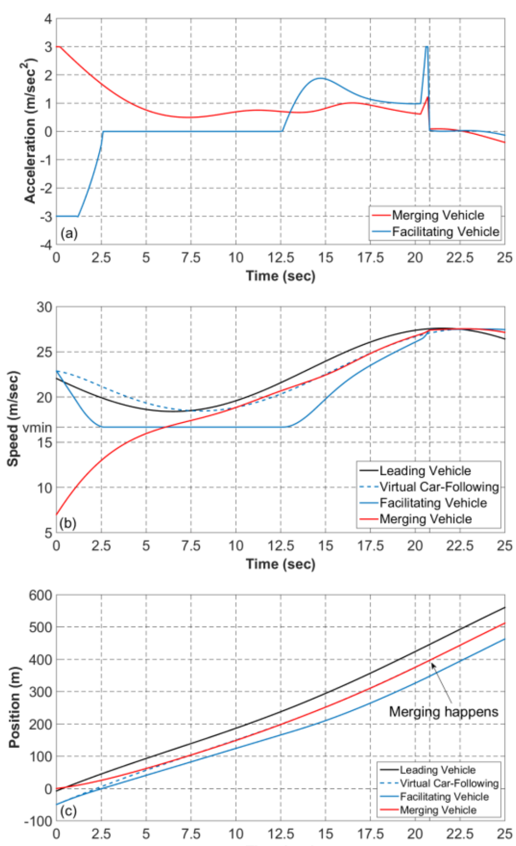

Fig. 5. Trajectories when the speed constraint is imposed: (a) accelerations; (b) speeds; (c) positions.

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