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STOCHASTIC LOGARITHMIC SCHRÖDINGER EQUATIONS: ENERGY REGULARIZED APPROACH*

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Abstract. In this paper, we prove the global existence and uniqueness of the solution of the stochastic logarithmic Schrödinger (SlogS) equation driven by either additive noise or multiplicative noise. The key ingredient lies in the regularized SlogS (RSlogS) equation with regularized energy and the strong convergence analysis of the solutions of the RSlogS equations. In addition, temporal Hölder regularity estimates and uniform estimates in energy space $\mathbb{H}^1(\mathcal{O})$ and weighted Sobolev space $L^2_\alpha(\mathcal{O})$ of the solutions for both the SlogS equation and RSlogS equation are also obtained.

Key words. stochastic Schrödinger equation, logarithmic nonlinearity, energy regularized approximation, strong convergence

MSC codes. 60H15, 35Q55, 47J05, 81Q05

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1. Introduction.

The deterministic logarithmic Schrödinger equation has wide applications in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, Bose–Einstein condensations, and so on (see, e.g., [5, 16, 18, 22, 23, 24, 25]). It takes the form of

$$\begin{aligned}&\partial_t u(t,x)=\mathbf{i}\Delta u(t,x)+\mathbf{i}\lambda u(t,x)\log(|u(t,x)|^2)+\mathbf{i}V(t,x,|u(t,x)|^2)u(t,x),\ t>0,\\ &u(0,x)=u_0(x),\end{aligned}$$

where Δ is the Laplacian operator on $\mathcal{O}\subset\mathbb{R}^d$ with $\mathcal{O}$ being either $\mathbb{R}^d$ or a bounded domain with homogeneous Dirichlet or periodic boundary condition, t is time, x is the spatial coordinate, $\lambda\in\mathbb{R}/\{0\}$ characterizes the strength of nonlinear interactions, and V is a real-valued function. While retaining many of the known features of the linear Schrödinger equation, Bialynicki-Birula and Mycielski show that only such a logarithmic nonlinearity satisfies the condition of separability of noninteracting systems (see [5]). The logarithmic nonlinearity makes the logarithmic Schrödinger equation unique among nonlinear wave equations due to its connection with nonlinear wave mechanics and nonlinear optics (see [7]). For instance, the longtime dynamics of the logarithmic Schrödinger equation is essentially different from the Schrödinger equation. There is a faster dispersive phenomenon when $\lambda<0$ and the convergence of the modulus of the solution to a universal Gaussian profile (see [7]), and no dispersive phenomenon when

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$\lambda > 0$ (see [8]). The logarithmic Schrödinger equation with a quadratic potential has also been recently studied (see [6] and references therein).

In this paper, we mainly focus on the well-posedness of the following stochastic logarithmic Schrödinger (SlogS) equation

$$\begin{aligned} du(t) &= \mathbf{i}\Delta u(t)dt + \mathbf{i}\lambda u(t)\log(|u(t)|^2)dt + \widetilde{g}(u(t)) \star dW(t),\ t>0, \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where $W(\cdot)$ is a standard Q-Wiener process with a bounded, self-adjoint, positive semidefinite operator $Q \in \mathcal{L}(\mathbb{H},\mathbb{H})$. The Karhunen–Loève expansion yields that $W(t) = \sum_{k\in\mathbb{N}^+} Q^{\frac{1}{2}} e_k \beta_k(t)$ with $\{e_k\}_{k\in\mathbb{N}^+}$ being an orthonormal basis of $L^2 := L^2(\mathcal{O};\mathbb{C})$ and $\{\beta_k\}_{k\in\mathbb{N}^+}$ being a sequence of independent Brownian motions on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Here $\widetilde{g}$ is a continuous function and $\widetilde{g}(u) \star dW(t)$ is defined by

$$\begin{aligned} \widetilde{g}(u(t)) \star dW(t) = &-\frac{1}{2}\sum_{k\in\mathbb{N}^+} |Q^{\frac{1}{2}}e_k|^2\Big(|g(|u(t)|^2)|^2 u(t)\Big)dt \\ &- \mathbf{i}\sum_{k\in\mathbb{N}^+} g(|u(t)|^2)g'(|u(t)|^2)|u(t)|^2 u(t) Im(Q^{\frac{1}{2}}e_k)Q^{\frac{1}{2}}e_k dt \\ &+ \mathbf{i}g(|u(t)|^2)u(t)dW(t) \end{aligned}$$

if $\widetilde{g}(x) = \mathbf{i}g(|x|^2)x$, and by

$$\widetilde{g}(u(t)) \star dW(t) = dW(t)$$

if $\widetilde{g} = 1$. We would like to remark that when $W(\cdot)$ is $L^2(\mathcal{O};\mathbb{R})$-valued and $\widetilde{g}(x) = \mathbf{i}g(|x|^2)x$, $\widetilde{g}(u) \star dW(t)$ is just the classical Stratonovich integral.

Introduce the weighted L^2-space

$$L^2_\alpha := \{v \in L^2 |\, x \mapsto (1+|x|^2)^{\frac{\alpha}{2}} v(x) \in L^2\}, \quad \alpha \in [0,1],$$

with the norm $\|v\|_{L^2_\alpha} := \|(1+|x|^2)^{\frac{\alpha}{2}} v(x)\|_{L^2}$. Denote $\mathbb{H} := L^2$ equipped with the L^2-norm $\|u\|^2 := \int_{\mathcal{O}} |u|^2 dx$ and the product $\langle u, v\rangle := \int_{\mathcal{O}} Re(u\bar{v})dx$ for $u, v \in \mathbb{H}$. Throughout this paper, we assume that

- $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}}e_i\|_{L^2_\alpha} + \|Q^{\frac{1}{2}}e_i\|^2_{\mathbb{H}^l} < \infty$ for some $l \in \mathbb{N}$ and $\alpha \in [0,1]$ when $\widetilde{g} = 1$;
- $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}}e_i\|^2_{\mathbb{H}^l} + \|Q^{\frac{1}{2}}e_i\|^2_{W^{l,\infty}} < \infty$ for some $l \in \mathbb{N}$ when $\widetilde{g}(x) = \mathbf{i}g(|x|^2)x$,

where $\mathbb{H}^l$ and $W^{l,\infty}$ are standard Sobolev spaces.

The SlogS equation (1.1) could be derived from the deterministic model by using Nelson's mechanics ([20]). By applying the Madelung transformation $u(t,x) = \sqrt{\rho(t,x)}e^{\mathbf{i}S(t,x)}$, [19] obtains a fluid expression of the solution as follows,

$$\begin{aligned} \partial_t S(t,x) &= -|\nabla S(t,x)|^2 - \frac{1}{4}\frac{\delta I}{\delta\rho}(\rho(t,x)) + \lambda\log(\rho) + V(t,x,\rho(t,x)), \\ \partial_t \rho(t,x) &= -2div(\rho(t,x)\nabla S(t,x)),\ S(0,x) = S_0(x)\,, \rho(0,x) = \rho_0(x), \end{aligned}$$

where $I(\rho) = \int_{\mathcal{O}} |\nabla\log(\rho)|^2 \rho dx$ is the Fisher information. If V is random and fluctuates rapidly, the term $\mathbf{i}Vu$ can be approximated by some multiplicative Gaussian noise $\widetilde{g}(u)\dot{W}$, which plays an important role in the theory of measurements continuous in time in open quantum systems (see, e.g., [4]). Then we could use the inverse of the Madelung transformation and formally obtain the SlogS equation

$$\begin{aligned} \partial_t u(t,x) &= \mathbf{i}\Delta u(t,x) + \mathbf{i}\lambda u(t,x)\log(|u(t,x)|^2) + \widetilde{g}(u(t,x))\dot{W}(t,x),\ x \in \mathcal{O},\ t>0, \\ u(0,x) &= u_0(x), \quad x \in \overline{\mathcal{O}}. \end{aligned}$$

The main assumptions on W and $\widetilde{g}$ are stated as follows.

Assumption 1. The diffusion operator is the Nemystkii operator of $\widetilde{g}$. W and $\widetilde{g}$ satisfy one of the following conditions:

Case 1. $\{W(t)\}_{t\geq 0}$ is $L^2(\mathcal{O};\mathbb{C})$-valued and $\widetilde{g}=1$;

Case 2. $\{W(t)\}_{t\geq 0}$ is $L^2(\mathcal{O};\mathbb{C})$-valued, $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x$, and $g\in\mathcal{C}_b^2([0,\infty))$ satisfies the growth condition

$$\sup_{x\in[0,\infty)}|g(x)|+\sup_{x\in[0,\infty)}|g'(x)x|+\sup_{x\in[0,\infty)}|g''(x)x^2|\leq C_g;$$

Case 3. $\{W(t)\}_{t\geq 0}$ is $L^2(\mathcal{O};\mathbb{R})$-valued, and $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x$, $g\in\mathcal{C}_b^1([0,\infty))$ satisfies the growth condition

$$\sup_{x\in[0,\infty)}|g(x)|+\sup_{x\in[0,\infty)}|g'(x)x|\leq C_g.$$

Here C_g is a positive constant.

Assumption 2. Assume that g satisfies

$$(x+y)(g(|x|^2)-g(|y|^2))\leq C_g|x-y|,\quad x,y\in[0,\infty). \tag{1.2}$$

When $\{W(t)\}_{t\geq 0}$ is $L^2(\mathcal{O};\mathbb{C})$-valued, we in addition assume that g satisfies the following one-sided Lipschitz continuity

$$|(\bar{y}-\bar{x})(g'(|x|^2)g(|x|^2)|x|^2x-g'(|y|^2)g(|y|^2)|y|^2y)|\leq C_g|x-y|^2,\quad x,y\in\mathbb{C}. \tag{1.3}$$

Here C_g is a positive constant.

A typical example is $\widetilde{g}(u)=\mathbf{i}u$, and then (1.1) becomes the SlogS equation driven by linear multiplicative noise in [3]. We would like to mention that the current study does not provide a specific physical motivation for our choice of $\widetilde{g}$ and that studying (1.1) is probably a purely mathematical question at this stage.

There are two main difficulties in proving the well-posedness of the SlogS equation. On the one hand, the random perturbation in the SlogS equation destroys a lot of physical conservation laws, like the mass and energy conservation laws in Cases 1 and 2, and the energy conservation law in Case 3. A similar phenomenon has been observed in the stochastic nonlinear Schrödinger equation with polynomial nonlinearity in $\mathbb{R}^d$ (see, e.g., [15]) and the one with cubic nonlinearity in $\mathbb{T}^d$ (see, e.g., [9]). On the other hand, the logarithmic nonlinearity in the SlogS equation is not locally Lipschitz continuous. The contraction mapping arguments via Strichartz estimates or the Fourier restriction norm method for the stochastic nonlinear Schrödinger equation with smooth nonlinearity play an important role in [2, 9, 15, 17] and references therein, but it cannot be applied directly here. Recently, when the driving noise is a linear multiplicative noise in Case 2 ($\widetilde{g}(u)=\mathbf{i}u$), [3] used a rescaling technique, together with maximal monotone operator theory, to obtain a unique global mild solution in some Orlicz space. However, it is unclear that such an approach could work for general $\widetilde{g}$. As far as we know, there are no results concerning the well-posedness of the SlogS equation driven by additive noise or general multiplicative noise. This is one main motivation of the current study.

To show the well-posedness of the considered model, we introduce an energy regularized problem inspired by [1] where the authors use the regularized problem to study error estimates of numerical methods for the deterministic logarithmic Schrödinger equation. The main idea is first to construct a proper approximation of $\log(|x|^2)$ denoted by $f_\epsilon(|x|^2)$. This induces the regularized entropy F_ϵ which is an approximation of the entropy $F(\rho)=\int_{\mathcal{O}}(\rho\log(\rho)-\rho)dx$, where $\rho=|u|^2$. The regularized Slogs (RSlogS) equation is defined by

$$du^\epsilon(t)=\mathbf{i}\Delta u^\epsilon(t)dt+\mathbf{i}u^\epsilon(t)\lambda f_\epsilon(|u^\epsilon(t)|^2)dt+\widetilde{g}(u^\epsilon(t))\star dW(t),\quad u^\epsilon(0)=u_0. \tag{1.4}$$

The corresponding regularized energy is $\frac{1}{2}\|\nabla(\cdot)\|^2+\frac{\lambda}{2}F_\epsilon(|(\cdot)|^2)$. Our approach to showing the well-posedness of the SlogS equation lies in two aspects. First, we show the local and global well-posedness of the RSlogS equation by using truncation arguments in [15]. To deal with the singularity caused by the logarithmic coefficient near the vacuum, we adopt the stochastic version of the functional setting in [7] where the well-posedness of the deterministic logarithmic Schrödinger equation with $\lambda<0$ is established. More precisely, we prove an ϵ-independent estimate in $\mathbb{H}^1:=W^{1,2}$ and the weighted L^2-space of the RSlogS equation (1.4). Then we are able to prove that the limit of $\{u^\epsilon\}_{\epsilon>0}$ is convergent to a unique stochastic process u which is shown to be the unique mild solution (see Appendix A for its definition) of (1.1). Furthermore, the sharp convergence rate of $\{u^\epsilon\}_{\epsilon>0}$ is obtained when $\mathcal{O}=\mathbb{R}^d$, or a bounded domain in $\mathbb{R}^d$ equipped with homogenous Dirichlet or periodic boundary condition. From this viewpoint, the functional setting and estimates introduced in [7], together with their stochastic version in this study, allow for a more robust approach to investigate the well-posedness for general SlogS equations. Our main result is formulated as follows.

THEOREM 1. *Let $T>0$, Assumptions 1 and 2 hold, and u_0 be $\mathbb{H}^1\cap L^2_\alpha$-valued and $\mathcal{F}_0$-measurable with any finite pth moment, i.e., $\mathbb{E}[\|u_0\|^p_{\mathbb{H}^1}]+\mathbb{E}[\|u_0\|^p_{L^2_\alpha}]<\infty$ for any $p\in\mathbb{N}^+$. Assume that $\sum_{i\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_i\|^2_{L^2_\alpha}+\|Q^{\frac{1}{2}}e_i\|^2_{\mathbb{H}^1}<\infty$ when $\widetilde{g}=1$ and that $\sum_{i\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_i\|^2_{\mathbb{H}^1}+\|Q^{\frac{1}{2}}e_i\|^2_{W^{1,\infty}}<\infty$ when $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x$. Then there exists a unique mild solution $u\in C([0,T];\mathbb{H})$ a.s. of (1.1). Moreover, for $p\geq 2$, there exists $C(Q,T,\lambda,p,u_0)>0$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)\|^p_{\mathbb{H}^1}\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)\|^p_{L^2_\alpha}\Big]\leq C(Q,T,\lambda,p,u_0).$$

When $W(t)$ is $L^2(\mathcal{O};\mathbb{R})$-valued, the well-posedness of the SlogS equation with a superlinearly growing diffusion coefficient, e.g., $\widetilde{g}(x)=\mathbf{i}\log(|x|^2+c)x$, is also proven (see Theorem 2).

The reminder of this article is organized as follows. In section 2, we introduce the RSlogS equation and show the well-posedness of the RSlogS equation driven by either additive or multiplicative noise. Section 3 is devoted to the ϵ-independent estimate of the mild solution in $\mathbb{H}^1$ and L^2_α of the RSlogS equation. In section 4, we prove the main result by passing the limit of the sequence of the regularized mild solutions and providing the sharp strong convergence rate. Several technique details are postponed to Appendix A. Throughout this article, C denotes various constants which may change from line to line.

2. Regularized SLogS equation. In this section, we show the well-posedness of the solution for (1.4) (see Appendix A for the definition of the solution). We would like to remark that there are several choices of the regularization function $f_\epsilon(|x|^2)$. For instance, one may take $f_\epsilon(|x|^2)=\log(\frac{|x|^2+\epsilon}{1+\epsilon|x|^2})$ (see Lemma 1 for the necessary properties) or $f_\epsilon(|x|^2)=\log(|x|^2+\epsilon)$ (see, e.g., [1] and references therein for more choices of regularization functions). If the regularization function f_ϵ enjoys the same properties of $\log(\frac{|x|^2+\epsilon}{1+\epsilon|x|^2})$, then one can follow our approach to obtain the well-posedness of (1.1). In the following, we first present the well-posedness of (1.4), and then derive global existence and a uniform estimate of its solution. For simplicity, we always assume that $0<\epsilon\ll 1$.

2.1. Well-posedness of Regularized SLogS equation. In this part, we give the detailed estimates to get the well-posedness in $\mathbb{H}$ of (1.4) by using the regularization function like $f_\epsilon(|x|^2)=\log(\frac{|x|^2+\epsilon}{1+\epsilon|x|^2})$. We would like to mention that one may also follow the approach in this section and the Sobolev embedding theorem, and use $\log(|x|^2+\epsilon)$ to get the local well-posedness in $\mathbb{H}^2$ when $d\le 3$ and $u_0\in\mathbb{H}^2$. Below we summarize the useful properties of $f_\epsilon(|x|^2)=\log(\frac{|x|^2+\epsilon}{1+\epsilon|x|^2})$.

LEMMA 1. *Let $\epsilon\in(0,1)$. Then $f_\epsilon(|x|^2)=\log(\frac{|x|^2+\epsilon}{1+\epsilon|x|^2})$ satisfies*

$$|f_\epsilon(|x|^2)|\le|\log(\epsilon)|, \tag{2.1}$$

$$|d_{|x|}f_\epsilon(|x|^2)|\le\frac{2(1-\epsilon^2)|x|}{(\epsilon+|x|^2)(1+\epsilon|x|^2)}, \tag{2.2}$$

$$|Im[(f_\epsilon(|x_1|^2)x_1-f_\epsilon(|x_2|^2)x_2)(\bar{x}_1-\bar{x}_2)]|\le 4(1-\epsilon^2)|x_1-x_2|^2. \tag{2.3}$$

Proof. The proof of the first and second estimates are derived by the property of $\log(\cdot)$. The last estimate is proven by similar arguments as in the proof of Lemma 8. ∎

Denote by $\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,T];\mathbb{H}))$ with $p\in[1,\infty)$ the space of process $v:[0,T]\times\Omega\to\mathbb{H}$ with continuous paths in $\mathbb{H}$ which is $\mathcal{F}_t$-adapted and satisfies

$$\|v\|^p_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,T];\mathbb{H}^1))}:=\mathbb{E}\Big[\sup_{t\in[0,T]}\|v(t)\|^p_{\mathbb{H}^1}\Big]<\infty.$$

Let $\tau\le T$ be an $\mathcal{F}_t$-stopping time. And we call $v\in\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,\tau);\mathbb{H}))$, if there exists $\{\tau_n\}_{n\in\mathbb{N}^+}$ with $\tau_n\nearrow\tau$ as $n\to\infty$ a.s., such that $v\in\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,\tau_n];\mathbb{H}))$ for $n\in\mathbb{N}^+$. Next we show the existence and uniqueness of the local mild solution (see Definition 1 in Appendix A).

For the sake of simplicity, let us ignore the dependence on ϵ and write $u_R:=u^\epsilon_R$, where u_R is the solution of the truncated equation

$$\begin{aligned} du_R&=\mathbf{i}\Delta u_R dt+\mathbf{i}\lambda\Theta_R(u_R,t)u_R f_\epsilon(|u_R|^2)dt\\ &\quad-\frac{1}{2}\Theta_R(u_R,t)\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2\Big(|g(|u_R|^2)|^2u_R\Big)dt\\ &\quad-\mathbf{i}\Theta_R(u_R,t)\sum_{k\in\mathbb{N}^+}g(|u_R|^2)g'(|u_R|^2)|u_R|^2u_R Im(Q^{\frac12}e_k)Q^{\frac12}e_k dt\\ &\quad+\Theta_R(u_R,t)\mathbf{i}g(|u_R|^2)u_R dW(t). \end{aligned} \tag{2.4}$$

Here, $\Theta_R(u,t):=\theta(\frac{\|u\|_{C([0,t];\mathbb{H})}}{R}), R>0$, with a cutoff function θ, that is, a positive C^∞ function on $\mathbb{R}^+$ which has a compact support and satisfies

$$\theta(x)=\begin{cases}0 & \text{for}\quad x\ge 2,\\ 1 & \text{for}\quad x\in[0,1].\end{cases}$$

In the following, adopting the truncation arguments in [15], we study the local and global well-posedness of RSlogS equations.

LEMMA 2. *Let Assumption* 1 *hold, and* $f_\epsilon(|x|^2) = \log(\frac{|x|^2+\epsilon}{1+\epsilon|x|^2})$*, and* u_0 *be* $\mathbb{H}$*-valued and* $\mathcal{F}_0$*-measurable with any finite pth moment,* $p \in \mathbb{N}^+$*. Assume that* $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}} e_i\|^2 < \infty$ *when* $\widetilde{g} = 1$ *and that* $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}} e_i\|^2_{L^\infty} < \infty$ *when* $\widetilde{g}(x) = \mathbf{i}g(|x|^2)x$*. Then there exists a unique global solution to* (2.4) *with a continuous* $\mathbb{H}$*-valued path.*

Proof. We only present the proof of the multiplicative noise case since the additive noise case is similar and simpler. Let $S(t) = \exp(\mathbf{i}\Delta t)$ be the C_0-group generated by $\mathbf{i}\Delta$. For fixed $R > 0$, we use the following notations, for $t \in [0,T]$,

$$\begin{aligned}
\Gamma^R_{det} u(t) &:= \mathbf{i}\int_0^t S(t-s)\Big(\Theta_R(u,s)\lambda f_\epsilon(|u(s)|^2)u(s)\Big)ds,\\
\Gamma^R_{mod} u(t) &:= -\frac{1}{2}\int_0^t S(t-s)\Big(\Theta_R(u,s)\sum_{j\in\mathbb{N}^+}|Q^{\frac{1}{2}}e_j|^2\Big(|g(|u(s)|^2)|^2 u(s)\Big)\Big)ds\\
&\quad -\mathbf{i}\int_0^t S(t-s)\Big(\Theta_R(u,s)\sum_{j\in\mathbb{N}^+} g(|u(s)|^2)g'(|u(s)|^2)|u(s)|^2 u(s) Im(Q^{\frac{1}{2}}e_j)Q^{\frac{1}{2}}e_j\Big)ds,\\
\Gamma^R_{Sto} u(t) &:= \mathbf{i}\int_0^t S(t-s)\Big(\Theta_R(u,s)g(|u(s)|^2)u(s)\Big)dW(s).
\end{aligned}$$

We look for a fixed point of the following operator given by

$$\Gamma^R u(t) := S(t)u_0 + \Gamma^R_{det}u(t) + \Gamma^R_{mod}u(t) + \Gamma^R_{Sto}u(t), u \in \mathbb{M}^p_{\mathcal{F}}(\Omega; C([0,r];\mathbb{H})),$$

where r will be chosen later. The unitary property of $S(\cdot)$ yields that

$$\|S(\cdot)u_0\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))} \le \|u_0\|_{\mathbb{H}}.$$

Now, we define a stopping time $\tau = \inf\{t \in [0,T] : \|u\|_{C([0,t];\mathbb{H})} \ge 2R\} \wedge r$. By (2.1) in Lemma 1, the definition of Θ_R and τ, and Assumption 1, we have that

$$\begin{aligned}
\|\Gamma^R_{det}u\|_{C([0,r];\mathbb{H})} &\le C|\lambda\log(\epsilon)|\int_0^r \Theta_R(u,s)\|u(s)\|ds \le C|\lambda\log(\epsilon)|\int_0^\tau \|u(s)\|ds\\
&\le C|\lambda\log(\epsilon)|\tau R \le C(\lambda,\epsilon)rR
\end{aligned}$$

and

$$\begin{aligned}
\|\Gamma^R_{mod}u\|_{C([0,r];\mathbb{H})} &\le C(g)\sum_{k\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_k\|^2_{L^\infty}\int_0^\tau \|u(s)\|ds\\
&\le C(g)r\sum_{k\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_k\|^2_{L^\infty}R.
\end{aligned}$$

Integrating over Ω yields that

$$\begin{aligned}
&\|\Gamma^R_{det}u\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))} + \|\Gamma^R_{mod}u\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))}\\
&\quad \le C(\lambda,\epsilon,g)r\Big(1+\sum_{j\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_j\|^2_{L^\infty}\Big)R.
\end{aligned}$$

The Burkholder inequality, the definition of Θ_R and τ, as well as Assumption 1 yield that for $p \ge 2$,

$$\begin{aligned}\|\Gamma^R_{sto}u\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))} \le& C\Big(\mathbb{E}\Big[\Big(\int_0^r \sum_{j\in\mathbb{N}^+}\Theta_R^2(u,s)\|g(|u(s)|^2)u(s)Q^{\frac12}e_j\|^2_{\mathbb{H}}ds\Big)^{\frac p2}\Big]\Big)^{\frac1p}\\ \le& C\Big(\mathbb{E}\Big[\Big(\int_0^\tau \sum_{j\in\mathbb{N}^+}\|g(|u(s)|^2)u(s)Q^{\frac12}e_j\|^2_{\mathbb{H}}ds\Big)^{\frac p2}\Big]\Big)^{\frac1p}\\ \le& C(g)r^{\frac12}\Big(\sum_{j\in\mathbb{N}^+}\|Q^{\frac12}e_j\|^2_{L^\infty}\Big)^{\frac12}R.\end{aligned}$$

Therefore, Γ^R is well-defined on $\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))$.

Now we turn to the contractivity of Γ^R. Let $u_1,u_2\in\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))$, and define stopping times $\tau_i=\inf\{t\in[0,T]:\|u_i\|_{C([0,r];\mathbb{H})}\ge 2R\}\wedge r, i=1,2$. For a fixed $\omega\in\Omega$, let us assume that $\tau_1\le\tau_2$ without loss of generality. Then a direct calculation, together with (2.1) in (2.2) in Lemma 1, leads to

$$\begin{aligned}&\|\Gamma^R_{det}u_1-\Gamma^R_{det}u_2\|_{C([0,r];\mathbb{H})}\\ &\quad\le C(\lambda)r\Big\|\Big(\Theta_R(u_2,\cdot)-\Theta_R(u_1,\cdot)\Big)f_\epsilon(|u_2|^2)u_2\Big\|_{C([0,r];\mathbb{H})}\\ &\qquad+C(\lambda)r\Big\|\Theta_R(u_1,\cdot)\Big(f_\epsilon(|u_1|^2)u_1-f_\epsilon(|u_2|^2)u_2\Big)\Big\|_{C([0,r];\mathbb{H})}\\ &\quad\le C(\lambda)r\|u_1-u_2\|_{C([0,r];\mathbb{H})}\|f_\epsilon(|u_2|^2)u_2\|_{C([0,\tau_2];\mathbb{H})}\\ &\qquad+C(\lambda)r\|f_\epsilon(|u_2|^2)u_2-f_\epsilon(|u_1|^2)u_1\|_{C([0,\tau_1];\mathbb{H})}\\ &\quad\le C(\lambda,\epsilon)r\|u_1-u_2\|_{C([0,r];\mathbb{H})}(1+R)\end{aligned}$$

and

$$\|\Gamma^R_{mod}u_1-\Gamma^R_{mod}u_2\|_{C([0,r];\mathbb{H})}\le C(g)\sum_{j\in\mathbb{N}^+}\|Q^{\frac12}e_j\|^2_{L^\infty}r\|u_1-u_2\|_{C([0,r];\mathbb{H})}(1+R).$$

By applying the Burkholder inequality and Assumption 1, we obtain

$$\begin{aligned}&\|\Gamma^R_{Sto}u_1-\Gamma^R_{Sto}u_2\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))}\\ &\quad\le C\Big\|\Theta_R(u_1,\cdot)(g(|u_1|^2)u_1-g(|u_2|^2)u_2)\Big\|_{L^p(\Omega;L^2([0,r];\mathbb{L}_2^0))}\\ &\qquad+C\Big\|(\Theta_R(u_1,\cdot)-\Theta_R(u_2,\cdot))g(|u_2|^2)u_2\Big\|_{L^p(\Omega;L^2([0,r];\mathbb{L}_2^0))}\\ &\quad\le C(g)\Big(\sum_{j\in\mathbb{N}^+}\|Q^{\frac12}e_j\|^2_{L^\infty}\Big)^{\frac12}r^{\frac12}\|u_1-u_2\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))}(1+R),\end{aligned}$$

where $\mathbb{L}_2^0$ is the space of Hilbert–Schmidt operators from $U_0=Q^{\frac12}(\mathbb{H})$ to $\mathbb{H}$. Combining all the above estimates, we have

$$\begin{aligned}&\|\Gamma^Ru_1-\Gamma^Ru_2\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))}\\ &\quad\le C(\lambda,\epsilon,g)\Big(1+\sum_{j\in\mathbb{N}^+}\|Q^{\frac12}e_j\|^2_{L^\infty}\Big)(r^{\frac12}+r)\|u_1-u_2\|_{\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))}(1+R),\end{aligned}$$

which implies that there exists a small $r>0$ depending on Q,R,λ,ϵ such that Γ^R is a strict contraction in $\mathbb{M}^p_{\mathcal{F}}(\Omega;C([0,r];\mathbb{H}))$ and has a fixed point $u^{R,1}$ satisfying $\Gamma^R(u^{R,1})=u^{R,1}$.

Assume that we have found the fixed point on each interval $[(l-1)r, lr]$, $l \leq k$, for some $k \geq 1$. Define

$$u^{R,k} = S(\cdot)u_0 + \Gamma^R_{det}u^{R,k} + \Gamma^R_{mod}u^{R,k} + \Gamma^R_{sto}u^{R,k}$$

on $[0, kr]$. In order to extend $u^{R,k}$ to $[kr, (k+1)r]$, we repeat previous arguments to show that on the interval $[kr, (k+1)r]$, there exists a fixed point of the map Γ^R defined by

$$\Gamma^R u(t) := S(t)u^{R,k}(kr) + \Gamma^{R,k}_{det}u(t) + \Gamma^{R,k}_{mod}u(t) + \Gamma^{R,k}_{Sto}u(t), u \in \mathbb{M}^p_{\mathcal{F}}(\Omega; C([0,r]; \mathbb{H})).$$

Here we use the following notations,

$$\Gamma^{R,k}_{det}u(t) := \mathbf{i}\int_0^t S(t-s)\Big(\widetilde{\Theta}_R(u,k,s)\lambda f_\epsilon(|u(s)|^2)u(s)\Big)ds,$$

$$\Gamma^{R,k}_{mod}u(t) := -\frac{1}{2}\int_0^t S(t-s)\Big(\widetilde{\Theta}_R(u,k,s)\sum_{j\in\mathbb{N}^+}|Q^{\frac{1}{2}}e_j|^2\Big(|g(|u(s)|^2)|^2u(s)\Big)\Big)ds$$

$$-\mathbf{i}\int_0^t S(t-s)\Big(\widetilde{\Theta}_R(u,k,s)\sum_{j\in\mathbb{N}^+}g(|u(s)|^2)|g'(|u(s)|^2)|u(s)|^2u(s)Im(Q^{\frac{1}{2}}e_j)Q^{\frac{1}{2}}e_j\Big)ds,$$

$$\Gamma^{R,k}_{Sto}u(t) := \mathbf{i}\int_0^t S(t-s)\Big(\widetilde{\Theta}_R(u,k,s)g(|u(s)|^2)u(s)\Big)dW^k(s),$$

where $t \in [0,r]$, $W^k(s) = W(s+kr) - W(kr)$, $u \in \mathbb{M}^p_{\mathcal{F}_{kr}}(\Omega; C([0,r]; \mathbb{H}))$, and

$$\widetilde{\Theta}_R(u,k,s) = \theta\left(\frac{\|u^{R,k}\|_{C([0,kr];\mathbb{H})} + \|u\|_{C([0,s];\mathbb{H})}}{R}\right).$$

For any different $v_1, v_2 \in \mathbb{M}^p_{\mathcal{F}_{kr}}(\Omega; C([0,r]; \mathbb{H}))$, we define the stopping times $\tau_i = \inf\{t \in [0, T-kr] : \|u^{R,k}\|_{C([0,kr];\mathbb{H})} + \|v_i\|_{C([0,t];\mathbb{H})} \geq 2R\} \wedge r$, $i = 1, 2$, and assume that $\tau_1 \leq \tau_2$ for convenience. Then the same procedures yield that this map is a strict contraction and has a fixed point $u^{R,k+1}$ for a small $r > 0$. Note that here $r > 0$ is actually the same as above. Now, we define a process u^R as $u^R(t) := u^{R,k}(t)$ for $t \in [0, kr]$ and $u^R(t) := u^{R,k+1}(t-kr)$ for $t \in [kr, (k+1)r]$. It can be checked that u^R satisfies (2.4) by the induction assumption and the definition of Γ^R. Meanwhile, the uniqueness of the mild solution can be obtained by repeating previous arguments. ☐

Proposition 1. *Let the hypotheses of Lemma* 2 *hold. There exists a unique local mild solution to* (1.4) *with a continuous* $\mathbb{H}$*-valued path. Furthermore, the solution is defined on a random interval* $[0, \tau^*_\epsilon)$*, where* $\tau^*_\epsilon := \tau^*_\epsilon(u_0, \omega)$ *is a stopping time such that* $\tau^*_\epsilon = +\infty$ *or* $\lim_{t\to\tau^*_\epsilon} \|u^\epsilon(t)\|_{\mathbb{H}} = +\infty$.

Proof. When $\widetilde{g} = 1$, one can follow the same steps as in the proof of [15, Theorem 3.1] to complete the proof. We only present the proof of the multiplicative noise case. Let $T > 0$ and $\{u^R\}_{R\in\mathbb{N}^+} \in \mathbb{M}^p_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{H}))$ be a sequence of the solution constructed in Lemma 2. Define a stopping time sequence $\tau_R := \inf\{t \geq 0 : \|u^R\|_{C([0,t];\mathbb{H})} \geq R\} \wedge T$. Then $\tau_R > 0$ is well-defined since $\|u^R\|_{C([0,t];\mathbb{H})}$ is an increasing, continuous, and $\mathcal{F}_t$-adapted process. We claim that if $R_1 \leq R_2$, then $\tau_{R_1} \leq \tau_{R_2}$ and $u^{R_1} = u^{R_2}$ *a.s.* on $[0, \tau_{R_1}]$. Let $\tau_{R_2,R_1} := \inf\{t \geq 0 : \|u^{R_2}\|_{C([0,t];\mathbb{H})} \geq R_1\} \wedge T$. Then it holds that $\tau_{R_2,R_1} \leq \tau_{R_2}$ and $\Theta_{R_2}(u^{R_2}, t) = \Theta_{R_1}(u^{R_2}, t)$ on $t \in [0, \tau_{R_2,R_1}]$. This implies that $\{(u^{R_2}, \tau_{R_2,R_1})\}$ is a solution of (2.4) and that $u^{R_1} = u^{R_2}$ *a.s.* on $\{t \leq \tau_{R_2,R_1}\}$. Thus we conclude that $\tau_{R_1} = \tau_{R_2,R_1}$ *a.s.* and that $u^{R_1} = u^{R_2}$ for $\{t \leq \tau_{R_1}\}$.

Now consider the triple $\{u,(\tau_R)_{R\in\mathbb{N}^+},\tau_\infty\}$ defined by $u(t):=u^R(t)$ for $t\in[0,\tau^R]$ and $\tau_\infty=\sup_{R\in\mathbb{N}^+}\tau_R$. From Lemma 2, we know that $u\in\mathbb{M}^p_{\mathcal{F}}(\Omega,C([0,t];\mathbb{H}))$ satisfies (1.4) for $t\le\tau_R$. The uniqueness of the local solution also holds. If we assume that (u,τ) and (v,σ) are local mild solutions of (1.4), then $u(t)=v(t)$ *a.s.* on $\{t<\sigma\wedge\tau\}$. Let $R_1,R_2\in\mathbb{N}^+$. Set $\widetilde{\tau}_{R_1,R_2}:=\inf\{t\in[0,T]:\max(\|u\|_{C([0,t];\mathbb{H})},\|v\|_{C([0,t];\mathbb{H})})\ge R_1\}\wedge\sigma_{R_2}\wedge\tau_{R_2}$. Then we have that on $\{t\le\widetilde{\tau}_{R_1,R_2}\}$, $(u,\widetilde{\tau}_{R_1,R_2})$, $(v,\widetilde{\tau}_{R_1,R_2})$ are local mild solutions of (1.4). The uniqueness in Lemma 2 leads to $u=v$ on $\{t\le\widetilde{\tau}_{R_1,R_2}\}$. Letting $R_1,R_2\to\infty$, we complete the proof. ∎

LEMMA 3. *Let $T>0$. Under the condition of Proposition 1, assume that $(u^\epsilon,\tau^*_\epsilon)$ is a local mild solution of (1.4) in $\mathbb{H}$. Then for any $p\ge2$, there exists a constant $C(Q,T,\lambda,p,u_0)>0$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,\tau^*_\epsilon\wedge T)}\|u^\epsilon(t)\|^p\Big]\le C(Q,T,\lambda,p,u_0).$$

Proof. Take any stopping time $\tau<\tau^*_\epsilon\wedge T$ *a.s.* Applying the Itô formula to $M^k(u^\epsilon(t))$, where $M(u^\epsilon(t)):=\|u^\epsilon(t)\|^2$ and $k\in\mathbb{N}^+$ or $k\ge2$, we obtain that for $t\in[0,\tau]$ and the case $\widetilde{g}=1$,

$$\begin{aligned}M^k(u^\epsilon(t))&=M^k(u_0)+2k(k-1)\int_0^t M^{k-2}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\langle u^\epsilon(s),Q^{\frac12}e_i\rangle^2ds\\&\quad+k\int_0^t M^{k-1}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2ds+2k\int_0^t M^{k-1}(u^\epsilon(s))\langle u^\epsilon(s),dW(s)\rangle,\end{aligned}$$

and for $t\in[0,\tau]$ and the case $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x$,

$$\begin{aligned}&M^k(u^\epsilon(t))\\&=M^k(u_0)+2k(k-1)\int_0^t M^{k-2}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\langle u^\epsilon(s),\mathbf{i}g(|u^\epsilon(s)|^2)u^\epsilon(s)Q^{\frac12}e_i\rangle^2ds\\&\quad+2k\int_0^t M^{k-1}(u^\epsilon(s))\langle u^\epsilon(s),\mathbf{i}g(|u^\epsilon(s)|^2)u^\epsilon(s)dW(s)\rangle+2k\int_0^t M^{k-1}(u^\epsilon(s))\\&\quad\times\Big\langle u^\epsilon(s),-\mathbf{i}\sum_{i\in\mathbb{N}^+}Im(Q^{\frac12}e_i)Q^{\frac12}e_ig'(|u^\epsilon(s)|^2)g(|u^\epsilon(s)|^2)|u^\epsilon(s)|^2u^\epsilon(s)\Big\rangle ds.\end{aligned}$$

In particular, if $W(t,x)$ is real valued and $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x$, we have $M^k(u^\epsilon(t))=M^k(u_0)$ for $t\in[0,T]$ a.s. By Assumption 1, using the martingale inequality, Hölder's inequality, Young's inequality, and Gronwall's inequality, we achieve that for all $k\ge1$,

$$\mathbb{E}\Big[M^k(u^\epsilon(t))\Big]\le C(T,k,u_0,Q).$$

Next, taking the supremum over t and repeating the above procedures, we have that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,\tau]}M^k(u^\epsilon(t))\Big]\\&\le\mathbb{E}\Big[M^k(u_0)\Big]+C(k)\mathbb{E}\Big[\int_0^\tau M^{k-2}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}(1+\|u^\epsilon(s)\|^2)\|Q^{\frac12}e_i\|_U^2ds\Big]\\&\quad+C(k)\mathbb{E}\Big[\Big(\int_0^\tau M^{2k-2}(u^\epsilon(s))(1+\|u^\epsilon(s)\|^4)\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|_U^2ds\Big)^{\frac12}\Big],\end{aligned}$$

where $U=\mathbb{H}$ for the additive noise case and $U=L^\infty$ for the multiplicative noise case. Applying the estimate of $\mathbb{E}[M^k(u^\epsilon(t))]$, we complete the proof by taking $p=2k$. ◻

The above lemma indicates the global existence of the mild solution, i.e., $\tau_\epsilon^*=+\infty$ a.s.

COROLLARY 1. *Let $T>0$. Under the condition of Proposition 1, there exists a unique mild solution u^ϵ of (1.4). Furthermore, for any $p\geq 2$, there exists a positive constant $C(Q,T,\lambda,p,u_0)>0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|u^\epsilon(t)\|^p\right]\leq C(Q,T,\lambda,p,u_0).$$

3. Uniform estimates of the RSlogS equation. In this section, we present several a priori estimates in the strong sense. Throughout this section, we assume that $u_0\in\mathbb{H}^1$ has a finite p-moment for all $p\geq 1$, $d\in\mathbb{N}^+$, and $f_\epsilon(x)=\log(\frac{x+\epsilon}{1+\epsilon x})$. We also suppose that $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{L^2_\alpha}+\|Q^{\frac12}e_i\|^2_{\mathbb{H}^1}<\infty$ when $\widetilde{g}=1$ and that $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{\mathbb{H}^1}+\|Q^{\frac12}e_i\|^2_{W^{1,\infty}}<\infty$ when $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x$. The introduction of the weighted Sobolev space in [7] is useful for proving uniform estimates when $\mathcal{O}=\mathbb{R}^d$.

To simplify the presentation, we omit some procedures like mollifying the unbounded operator Δ and taking the limit on the regularization parameter. More precisely, the mollifier $\widetilde{\Theta}_m$, $m\in\mathbb{N}^+$, may be defined by the Fourier transformation (see, e.g., [15])

$$\mathbb{F}(\widetilde{\Theta}_m v)(\xi)=\widetilde{\theta}\left(\frac{|\xi|}{m}\right)\widehat{v}(\xi),\ \xi\in\mathbb{R}^d,$$

where $\widetilde{\theta}$ is a positive $\mathcal{C}^\infty$ function on $\mathbb{R}^+$ and has a compact support satisfying $\widetilde{\theta}(x)=0$ for $x\geq 2$ and $\widetilde{\theta}(x)=1$ for $0\leq x\leq 1$. Another choice of mollifier is via the Yosida approximation $\Theta_m:=m(m-\Delta)^{-1}$ for $m\in\mathbb{N}^+$ (see, e.g., [17]). This kind of procedure is introduced to make sure that the Itô formula can be applied rigorously to deduce several a priori estimates. If $\mathcal{O}$ becomes a bounded domain equipped with a periodic or homogenous Dirichlet boundary condition, the mollifier can be chosen as the Galerkin projection, and the approximated equation becomes the Galerkin approximation or other spatial discretization (see, e.g., [10, 11, 12, 14]).

3.1. A priori estimates in $\mathbb{H}^1$. In this part, we show that the mild solution $u^\epsilon\in\mathbb{H}^1$ a.s. under suitable conditions on u_0 and Q. To this end, we introduce the stopping time $\tau_{\epsilon,R}=\inf\{t\geq 0|\|u^\epsilon(t)\|_{\mathbb{H}^1}\geq R\}$ and denote $\widetilde{\tau}_\epsilon^*=\lim_{R\to\infty}\tau_{\epsilon,R}$.

LEMMA 4. *Let $T>0$ and $u_0\in\mathbb{H}^1$ have a finite p-moment for $p\geq 1$, $d\in\mathbb{N}^+$, and $f_\epsilon(x)=\log(\frac{x+\epsilon}{1+\epsilon x})$. Assume that $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{L^2_\alpha}+\|Q^{\frac12}e_i\|^2_{\mathbb{H}^1}<\infty$ when $\widetilde{g}=1$ and that $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{\mathbb{H}^1}+\|Q^{\frac12}e_i\|^2_{W^{1,\infty}}<\infty$ when $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x$. Then for every stopping time τ such that $\tau<\widetilde{\tau}_\epsilon^*\wedge T$ a.s., there exists $C(Q,T,\lambda,p,u_0)>0$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,\tau)}\|u^\epsilon(t)\|^p_{\mathbb{H}^1}\Big]\leq C(Q,T,\lambda,p,u_0).$$

Proof. Take any stopping time $\tau<\widetilde{\tau}_\epsilon^*\wedge T$ *a.s.* Applying the Itô formula to the kinetic energy $K(u^\epsilon(t)):=\frac12\|\nabla u^\epsilon(t)\|^2$, and using the integration by parts, we obtain that for $k\in\mathbb{N}^+$ and $t\in[0,\tau]$,

$$\begin{aligned}
&K^k(u^\epsilon(t))\\
&= K^k(u_0) + k\int_0^t K^{k-1}(u^\epsilon(s))\langle \nabla u^\epsilon(s), \mathbf{i}2\lambda f_\epsilon'(|u^\epsilon(s)|^2) Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)\rangle ds\\
&\quad + \frac{1}{2}k(k-1)\int_0^t K^{k-2}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\langle \nabla u^\epsilon(s), \nabla Q^{\frac{1}{2}}e_i\rangle^2 ds\\
&\quad + \frac{k}{2}\int_0^t K^{k-1}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\langle \nabla Q^{\frac{1}{2}}e_i, \nabla Q^{\frac{1}{2}}e_i\rangle ds\\
&\quad + k\int_0^t K^{k-1}(u^\epsilon(s))\langle \nabla u^\epsilon(s), \nabla dW(s)\rangle
\end{aligned}$$

for the additive noise case, and

$$\begin{aligned}
&K^k(u^\epsilon(t))\\
&= K^k(u_0) + k\int_0^t K^{k-1}(u^\epsilon(s))\langle \nabla u^\epsilon(s), \mathbf{i}2\lambda f_\epsilon'(|u^\epsilon(s)|^2) Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)\rangle ds\\
&\quad + \frac{1}{2}k(k-1)\int_0^t K^{k-2}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\langle \nabla u^\epsilon(s), \mathbf{i}\nabla(g(|u^\epsilon(s)|^2)u^\epsilon(s)Q^{\frac{1}{2}}e_i)\rangle^2 ds\\
&\quad + \frac{k}{2}\int_0^t K^{k-1}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\langle \nabla(g(|u^\epsilon(s)|^2)u^\epsilon(s)Q^{\frac{1}{2}}e_i), \nabla(g(|u^\epsilon(s)|^2)u^\epsilon(s)Q^{\frac{1}{2}}e_i)\rangle ds\\
&\quad + k\int_0^t K^{k-1}(u^\epsilon(s))\langle \nabla u^\epsilon(s), \mathbf{i}\nabla g(|u^\epsilon(s)|^2)u^\epsilon(s)dW(s)\rangle\\
&\quad + \frac{k}{2}\int_0^t K^{k-1}(u^\epsilon(s))\Big\langle \Delta u^\epsilon(s), \sum_{i\in\mathbb{N}^+}|Q^{\frac{1}{2}}e_i|^2(g(|u^\epsilon(s)|^2))^2u^\epsilon(s)\Big\rangle ds\\
&\quad - k\int_0^t K^{k-1}(u^\epsilon(s))\\
&\quad \times \Big\langle \Delta u^\epsilon(s), \mathbf{i}\sum_{i\in\mathbb{N}^+} g(|u^\epsilon(s)|^2)g'(|u^\epsilon(s)|^2)|u^\epsilon(s)|^2u^\epsilon(s)Im(Q^{\frac{1}{2}}e_i)Q^{\frac{1}{2}}e_i\Big\rangle ds
\end{aligned}$$

for the multiplicative noise case. Applying integration by parts in the multiplicative noise case, we further obtain

$$\begin{aligned}
&K^k(u^\epsilon(t))\\
&\le K^k(u_0) + k\int_0^t K^{k-1}(u^\epsilon(s))\langle \nabla u^\epsilon(s), \mathbf{i}2\lambda f_\epsilon'(|u^\epsilon(s)|^2) Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)\rangle ds\\
&\quad + C_k\int_0^t K^{k-2}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\Big(\langle \nabla u^\epsilon(s), \mathbf{i}g(|u^\epsilon(s)|^2)u^\epsilon(s)\nabla Q^{\frac{1}{2}}e_i)\rangle^2\\
&\quad + \langle \nabla u^\epsilon(s), \mathbf{i}g'(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)Q^{\frac{1}{2}}e_i)\rangle^2\Big)ds\\
&\quad + C_k\int_0^t K^{k-1}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\Big(\|g(|u^\epsilon(s)|^2)u^\epsilon(s)\nabla Q^{\frac{1}{2}}e_i\|^2\\
&\quad + \|g(|u^\epsilon(s)|^2)\nabla u^\epsilon(s)Q^{\frac{1}{2}}e_i\|^2 + \|g'(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)Q^{\frac{1}{2}}e_i\|^2\Big)ds\\
&\quad + k\int_0^t K^{k-1}(u^\epsilon(s))\langle \nabla u^\epsilon, \mathbf{i}\nabla g(|u^\epsilon(s)|^2)u^\epsilon(s)dW(s)\rangle
\end{aligned}$$

$$+ C_k \int_0^t K^{k-1}(u^\epsilon(s)) \sum_{i\in\mathbb{N}^+} \Big(|\langle \nabla u^\epsilon(s), \mathbf{i}(\nabla Im(Q^{\frac12}e_i)Q^{\frac12}e_i$$
$$+ \nabla Q^{\frac12}e_i Im(Q^{\frac12}e_i))g'(|u^\epsilon(s)|^2)g(|u^\epsilon(s)|^2)|u^\epsilon(s)|^2u^\epsilon(s)\rangle|$$
$$+ |\langle \nabla u^\epsilon(s), \mathbf{i}Im(Q^{\frac12}e_i)Q^{\frac12}e_i g''(|u^\epsilon(s)|^2)g(|u^\epsilon(s)|^2)|u^\epsilon(s)|^2u^\epsilon(s)Re(\bar u^\epsilon(s)\nabla u^\epsilon(s))\rangle|$$
$$+ |\langle \nabla u^\epsilon(s), \mathbf{i}Im(Q^{\frac12}e_i)Q^{\frac12}e_i |g'(|u^\epsilon(s)|^2)|^2|u^\epsilon(s)|^2u^\epsilon(s)Re(\bar u^\epsilon(s)\nabla u^\epsilon(s))\rangle|$$
$$+ |\langle \nabla u^\epsilon(s), \mathbf{i}Im(Q^{\frac12}e_i)Q^{\frac12}e_i g'(|u^\epsilon(s)|^2)g(|u^\epsilon(s)|^2)(2Re(\bar u^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)$$
$$+ |u^\epsilon(s)|^2\nabla u^\epsilon(s))\rangle|\Big)ds.$$

By using the property of g in Assumption 1 and conditions on Q, and applying Hölder's and Burkholder's inequalities, as well as the property $xf'_\epsilon(x) \le 2$ derived from (2.2), we achieve that

$$\mathbb{E}[\sup_{r\in[0,t]} K^k(u^\epsilon(r))]$$
$$\le \mathbb{E}[K^k(u_0)] + C_k|\lambda|\mathbb{E}\Big[\int_0^t \sup_{r\in[0,s]} K^{k-1}(u^\epsilon(r))\|\nabla u^\epsilon(s)\|^2 ds\Big]$$
$$+ C_k\mathbb{E}\Big[\int_0^t \sup_{r\in[0,s]} K^{k-2}(u^\epsilon(r)) \sum_{i\in\mathbb{N}^+} \|\nabla u^\epsilon(s)\|^2(1+\|\nabla u^\epsilon(s)\|^2$$
$$+ \|u^\epsilon(s)\|^2)\|Q^{\frac12}e_i\|_U^2 ds\Big]$$
$$+ C_k\mathbb{E}\Big[\Big(\int_0^t K^{2k-2}(u^\epsilon(s)) \sum_{i\in\mathbb{N}^+} \|\nabla u^\epsilon(s)\|^2(1+\|\nabla u^\epsilon(s)\|^2$$
$$+ \|u^\epsilon(s)\|^2)\|\nabla Q^{\frac12}e_i\|_U^2 ds\Big)^{\frac12}\Big],$$

where $\|Q^{\frac12}e_i\|_U^2 = \|Q^{\frac12}e_i\|_{\mathbb{H}^1}^2$ for the additive noise case and $\|Q^{\frac12}e_i\|_U^2 = \|Q^{\frac12}e_i\|_{\mathbb{H}^1}^2 + \|Q^{\frac12}e_i\|_{W^{1,\infty}}^2$ for the multiplicative noise case. From the definition of $K(u^\epsilon)$, Hölder's and Young's inequalities, and Corollary 1, it follows that for a small $\epsilon_1 > 0$,

$$\mathbb{E}[\sup_{r\in[0,t]} K^k(u^\epsilon(r))]$$
$$\le \mathbb{E}[K^k(u_0)] + C(k,\lambda,Q)\mathbb{E}\Big[\int_0^t \sup_{r\in[0,s]} K^k(u^\epsilon(r))ds\Big]$$
$$+ C(k,\lambda,Q)\Big(1+\mathbb{E}\Big[\int_0^t \|u^\epsilon(s)\|^{2k}ds\Big]\Big) + C(k,\lambda,Q)\mathbb{E}\Big[\Big(\int_0^t K^{2k}(u^\epsilon(s))ds\Big)^{\frac12}\Big]$$
$$\le \mathbb{E}[K^k(u_0)] + C(k,\lambda,Q)\mathbb{E}\Big[\int_0^t \sup_{r\in[0,s]} K^k(u^\epsilon(r))ds\Big] + C(k,\lambda,Q,u_0)$$
$$+ C(k,\lambda,Q)\mathbb{E}\Big[\sup_{r\in[0,t]} K^{\frac{k}{2}}(u^\epsilon(r))\Big(\int_0^t K^k(u^\epsilon(s))ds\Big)^{\frac12}\Big]$$
$$\le \mathbb{E}[K^k(u_0)] + C(k,\lambda,Q)\mathbb{E}\Big[\int_0^t \sup_{r\in[0,s]} K^k(u^\epsilon(r))ds\Big] + C(k,\lambda,Q,u_0)$$
$$+ C(k,\lambda,Q)\epsilon_1\mathbb{E}\Big[\sup_{r\in[0,t]} K^k(u^\epsilon(r))\Big] + C(k,\lambda,Q,\epsilon_1)\mathbb{E}\Big[\int_0^t \sup_{r\in[0,s]} K^k(u^\epsilon(r))ds\Big].$$

Applying Gronwall's inequality and letting $C(k,\lambda,Q)\epsilon_1 < 1$, we complete the proof by taking $p=2k$. ☐

Thanks to the definition of $\widetilde{\tau}_\epsilon^*$ and Chebyshev's inequality, by using Lemma 4, one can get

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^\epsilon(t)\|_{\mathbb{H}^1}^p\Big] \le C(Q,T,\lambda,p,u_0).$$

From the above proofs of Lemmas 3 and 4, it is not hard to see that to obtain ϵ-independent estimates, the boundedness restriction $\sup_{x\ge 0}|g(x)| < \infty$ may not be necessary in the case that $W(t,x)$ is real valued. We present such result in the following which is the key to the global well-posedness of an SlogS equation with superlinear growth diffusion in the next section.

LEMMA 5. *Let $T>0$ and u^ϵ be a local mild solution in $\mathbb{H}^1$ for any $p\ge 1$. Assume that $u_0 \in \mathbb{H}^1 \cap L^2_\alpha$, for some $\alpha\in(0,1]$, is $\mathcal{F}_0$-measurable and has any finite pth moment, and $W(t,x)$ is real valued with $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{W^{1,\infty}} < \infty$. Let $\widetilde{g}(x) = \mathbf{i}g(|x|^2)x$, $g\in\mathcal{C}^1_b([0,\infty))\cap\mathcal{C}([0,\infty))$ satisfy the growth condition, and the embedding condition,*

$$\sup_{x\in[0,\infty)} |g'(x)x| \le C_g, \tag{3.1}$$

$$\|vg(|v|^2)\|_{L^q} \le C_d(1+\|v\|_{\mathbb{H}^1} + \|v\|_{L^2_\alpha}), \tag{3.2}$$

for some $q\ge 2$, where $C_g>0$ depends on g, and $C_d>0$ depends on $\mathcal{O}$, d, $\|v\|$. Then it holds that $M(u^\epsilon(t)) = M(u_0)$ for $t\in[0,\widetilde{\tau}_\epsilon^*\wedge T]$. Furthermore, for every stopping time τ such that $\tau<\widetilde{\tau}_\epsilon^*\wedge T$ a.s., there exists $C(Q,T,\lambda,p,u_0)>0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,\tau)} \|u^\epsilon(t)\|_{\mathbb{H}^1}^p\Big] \le C(Q,T,\lambda,p,u_0).$$

Proof. The proof is similar to those of Lemmas 3 and 4. We only need to modify the estimation involved with g. The mass conservation is not hard to obtain since the calculations in Lemma 3 only use the assumptions that $g\in\mathcal{C}([0,\infty))$ and $W(t,x)$ is real valued. Therefore, we focus on the estimate in $\mathbb{H}^1$. We only show estimation about $\mathbb{E}[K^k(u^\epsilon(t))]$ since the proof on $\mathbb{E}[\sup_{t\in[0,T]} K^k(u^\epsilon(t))]$ is similar. Then following the same steps in Lemma 4, we get that for $\frac{1}{q}+\frac{1}{q'}=\frac12$,

$$\begin{aligned}
&\mathbb{E}\Big[K^k(u^\epsilon(t))\Big]\\
&\quad\le \mathbb{E}\Big[K^k(u_0)\Big]\\
&\qquad + k\mathbb{E}\Big[\int_0^t K^{k-1}(u^\epsilon(s))\langle\nabla u^\epsilon(s), \mathbf{i}2\lambda f_\epsilon'(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)\rangle ds\Big]\\
&\qquad + \mathbb{E}\Big[C_k\int_0^t K^{k-2}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\Big(\langle\nabla u^\epsilon(s), \mathbf{i}g(|u^\epsilon(s)|^2)u^\epsilon(s)\nabla Q^{\frac12}e_i)\rangle^2\\
&\qquad + \langle\nabla u^\epsilon(s), \mathbf{i}g'(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)Q^{\frac12}e_i)\rangle^2\Big)ds\Big]\\
&\qquad + C_k\mathbb{E}\Big[\int_0^t K^{k-1}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\Big(\|g(|u^\epsilon(s)|^2)u^\epsilon(s)\nabla Q^{\frac12}e_i\|^2\\
&\qquad + \|g'(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)Q^{\frac12}e_i\|^2\Big)ds\Big]
\end{aligned}$$

$$\le \mathbb{E}\Big[K^k(u_0)\Big] + C_k\mathbb{E}\Big[\int_0^t K^k(u^\epsilon(s))ds\Big]$$
$$+ \mathbb{E}\Big[C_k\int_0^t K^{k-1}(u^\epsilon(s))\sum_{i\in\mathbb{N}^+}\Big(\|g(|u^\epsilon(s)|^2)u^\epsilon(s)\|_{L^q}^2\|\nabla Q^{\frac12}e_i\|_{L^{q'}}^2$$
$$+ \|\nabla u^\epsilon(s)\|^2\|Q^{\frac12}e_i\|^2\Big)ds\Big].$$

Applying the embedding condition (3.2) to g, the mass conservation law, and the procedures in the proof of Proposition 2, we complete the proof by taking $p=2k$ and using Gronwall's inequality. ∎

In the rest of this paper, we will frequently use the Gagliardo–Nirenberg interpolation inequality, i.e., for $\frac{1}{q}=\frac12-\frac{\gamma}{d}$ and $\gamma\in[0,1]$,

$$\|u\|_{L^q}\le C\|u\|^{1-\gamma}\|\nabla u\|^{\gamma}. \tag{3.3}$$

Here $\gamma\in[0,\frac12)$ if $d=1$ and $\gamma\in[0,1)$ if $d\ge 2$. In particular, we will take $q=2+2\eta'$ for some $\eta'>0$ such that $\gamma=\frac{\eta' d}{2+2\eta'}\in(0,1)$.

It can be seen that the embedding condition (3.2) depends on the assumption on Q and d. One unbounded example of g which satisfies the embedding condition (3.2) is $g(x)=\log(c+x)$ for $x\ge0$ and $c>0$. Let us verify this example on $\mathcal{O}=\mathbb{R}^d$. If the domain $\mathcal{O}$ is bounded, one can obtain a similar estimate. Applying the Gagliardo–Nirenberg interpolation inequality (3.3), the properties that $\log(1+y)\le C_\eta y^\eta$, $\eta>0$ for $y\ge0$, and $|\log(x)|\le C'_\eta x^{-\eta}$, $\eta>0$ for $x\in(0,1)$, we get that for $q>2$ and small enough $\eta>0$,

$$\begin{aligned}
&\|g(|v|^2)v\|_{L^q}^q\\
&\le \int_{c+|v|^2\ge1}|\log(c+|v|^2)|^q|v|^qdx+\int_{c+|v|^2\le1}|\log(c+|v|^2)|^q|v|^qdx\\
&\le C(1+c^q)\Big(\|v\|_{L^q}^q+\|v\|_{L^{q+q\eta}}^{q+q\eta}+\|v\|_{L^{q-q\eta}}^{q-q\eta}\Big)\\
&\le C(1+c^q)\Big(\|v\|^{1-\alpha_0}\|\nabla v\|^{\alpha_0}+(\|v\|^{1-\alpha_1}\|\nabla v\|^{\alpha_1})^{1+\eta}+(\|v\|^{1-\alpha_2}\|\nabla v\|^{\alpha_2})^{1-\eta}\Big)^q\\
&\le C(1+c^q)\Big(\|\nabla v\|^{q\alpha_0}+\|\nabla v\|^{(q+q\eta)\alpha_1}+\|\nabla v\|^{(q-q\eta)\alpha_2}\Big),
\end{aligned}\tag{3.4}$$

where $\alpha_0=\frac{d(q-2)}{2q},\alpha_1=\frac{d(q(1+\eta)-2)}{2q(1+\eta)}$, and $\alpha_2=\frac{d(q(1-\eta)-2)}{2q(1-\eta)}$ satisfy $\alpha_i\in(0,1),i\in\{0,1,2\}$. The assumption (3.2) on g is indeed satisfied for $g(x)=\log(c+x)$ only for some $q>2$ and $\eta>0$ small enough such that all the exponents on the norms in (3.4) are smaller than q. When $q=2$, similar calculations, together with the interpolation inequality in Lemma 6, yield that

$$\begin{aligned}
\|g(|v|^2)v\|^2 &\le C(1+c^2)\Big(\|v\|^2+\|v\|_{L^{2+2\eta}}^{2+2\eta}+\|v\|_{L^{2-2\eta}}^{2-2\eta}\Big)\\
&\le C(1+c^2)\Big(\|\nabla v\|^2+\|\nabla v\|^{(2+2\eta)\alpha_1}+\|v\|_{L^2_{\alpha_2}}^{\frac{d\eta}{\alpha_2}}\Big),
\end{aligned}\tag{3.5}$$

where $\alpha_1=\frac{d(2(1+\eta)-2)}{4(1+\eta)}\in(0,1)$ and $\alpha_2\in(\frac{d\eta}{2-2\eta},1)$. The assumption (3.2) on g is indeed satisfied for $g(x)=\log(c+x)$ only for some $q=2$ and $\eta>0$ small enough such that all the exponents on the norms in (3.5) are smaller than 2.

Remark 1. It can be seen that the constant term in (3.4) or (3.5) is uniformly bounded on $(0,1]$ with respect to c. This, together with similar steps in the proofs of Corollary 1, Lemma 5, and Proposition 2, implies that for any $(c,\epsilon)\in\mathbb{R}^+\times\mathbb{R}^+$, there exists a unique mild solution of

$$du^{\epsilon,c}=\mathbf{i}\Delta u^{\epsilon,c}dt+\mathbf{i}\lambda u^{\epsilon,c}f_\epsilon(|u^{\epsilon,c}|^2)dt-\frac{1}{2}\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2\Big(|\log^2(|u^{\epsilon,c}|^2+c)u^{\epsilon,c}\Big)dt \\ +\mathbf{i}\log(|u^{\epsilon,c}|^2+c)u^{\epsilon,c}dW(t) \tag{3.6}$$

with $u^{\epsilon,c}(0)=u_0$ and $W(\cdot)$ being $L^2(\mathcal{O};\mathbb{R})$ valued. Meanwhile, it holds that for any $p\in\mathbb{N}^+$,

$$\mathbb{E}[\sup_{t\in[0,T]}\|u^{\epsilon,c}(t)\|^p_{L^2_\alpha}]+\mathbb{E}[\sup_{t\in[0,T]}\|u^{\epsilon,c}(t)\|^p_{\mathbb{H}^1}]\le C(Q,T,\lambda,p,u_0).$$

The above result would be preliminary to studying the singular SLogS equation

$$du=\mathbf{i}\Delta udt+\mathbf{i}\lambda u\log(|u|^2)dt-\frac{1}{2}\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2|\log^2(|u|^2)udt \\ +\mathbf{i}\log(|u|^2)udW(t). \tag{3.7}$$

3.2. L^2_α-estimate and modified energy. Beyond the L^2 and $\mathbb{H}^1$ estimates, we also need the uniform boundedness in L^2_α, $\alpha>0$, to show the strong convergence of $\{u^\epsilon\}_{\epsilon>0}$ when $\mathcal{O}=\mathbb{R}^d$. To this end, the following useful weighted interpolation inequality is introduced. We would like to mention that when $\mathcal{O}$ is a bounded domain, such an estimate in L^2_α, $\alpha>0$ is not necessary.

LEMMA 6. *Let $d\in\mathbb{N}^+$ and $\eta\in(0,1)$. Then for $\alpha>\frac{d\eta}{2-2\eta}$, it holds that for some $C=C(d)>0$,*

$$\|v\|_{L^{2-2\eta}}\le C\|v\|^{1-\frac{d\eta}{2\alpha(1-\eta)}}\|v\|^{\frac{d\eta}{2\alpha(1-\eta)}}_{L^2_\alpha},\quad v\in L^2\cap L^2_\alpha.$$

Proof. Using the Cauchy–Schwarz inequality and $\alpha>\frac{d\eta}{2-2\eta}$, we have that for any $r>0$,

$$\begin{aligned}\|v\|^{2-2\eta}_{L^{2-2\eta}}&\le\int_{|x|\le r}|v(x)|^{2-2\eta}dx+\int_{|x|\ge r}\frac{|x|^{\alpha(2-2\eta)}|v(x)|^{2-2\eta}}{|x|^{\alpha(2-2\eta)}}dx\\&\le Cr^{d\eta}\|v\|^{2-2\eta}+C\|v\|^{2-2\eta}_{L^2_\alpha}\Big(\int_{|x|\ge r}\frac{1}{|x|^{\frac{\alpha(2-2\eta)}{\eta}}}dx\Big)^\eta\\&\le Cr^{d\eta}\|v\|^{2-2\eta}+Cr^{-\alpha(2-2\eta)+d\eta}\|v\|^{2-2\eta}_{L^2_\alpha}.\end{aligned}$$

Letting $r=\big(\frac{\|v\|_{L^2_\alpha}}{\|v\|}\big)^{\frac{1}{\alpha}}$, we complete the proof. ∎

PROPOSITION 2. *Let $T>0$, $\mathcal{O}=\mathbb{R}^d$, $d\in\mathbb{N}^+$, and $u_0\in L^2_\alpha\cap\mathbb{H}^1$ for some $\alpha\in(0,1]$, and $\mathcal{F}_0$-measurable with any finite pth moment, $p\in\mathbb{N}^+$. Suppose that $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{\mathbb{H}^1}<\infty$ for the additive noise and $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{W^{1,\infty}}<\infty$ for the multiplicative noise. Then the solution u^ϵ of the regularized problem satisfies, for $\alpha\in(0,1]$,*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^\epsilon(t)\|^{2p}_{L^2_\alpha}\Big]\le C(Q,T,\lambda,p,u_0). \tag{3.8}$$

Proof. We first introduce the stopping time

$$\widehat{\tau}_{\epsilon,R}=\inf\Big\{t\in[0,T]:\sup_{s\in[0,t]}\|u^\epsilon(s)\|_{L^2_\alpha}\geq R\Big\}\wedge T,$$

then show that $\mathbb{E}[\sup_{t\in[0,\widehat{\tau}_{\epsilon,R}]}\|u^\epsilon(t)\|^{2p}_{L^2_\alpha}]\leq C(T,u_0,Q,p)$ independent of R. After taking $R\to\infty$, we get $\widehat{\tau}_{\epsilon,R}\to T$ a.s., by Chebyshev's inequality. For simplicity, we only prove the uniform upper bound when $p=1$.

Taking $0<t\leq t_1\leq\widehat{\tau}_{\epsilon,R}$, and applying the Itô formula to $\|u^\epsilon\|^2_{L^2_\alpha}=\int_{\mathbb{R}^d}(1+|x|^2)^\alpha|u^\epsilon|^2dx$, we get

$$\begin{aligned}&\|u^\epsilon(t)\|^2_{L^2_\alpha}\\&=\|u_0\|^2_{L^2_\alpha}+\int_0^t 2\langle(1+|x|^2)^\alpha u^\epsilon(s),\mathbf{i}\Delta u^\epsilon(s)\rangle ds+\int_0^t 2\sum_{i\in\mathbb{N}^+}\langle(1+|x|^2)^\alpha Q^{\frac12}e_i,Q^{\frac12}e_i\rangle ds\\&\quad+\int_0^t 2\langle(1+|x|^2)^\alpha u^\epsilon(s),\mathbf{i}f_\epsilon(|u^\epsilon(s)|^2)u^\epsilon(s)\rangle ds+\int_0^t 2\langle(1+|x|^2)^\alpha u^\epsilon(s),dW(s)\rangle\end{aligned}$$

for the additive noise case, and

$$\begin{aligned}&\|u^\epsilon(t)\|^2_{L^2_\alpha}\\&=\|u_0\|^2_{L^2_\alpha}+2\int_0^t\langle(1+|x|^2)^\alpha u^\epsilon(s),\mathbf{i}\Delta u^\epsilon(s)+\mathbf{i}f_\epsilon(|u^\epsilon(s)|^2)u^\epsilon(s)\rangle ds\\&\quad-\int_0^t\Big\langle(1+|x|^2)^\alpha u^\epsilon(s),(g(|u^\epsilon(s)|^2))^2u^\epsilon(s)\sum_{i\in\mathbb{N}^+}|Q^{\frac12}e_i|^2\Big\rangle ds\\&\quad+\int_0^t\sum_{i\in\mathbb{N}^+}\langle(1+|x|^2)^\alpha g(|u^\epsilon(s)|^2)u^\epsilon(s)Q^{\frac12}e_i,g(|u^\epsilon(s)|^2)u^\epsilon(s)Q^{\frac12}e_i\rangle ds\\&\quad-2\int_0^t\langle(1+|x|^2)^\alpha u^\epsilon(s),\mathbf{i}\sum_{i\in\mathbb{N}^+}g(|u^\epsilon(s)|^2)g'(|u^\epsilon(s)|^2)|u^\epsilon(s)|^2u^\epsilon(s)Im(Q^{\frac12}e_i)Q^{\frac12}e_i\rangle ds\\&\quad+2\int_0^t\langle(1+|x|^2)^\alpha u^\epsilon(s),\mathbf{i}g(|u^\epsilon(s)|^2)u^\epsilon(s)dW(s)\rangle\end{aligned}$$

for the multiplicative noise case. Using the integration by parts, then taking the supremum over $t\in[0,t_1]$, and applying the Burkholder inequality and Assumption 1, we deduce

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,t_1]}\|u^\epsilon(t)\|^2_{L^2_\alpha}\Big]\\&\leq\mathbb{E}\Big[\|u_0\|^2_{L^2_\alpha}\Big]+C_\alpha\mathbb{E}\Big[\int_0^T\Big|\langle(1+|x|^2)^{\alpha-1}xu^\epsilon(s),\mathbf{i}\nabla u^\epsilon(s)\rangle\Big|ds\Big]+C\int_0^t\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{L^2_\alpha}ds\\&\quad+C\Big(\mathbb{E}\Big[\int_0^{t_1}\sum_{i\in\mathbb{N}^+}\|(1+|x|^2)^{\frac{\alpha}{2}}u^\epsilon(s)\|^2\|(1+|x|^2)^{\frac{\alpha}{2}}Q^{\frac12}e_i\|^2ds\Big]\Big)^{\frac12}\end{aligned}$$

for the additive noise case, and

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,t_1]}\|u^\epsilon(t)\|^2_{L^2_\alpha}\Big]&\leq\mathbb{E}\Big[\|u_0\|^2_{L^2_\alpha}\Big]+C_\alpha\mathbb{E}\Big[\int_0^T\Big|\langle(1+|x|^2)^{\alpha-1}xu^\epsilon(s),\mathbf{i}\nabla u^\epsilon(s)\rangle\Big|ds\Big]\\&\quad+C\Big(\mathbb{E}\Big[\int_0^{t_1}\sum_{i\in\mathbb{N}^+}\|(1+|x|^2)^{\frac{\alpha}{2}}u^\epsilon(s)\|^4\|Q^{\frac12}e_i\|^2_{L^\infty}ds\Big]\Big)^{\frac12}\end{aligned}$$

for the multiplicative noise case. Notice that for $\alpha \in (0,1]$,

$$\|(1+|x|^2)^{\alpha-1} x u^\epsilon\| \le C\|u^\epsilon\|_{L^2_\alpha}.$$

In the additive noise case, Young's and Gronwall's inequalities, together with an a priori estimate of u^ϵ in $\mathbb{H}^1$ and the assumption $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}} e_i\|^2_{L^2_\alpha} < \infty$ yield (3.8). Similar arguments, together with the assumption $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}} e_i\|^2_{L^\infty} < \infty$, lead to the desired result in the multiplicative noise case. ∎

Similarly, we could also obtain the following a priori estimate when g is not bounded. We define the stopping time

$$\widehat{\tau}^*_\epsilon = \lim_{R\to\infty} \widehat{\tau}_{\epsilon,R} = \lim_{R\to\infty} \inf\Big\{t \ge 0 : \sup_{s\in[0,t]} \|u^\epsilon(s)\|_{L^2_\alpha} \ge R\Big\}.$$

Corollary 2. *Under the condition of Lemma* 5*, the solution* u^ϵ *of the regularized problem satisfies, for* $\alpha \in (0,1]$,

$$\mathbb{E}\Big[\sup_{t\in[0,\tau)} \|u^\epsilon(t)\|^{2p}_{L^2_\alpha}\Big] \le C(Q,T,\lambda,p,u_0)$$

for every stopping time τ *such that* $\tau < \widehat{\tau}^*_\epsilon \wedge T$ *a.s.*

From the above uniform estimate, it follows that $\widehat{\tau}^*_\epsilon \wedge T = T$ a.s. However, it is not possible to obtain the uniform bound of the exact solution in L^2_α for $\alpha \in (1,2]$ like in the deterministic case. The main reason is that the rough driving noise leads to low Hölder regularity in time and loss of uniform estimate in $\mathbb{H}^2$ for the mild solution. We cannot expect that the mild solution of (1.4) enjoys an ϵ-independent estimate in $\mathbb{H}^2$. More precisely, we prove that applying the regularization $f_\epsilon(|x|^2) = \log(|x|^2+\epsilon)$ in Proposition 1, one may only expect an ϵ-dependent estimate in $\mathbb{H}^2$. We omit the tedious calculation and procedures, and present a sketch of the proof for Lemma 7 and Propostion 3 in Appendix A.

Lemma 7. *Let* $T > 0$, $d = 1$, $f_\epsilon(|x|^2) = \log(|x|^2+\epsilon)$, *and* $u_0 \in \mathbb{H}^2$ *have a finite* p*-moment for* $p \ge 1$. *Suppose that* $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}} e_i\|^2_{\mathbb{H}^2} < \infty$ *for the additive noise and* $\sum_{i\in\mathbb{N}^+} \|Q^{\frac{1}{2}} e_i\|^2_{W^{2,\infty}} < \infty$ *for the multiplicative noise. Then for any* $p \ge 2$, *there exists a positive constant* $C(Q,T,\lambda,p,u_0)$ *such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^\epsilon(t)\|^{2p}_{\mathbb{H}^2}\Big] \le C(Q,T,\lambda,p,u_0)(1+\epsilon^{-2p}).$$

Proposition 3. *Let* $\mathcal{O} = \mathbb{R}$. *Under the condition of Lemma* 7*, assume that* $u_0 \in L^2_\alpha$ *for some* $\alpha \in (1,2]$ *with a finite* p*-moment for* $p \ge 1$. *Then the solution* $u^\epsilon(t)$ *of the regularized problem satisfies, for* $\alpha \in (1,2]$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^\epsilon(t)\|^{2p}_{L^2_\alpha}\Big] \le C(Q,T,\lambda,p,u_0)(1+\epsilon^{-2p}).$$

The above results indicate that both spatial and temporal regularity for SLogS equation are rougher than deterministic LogS equation.

In the following, we present the behavior of the regularized energy for the RSlogS equation. When applying $f_\epsilon(|x|^2) = \log(|x|^2+\epsilon)$, the modified energy of (1.4) becomes

$$H_\epsilon(u^\epsilon(t)) := K(u^\epsilon(t)) - \frac{\lambda}{2} F_\epsilon(|u^\epsilon(t)|^2)$$

with $F_\epsilon(|u^\epsilon|^2)=\int_{\mathcal{O}}((\epsilon+|u^\epsilon|^2)\log(\epsilon+|u^\epsilon|^2)-|u^\epsilon|^2-\epsilon\log\epsilon)dx$. When using another regularization function $f_\epsilon(|u|^2)=\log(\frac{\epsilon+|u|^2}{1+\epsilon|u|^2})$, its regularized entropy in the modified energy H_ϵ becomes

$$F_\epsilon(|u^\epsilon|^2)=\int_{\mathcal{O}}\Big(|u^\epsilon|^2\log(\frac{|u^\epsilon|^2+\epsilon}{1+|u^\epsilon|^2\epsilon})+\epsilon\log(|u^\epsilon|^2+\epsilon)-\frac{1}{\epsilon}\log(\epsilon|u^\epsilon|^2+1)-\epsilon\log(\epsilon)\Big)dx.$$

In general, the modified energy is defined by the regularized entropy $\widetilde{F}_\epsilon(\rho)=\int_{\mathcal{O}}\int_0^\rho f_\epsilon(s)dsdx$, where $f_\epsilon(\cdot)$ is a suitable approximation of $\log(\cdot)$. We remark that the regularized energy is well-defined when $\mathcal{O}$ is a bounded domain. The additional constant term $\epsilon\log(\epsilon)$ ensures that the regularized energy is still well-defined when $\mathcal{O}=\mathbb{R}^d$. We leave the proof of Proposition 4 to Appendix A.

Proposition 4. *Let $T>0$. Under the conditions of Proposition 2, assume that u^ϵ is the mild solution in $\mathbb{H}^1\cap L^2_\alpha$. Then for any $p\geq 2$, there exists a positive constant $C(Q,T,\lambda,p,u_0)$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|H_\epsilon(u^\epsilon(t))|^p\Big]\leq C(Q,T,\lambda,p,u_0).$$

Below, we conclude the global existence of the unique mild solution for (1.4) based on Proposition 1, Lemma 7, and Lemma 4, as well as a standard argument in [15].

Proposition 5. *Let Assumption 1 hold and $(u^\epsilon,\widetilde{\tau}^*_\epsilon)$ be the mild solution in Lemma 4. Then the mild solution u^ϵ in $\mathbb{H}^1$ is global, i.e., $\widetilde{\tau}^*_\epsilon=+\infty$ a.s. In addition assume that the condition of Lemma 7 holds, then the mild solution $u^\epsilon\in\mathbb{H}^2$ a.s.*

4. Well-posedness for the SLogS equation. Based on the a priori estimates of the regularized problem, we are going to prove the strong convergence of any sequence of the solutions of the regularized problem. This immediately implies the existence and uniqueness of the mild solution for the SLogS equation.

4.1. Well-posedness for the SLogS equation via a strong convergence approximation. In this part, we not only show the strong convergence of a sequence of solutions of regularized problems, but also give the explicit strong convergence rate. The strong convergence rate of the RSLogS equation will make a great contribution to the numerical analysis of numerical schemes for the SLogS equation. Indeed, this topic will be studied in a companion paper (see [13]). For the strong convergence result, we only present the mean square convergence rate result since the proof of the strong convergence rate in $L^q(\Omega), q\geq 2$ is similar. In this section, the properties of regularization function f_ϵ in Lemmas 1 and 8 will be frequently used.

In the multiplicative noise case, Assumption 2 is needed to obtain the strong convergence rate of the solution of (1.4). We remark that the assumption can be weakened if one only wants to obtain the strong convergence instead of deriving a convergence rate. Some sufficient condition for (1.3) in Assumption 2 is

$$\Big|(g'(|x|^2)g(|x|^2)|x|^2-g'(|y|^2)g(|y|^2)|y|^2)(|x|^2-|y|^2)\Big|\leq C_g|x-y|^2, x,y\in\mathbb{C}$$

or

$$\Big|(g'(|x|^2)g(|x|^2)|x|^2-g'(|y|^2)g(|y|^2)|y|^2)(|x|+|y|)\Big|\leq C_g|x-y|, x,y\in\mathbb{C}.$$

Functions like $1,\frac{1}{c+x},\frac{x}{c+x},\frac{x}{c+x^2},\log(\frac{c+|x|^2}{1+c|x|^2})$ with $c>0$, etc., will satisfy Assumption 2.

The main idea of the proof lies in showing that for any decreasing sequence $\{\epsilon_n\}_{n\in\mathbb{N}^+}$ satisfying $\lim\limits_{n\to\infty}\epsilon_n=0$, $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ must be a Cauchy sequence in $C([0,T];L^p(\Omega;\mathbb{H}))$, $p\geq 2$. As a result, we obtain that there exists a limit process u in $C([0,T];L^p(\Omega;\mathbb{H}))$ which is shown to be independent of the sequence $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ and is the unique mild solution of (1.1).

Proof of Theorem 1. Based on Proposition 5, we can construct a sequence of mild solutions $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ of (1.4) with $f_{\epsilon_n}(|x|^2)=\log(\frac{\epsilon_n+|x|^2}{1+\epsilon_n|x|^2})$. Here the decreasing sequence $\{\epsilon_n\}_{n\in\mathbb{N}^+}$ satisfies $\lim_{n\to\infty}\epsilon_n=0$. We use the following steps to complete the proof. For simplicity, we only present the details for $p=2$ since the procedures for $p>2$ are similar.

Step 1*:* $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $L^2(\Omega;C([0,T];\mathbb{H}))$.

Fix different $n,m\in\mathbb{N}^+$ such that $n<m$. Subtracting the equation of u^{ϵ_n} from the equation of u^{ϵ_m}, we have that

$$d(u^{\epsilon_m}-u^{\epsilon_n})=\mathbf{i}\Delta(u^{\epsilon_m}-u^{\epsilon_n})dt+\mathbf{i}\lambda(f_{\epsilon_m}(|u^{\epsilon_m}|^2)u^{\epsilon_m}-f_{\epsilon_n}(|u^{\epsilon_n}|^2)u^{\epsilon_n})dt$$

for the additive noise case, and

$$\begin{aligned}
&d(u^{\epsilon_m}-u^{\epsilon_n})\\
&\quad=\mathbf{i}\Delta(u^{\epsilon_m}-u^{\epsilon_n})dt+\mathbf{i}\lambda(f_{\epsilon_m}(|u^{\epsilon_m}|^2)u^{\epsilon_m}-f_{\epsilon_n}(|u^{\epsilon_n}|^2)u^{\epsilon_n})dt\\
&\qquad-\frac{1}{2}\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2\Big(|g(|u^{\epsilon_m}|^2)|^2u^{\epsilon_m}-|g(|u^{\epsilon_n}|^2)|^2u^{\epsilon_n}\Big)dt\\
&\qquad-\mathbf{i}\sum_{k\in\mathbb{N}^+}Im(Q^{\frac12}e_k)Q^{\frac12}e_k\Big(g'(|u^{\epsilon_m}|^2)g(|u^{\epsilon_m}|^2)|u^{\epsilon_m}|^2u^{\epsilon_m}\\
&\qquad-g'(|u^{\epsilon_n}|^2)g(|u^{\epsilon_n}|^2)|u^{\epsilon_n}|^2u^{\epsilon_n}\Big)dt\\
&\qquad+\mathbf{i}\Big(g(|u^{\epsilon_m}|^2)u^{\epsilon_m}-g(|u^{\epsilon_n}|^2)u^{\epsilon_n}\Big)dW(t)
\end{aligned}$$

for the multiplicative noise case. Then using the Itô formula to $\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2$, (2.3) in Lemma 1, the property of logarithmic function $\log(1+x)\leq C_{\eta'}x^{\eta'}$, $\eta'>0$ for $x\geq 0$, we obtain that

$$\begin{aligned}
(4.1)\quad&\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\\
&=\int_0^t 2\langle u^{\epsilon_m}(s)-u^{\epsilon_n},\mathbf{i}\lambda(f_{\epsilon_m}(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)-f_{\epsilon_n}(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s))\rangle ds\\
&\leq\int_0^t 4|\lambda|\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds\\
&\quad+4|\lambda|\int_0^t|\langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s),\mathbf{i}(f_{\epsilon_m}(|u^{\epsilon_n}(s)|^2)-f_{\epsilon_n}(|u^{\epsilon_n}(s)|^2))u^{\epsilon_n}(s)\rangle|ds\\
&\leq\int_0^t 6|\lambda|\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds\\
&\quad+4|\lambda|\int_0^t\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|_{L^1}\Big\|\log(1+\frac{\epsilon_n-\epsilon_m}{\epsilon_m+|u^{\epsilon_n}(s)|^2})|u^{\epsilon_n}(s)|\Big\|_{L^\infty}ds\\
&\quad+2|\lambda|\int_0^t\|\log(1+\frac{(\epsilon_n-\epsilon_m)|u^{\epsilon_n}(s)|^2}{1+\epsilon_m|u^{\epsilon_n}(s)|^2})|u^{\epsilon_n}(s)|\|^2ds\\
&\leq\int_0^t 6|\lambda|\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds
\end{aligned}$$

$$
\begin{aligned}
&+4|\lambda|\int_0^t \|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|_{L^1}\Big\|\frac{\sqrt{\epsilon_n-\epsilon_m}}{\sqrt{\epsilon_m+|u^{\epsilon_n}(s)|^2}}|u^{\epsilon_n}(s)|\Big\|_{L^\infty}ds\\
&+2|\lambda|\int_0^t \|\log(1+\frac{(\epsilon_n-\epsilon_m)|u^{\epsilon_n}(s)|^2}{1+\epsilon_m|u^{\epsilon_n}(s)|^2})|u^{\epsilon_n}(s)|\|^2ds\\
\le&\int_0^t 6|\lambda|\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds+4|\lambda|\epsilon_n^{\frac12}\int_0^t\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|_{L^1}ds\\
&+2|\lambda|C\epsilon_n^{\eta'}\int_0^t\|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds
\end{aligned}
$$

for the additive noise case, and

$$
\begin{aligned}
&\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\\
&=\int_0^t 2\langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s),\mathbf{i}\lambda(f_{\epsilon_m}(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)-f_{\epsilon_n}(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s))\rangle ds\\
&\quad-\int_0^t\langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s),\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2\Big(|g(|u^{\epsilon_m}(s)|^2)|^2u^{\epsilon_m}(s)\\
&\quad-|g(|u^{\epsilon_n}(s)|^2)|^2u^{\epsilon_n}(s)\Big)\rangle ds\\
&\quad-2\int_0^t\langle u^{\epsilon_m}(s)\\
&\quad-u^{\epsilon_n}(s),\mathbf{i}\sum_{k\in\mathbb{N}^+}Im(Q^{\frac12}e_k)Q^{\frac12}e_k\Big(g'(|u^{\epsilon_m}(s)|^2)g(|u^{\epsilon_m}(s)|^2)|u^{\epsilon_m}(s)|^2u^{\epsilon_m}(s)\\
&\quad-g'(|u^{\epsilon_n}(s)|^2)g(|u^{\epsilon_n}(s)|^2)|u^{\epsilon_n}(s)|^2u^{\epsilon_n}(s)\Big)\rangle ds\\
&\quad+2\int_0^t\langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s),\mathbf{i}\Big(g(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)-g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)\Big)dW(s)\rangle\\
&\quad+\int_0^t\langle g(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)\\
&\quad-g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s),\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2\Big(g(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)-g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)\Big)\rangle ds\\
&\le\int_0^t 6|\lambda|\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds+4|\lambda|\epsilon_n^{\frac12}\int_0^t\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|_{L^1}ds\\
&\quad+2|\lambda|C\epsilon_n^{\eta'}\int_0^t\|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds\\
&\quad+2\int_0^t\langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s),\mathbf{i}\Big(g(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)-g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)\Big)dW(s)\rangle\\
&\quad+\int_0^t\langle(g(|u^{\epsilon_m}(s)|^2)-g(|u^{\epsilon_n}(s)|^2))u^{\epsilon_n}(s),\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2(g(|u^{\epsilon_m}(s)|^2)\\
&\quad-g(|u^{\epsilon_n}(s)|^2))u^{\epsilon_m}(s)\rangle ds\\
&\quad-2\int_0^t\langle u^{\epsilon_m}(s)\\
&\quad-u^{\epsilon_n}(s),\mathbf{i}\sum_{k\in\mathbb{N}^+}Im(Q^{\frac12}e_k)Q^{\frac12}e_k\Big(g'(|u^{\epsilon_m}(s)|^2)g(|u^{\epsilon_m}(s)|^2)|u^{\epsilon_m}(s)|^2u^{\epsilon_m}(s)\\
&-g'(|u^{\epsilon_n}(s)|^2)g(|u^{\epsilon_n}(s)|^2)|u^{\epsilon_n}(s)|^2u^{\epsilon_n}(s)\Big)\rangle ds
\end{aligned}
$$

for the multiplicative noise case. By using (1.2) and (1.3) in Assumption 2 and the assumptions on Q, we have

(4.2)
$$\begin{aligned}
&\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\\
&\quad\le \int_0^t (6|\lambda|+C(g,Q))\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2 ds + 4|\lambda|\epsilon_n^{\frac{1}{2}}\int_0^t \|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|_{L^1}ds\\
&\qquad + 2|\lambda|C\epsilon_n^{\eta'}\int_0^t \|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds\\
&\qquad + \int_0^t \langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s), \mathbf{i}\Big(g(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)-g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)\Big)dW(s)\rangle.
\end{aligned}$$

Next we show the strong convergence of the sequence $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ in the following different cases.

Case 1*:* $\mathcal{O}$ is a bounded domain. By using the Hölder inequality $\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|_{L^1}\le |\mathcal{O}|^{\frac{1}{2}}\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|$ on (4.1) and (4.2), and using Gronwall's inequality, we get

$$\sup_{t\in[0,T]}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2 \le C(\lambda,T,|\mathcal{O}|)(\epsilon_n+\epsilon_n^{\eta'})(1+\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)\|_{L^{2+2\eta'}}^{2+2\eta'})$$

for the additive noise case. Applying (3.3), Corollary 1, and Lemma 4, it holds that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\le C(\lambda,T,|\mathcal{O}|,u_0)(\epsilon_n+\epsilon_n^{\eta'}).$$

In the multiplicative noise case, taking the supremum over t, then taking the expectation on (4.2), and adopting (3.3), Corollary 1, and Lemma 4, together with the Burkholder and Young inequalities, we get that, for a small $\kappa>0$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\\
&\quad\le C(\lambda,T,|\mathcal{O}|,Q)(\epsilon_n+\epsilon_n^{\eta'})\\
&\qquad + C\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s),\mathbf{i}\Big(g(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)\\
&\qquad - g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)\Big)dW(s)\rangle\Big|\Big]\\
&\quad\le C(\lambda,T,|\mathcal{O}|,Q)(\epsilon_n+\epsilon_n^{\eta'}) + C\mathbb{E}\Big[\Big(\int_0^T\sum_{i\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_i\|_{L^\infty}^2\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^4ds\Big)^{\frac{1}{2}}\Big]\\
&\quad\le C(\lambda,T,|\mathcal{O}|,Q)(\epsilon_n+\epsilon_n^{\eta'})\\
&\qquad + C\mathbb{E}\Big[\sup_{s\in[0,T]}\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|\Big(\int_0^T\sum_{i\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_i\|_{L^\infty}^2\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds\Big)^{\frac{1}{2}}\Big]\\
&\quad\le C(\lambda,T,|\mathcal{O}|,Q)(\epsilon_n+\epsilon_n^{\eta'}) + \kappa\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\\
&\qquad + C(\kappa)\mathbb{E}\Big[\int_0^T\sum_{i\in\mathbb{N}^+}\|Q^{\frac{1}{2}}e_i\|_{L^\infty}^2\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds\Big].
\end{aligned}$$

Taking $\kappa < \frac{1}{2}$ and applying Gronwall's inequality, we have that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2\Big] \le C(Q,T,\lambda,p,u_0,|\mathcal{O}|)(\epsilon_n + \epsilon_n^{\eta'}).$$

Case 2: $\mathcal{O} = \mathbb{R}^d$. In this case, slightly different from (4.1) and (4.2), we bound the term $4|\lambda|\int_0^t |\langle u^{\epsilon_m}(s) - u^{\epsilon_n}(s), \mathbf{i}(f_{\epsilon_m}(|u^{\epsilon_n}(s)|^2) - f_{\epsilon_n}(|u^{\epsilon_n}(s)|^2))u^{\epsilon_n}(s)\rangle|ds$ as follows.

Using the fact that $\log(1+y) \le C_\kappa y^\kappa, y \ge 0$, for any $\kappa \in (0,1]$, taking $\kappa = \frac{\eta}{2} \in (0, \frac{\alpha}{2\alpha+d})$ and $\kappa = \eta' \in (0,1)$ such that $\frac{\eta' d}{2\eta'+2} \in [0,1)$, and applying Hölder's inequality, we have

$$\begin{aligned}
&4|\lambda|\int_0^t |\langle u^{\epsilon_m}(s) - u^{\epsilon_n}(s), \mathbf{i}(f_{\epsilon_m}(|u^{\epsilon_n}(s)|^2) - f_{\epsilon_n}(|u^{\epsilon_n}(s)|^2))u^{\epsilon_n}(s)\rangle|ds\\
&\le 4|\lambda|\int_0^t\int_{\mathcal{O}} |u^{\epsilon_m}(s) - u^{\epsilon_n}(s)||\log(1+\frac{\epsilon_n-\epsilon_m}{\epsilon_m+|u^{\epsilon_n}(s)|^2})u^{\epsilon_n}(s)|dxds\\
&\quad + 2|\lambda|\int_0^t \|\log(1+\frac{(\epsilon_n-\epsilon_m)|u^{\epsilon_n}(s)|^2}{1+\epsilon_m|u^{\epsilon_n}(s)|^2})|u^{\epsilon_n}(s)|\|^2 ds\\
&\quad + 2|\lambda|\int_0^t \|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 ds\\
&\le 4|\lambda|\int_0^t\int_{\mathcal{O}} \epsilon_n^{\frac{\eta}{2}}|u^{\epsilon_n}(s)|(\epsilon_m+|u^{\epsilon_n}(s)|^2)^{-\frac{\eta}{2}}|u^{\epsilon_n}(s) - u^{\epsilon_m}(s)|dsdx\\
&\quad + 2|\lambda|C\epsilon_n^{\eta'}\int_0^t \|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds + 2|\lambda|\int_0^t \|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 ds\\
&\le 4|\lambda|\int_0^t \epsilon_n^{\frac{\eta}{2}}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|\|u^{\epsilon_n}(s)\|_{L^{2-2\eta}}^{1-\eta}ds\\
&\quad + 2|\lambda|C\epsilon_n^{\eta'}\int_0^t \|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds + 2|\lambda|\int_0^t \|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 ds.
\end{aligned}$$

Since $u_0 \in L^2_\alpha, \alpha \in (0,1]$, using the interpolation inequality in Lemma 6 implies that for $\alpha > \frac{\eta d}{2(1-\eta)}$,

$$\begin{aligned}
&\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2\\
&\le \int_0^t 6|\lambda|\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 ds + 4|\lambda|\int_0^t \epsilon_n^{\frac{\eta}{2}}\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|\|u^{\epsilon_n}(s)\|_{L^{2-2\eta}}^{1-\eta}ds\\
&\quad + 2|\lambda|C\epsilon_n^{\eta'}\int_0^t \|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds\\
&\le \int_0^t 6|\lambda|\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 ds + 2|\lambda|\int_0^t \|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2\\
&\quad + 2|\lambda|\epsilon_n^{\eta}\int_0^t \|u^{\epsilon_n}(s)\|_{L^{2-2\eta}}^{2-2\eta}ds + 2|\lambda|C\epsilon_n^{\eta'}\int_0^t \|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds\\
&\le \int_0^t 8|\lambda|\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 ds + 2|\lambda|C\epsilon_n^{\eta}\int_0^t \|u^{\epsilon_n}(s)\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}}\|u^{\epsilon_n}(s)\|^{2-2\eta-\frac{d\eta}{\alpha}}ds\\
&\quad + 2|\lambda|C\epsilon_n^{\eta'}\int_0^t \|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds
\end{aligned}$$

for the additive noise case, and

$$\begin{aligned}&\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\\&\quad\le \int_0^t C\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2ds + C\epsilon_n^{\eta}\int_0^t \|u^{\epsilon_n}(s)\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}}\|u^{\epsilon_n}(s)\|^{2-2\eta-\frac{d\eta}{\alpha}}ds\\&\qquad+2|\lambda|C\epsilon_n^{\eta'}\int_0^t\|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}ds\\&\qquad+\Big|\int_0^t\langle u^{\epsilon_m}(s)-u^{\epsilon_n}(s),\mathbf{i}(g(|u^{\epsilon_m}(s)|^2)u^{\epsilon_m}(s)-g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s))dW(s)\rangle\Big|\end{aligned}$$

for the multiplicative noise case. Then taking the supremum over t, taking the expectation, using (1.2), Lemma 3, and Proposition 2, and applying Gronwall's inequality, as well as Lemma 6 and (3.3), we have that, for $\alpha\in(0,1]$, $\eta\in(0,\frac{2\alpha}{2\alpha+d})$, and $\frac{\eta' d}{2\eta'+2}\in(0,1)$,

(4.3)
$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\\&\quad\le C(T,Q,u_0,g)\mathbb{E}\Big[\sup_{s\in[0,T]}\Big(\|u^{\epsilon_n}(s)\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}}\|u^{\epsilon_n}(s)\|^{2-2\eta-\frac{d\eta}{\alpha}}+\|u^{\epsilon_n}(s)\|_{L^{2+2\eta'}}^{2+2\eta'}\Big)\Big](\epsilon_n^{\eta}+\epsilon_n^{\eta'})\\&\quad\le C(T,Q,u_0,g,\alpha,\eta)\epsilon_n^{\min(\eta,\eta')}.\end{aligned}$$

Step 2*:* The limit process u of $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ in $\mathbb{M}^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{H}))$ satisfies (1.1) in mild form. We use the multiplicative noise case to present all the detailed procedures. It suffices to prove that each term in the mild form of the RSlogS equation (1.4) converges to the corresponding part in

$$\begin{aligned}&S(t)u_0+\mathbf{i}\lambda\int_0^t S(t-s)\log(|u(s)|^2)u(s)ds\\&\quad-\frac{1}{2}\int_0^t S(t-s)(g(|u(s)|^2))^2u(s)\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2ds\\&\quad-\mathbf{i}\int_0^t S(t-s)g'(|u(s)|^2)g(|u(s)|^2)|u(s)|^2u(s)\sum_{k\in\mathbb{N}^+}Im(Q^{\frac12}e_k)Q^{\frac12}e_kds\\&\quad+\mathbf{i}\int_0^t S(t-s)g(|u(s)|^2)u(s)dW(s)\\&=:S(t)u_0+V_1+V_2+V_3+V_4.\end{aligned}$$

We first claim that all the terms V_1–V_4 make sense. By Lemma 4 and Proposition 2, we have that for $p\ge 2$,

$$\sup_n\mathbb{E}\left[\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)\|_{\mathbb{H}^1}^p\right]+\sup_n\mathbb{E}\left[\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)\|_{L^2_\alpha}^p\right]\le C(u_0,T,Q).$$

By applying the Fourier transform and Parseval's theorem, using the Fatou lemma, and strong convergence of $(u^{\epsilon_n})_{n\in\mathbb{N}^+}$ in $\mathbb{M}^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{H}))$, we obtain

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)\|_{\mathbb{H}^1}^2\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)\|_{L^2_\alpha}^2\Big]\\&\quad\le\sup_n\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)\|_{\mathbb{H}^1}^2\Big]+\sup_n\mathbb{E}[\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)\|_{L^2_\alpha}^2]\le C(u_0,T,Q).\end{aligned}$$

Then (3.3) yields that for small $\eta', \eta > 0$,

$$\begin{aligned}\|\log(|u|^2)u\|^2 &= \int_{|u|^2\geq 1} (\log(|u|^2))^2|u|^2dx + \int_{|u|^2\leq 1} (\log(|u|^2))^2|u|^2dx \\ &\leq \int_{|u|^2\geq 1} |u|^{2+2\eta'}dx + \int_{|u|^2\leq 1} |u|^{2-2\eta}dx \\ &\leq C(\|u\|_{L^{2+2\eta'}}^{2+2\eta'} + \|u\|_{L^{2-2\eta}}^{2-2\eta}) \\ &\leq C(\|u\|_{L^{2-2\eta}}^{2-2\eta} + \|\nabla u\|^{d\eta'}\|u\|^{2\eta'+2-d\eta'}).\end{aligned}$$

When $\mathcal{O} = \mathbb{R}^d$, we use the weighted version of the interpolation inequality in Lemma 6 to deal with the term $\|u\|_{L^{2-2\eta}}^{2-2\eta}$, and have that for small $\eta < \frac{2\alpha}{2\alpha+d}$,

$$\|\log(|u|^2)u\|^2 \leq C(\|\nabla u\|^{d\eta'}\|u\|^{2\eta'+2-d\eta'} + \|u\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}}\|u\|^{2-2\eta-\frac{d\eta}{\alpha}}).$$

This implies that V_1 makes sense in $\mathbb{M}^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{H}))$ by Proposition 2 and Lemmas 3 and 4. Meanwhile, we can show that V_2–$V_4 \in \mathbb{M}^2_{\mathcal{F}}(\Omega;C([0,T];\mathbb{H}))$ by using the Minkowski and Burkholder inequalities due to our assumptions on g and Q.

Next, we show that the mild form of u^{ϵ_n} converges to $S(t)u_0 + V_1 + V_2 + V_3 + V_4$. To prove that

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} \Big\| \int_0^t S(t-s)\mathbf{i}\lambda f_{\epsilon_n}(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)ds - V_1\Big\|^2\Big] = 0,$$

we use the following decomposition of $f_{\epsilon_n}(|u^{\epsilon_n}|^2)u^{\epsilon_n} - \log(|u|^2)u$. When $|u| > |u^{\epsilon_n}|$,

$$\begin{aligned}&f_{\epsilon_n}(|u^{\epsilon_n}|^2)u^{\epsilon_n} - \log(|u|^2)u \\ &\quad = (f_{\epsilon_n}(|u^{\epsilon_n}|^2) - f_{\epsilon_n}(|u|^2))u^{\epsilon_n} + f_{\epsilon_n}(|u|^2)(u^{\epsilon_n} - u) + (f_{\epsilon_n}(|u|^2) - \log(|u|^2))u,\end{aligned}$$

and when $|u| < |u^{\epsilon_n}|$,

$$\begin{aligned}&f_{\epsilon_n}(|u^{\epsilon_n}|^2)u^{\epsilon_n} - \log(|u|^2)u \\ &\quad = (\log(|u^{\epsilon_n}|^2) - \log(|u|^2))u + \log(|u^{\epsilon_n}|^2)(u^{\epsilon_n} - u) + (f_{\epsilon_n}(|u^{\epsilon_n}|^2) - \log(|u^{\epsilon_n}|^2))u^{\epsilon_n}.\end{aligned}$$

For convenience, let us show the estimate for $|u| > |u^{\epsilon_n}|$; the other case will be estimated in a similar way. By using the Hölder inequality, the mean-value theorem, as well as (2.2), we have that for small $\gamma > 0$,

$$\begin{aligned}&\Big|(f_{\epsilon_n}(|u^{\epsilon_n}|^2) - f_{\epsilon_n}(|u|^2))u^{\epsilon_n}\Big| \\ &\quad\leq \Big|(f_{\epsilon_n}(|u|^2) - f_{\epsilon_n}(|u^{\epsilon_n}|^2))^{\frac12-\frac12\gamma}(f_{\epsilon_n}(|u|^2) - f_{\epsilon_n}(|u^{\epsilon_n}|^2))^{\frac12+\frac12\gamma}u^{\epsilon_n}\Big| \\ &\quad\leq C\Big|(|u| - |u^{\epsilon_n}|)^{\frac12-\frac12\gamma}(|u| + |u^{\epsilon_n}|)^{\frac12-\frac12\gamma}\frac{|u^{\epsilon_n}|}{(\epsilon_n + |u^{\epsilon_n}|^2)^{\frac12-\frac12\gamma}}(|f_{\epsilon_n}(|u^{\epsilon_n}|^2)| \\ &\qquad + |f_{\epsilon_n}(|u|^2)|)^{\frac12+\frac12\gamma}\Big|.\end{aligned}$$

This implies that for small enough $\eta > 0$,

$$\begin{aligned}
&\int_{|u|>|u^{\epsilon_n}|} \Big|(f_{\epsilon_n}(|u^{\epsilon_n}|^2) - f_{\epsilon_n}(|u|^2))u^{\epsilon_n}\Big|^2 dx\\
&\le C\int_{|u|>|u^{\epsilon_n}|} \frac{|u^{\epsilon_n}|^2}{(\epsilon_n+|u^{\epsilon_n}|^2)^{1-\gamma}}\\
&\quad \times |u-u^{\epsilon_n}|^{1-\gamma}(|u|+|u^{\epsilon_n}|)^{1-\gamma}(|f_{\epsilon_n}(|u^{\epsilon_n}|^2)|+|f_{\epsilon_n}(|u|^2)|)^{1+\gamma}dx\\
&\le C\int_{|u|>|u^{\epsilon_n}|,\epsilon_n+|u^{\epsilon_n}|^2\le 1} \frac{|u^{\epsilon_n}|^2}{(\epsilon_n+|u^{\epsilon_n}|^2)^{1-\gamma}}|u-u^{\epsilon_n}|^{1-\gamma}\\
&\quad \times (|u|+|u^{\epsilon_n}|)^{1-\gamma}(|f_{\epsilon_n}(|u^{\epsilon_n}|^2)|+|f_{\epsilon_n}(|u|^2)|)^{1+\gamma}dx\\
&\quad + C\int_{|u|>|u|^{\epsilon_n},\epsilon_n+|u^{\epsilon_n}|^2\ge 1} \frac{|u^{\epsilon_n}|^2}{(\epsilon_n+|u^{\epsilon_n}|^2)^{1-\gamma}}\\
&\quad \times |u-u^{\epsilon_n}|^{1-\gamma}(|u|+|u^{\epsilon_n}|)^{1-\gamma}(|f_{\epsilon_n}(|u^{\epsilon_n}|^2)|+|f_{\epsilon_n}(|u|^2)|)^{1+\gamma}dx\\
&\le C\int_{|u|>|u^{\epsilon_n}|,\epsilon_n+|u^{\epsilon_n}|^2\le 1} \frac{|u^{\epsilon_n}|^2}{(\epsilon_n+|u^{\epsilon_n}|^2)^{1-\gamma}}\\
&\quad \times |u-u^{\epsilon_n}|^{1-\gamma}(|u|+|u^{\epsilon_n}|)^{1-\gamma}((\epsilon_n+|u^{\epsilon_n}|^2)^{-\eta}+(\epsilon_n+|u|^2)^{\eta})dx\\
&\quad + C\int_{|u|>|u|^{\epsilon_n},\epsilon_n+|u^{\epsilon_n}|^2\ge 1} \frac{|u^{\epsilon_n}|^2}{(\epsilon_n+|u^{\epsilon_n}|^2)^{1-\gamma}}(|u|+|u^{\epsilon_n}|)^{1-\gamma}|u-u^{\epsilon_n}|^{1-\gamma}(\epsilon_n+|u|^2)^{\eta}dx.
\end{aligned}$$

Now choosing $1-\gamma+2\eta \le 1$, using the Hölder inequality, and the weighted interpolation inequality in Lemma 6, as well as the Gagliardo–Nirenberg interpolation inequality (3.3), we have that for $\alpha_1 \in (\frac{(\gamma-\eta)d}{1+\gamma-2\eta}, 1)$, $\alpha_2 \in (\frac{d\gamma}{1-\gamma}, 1)$, and $\alpha_3 = \frac{\gamma+2\eta}{1+\gamma+2\eta}d \in (0,1)$,

$$\begin{aligned}
&\int_{|u|>|u^{\epsilon_n}|} \Big|(f_{\epsilon_n}(|u^{\epsilon_n}|^2) - f_{\epsilon_n}(|u|^2))u^{\epsilon_n}\Big|^2 dx\\
&\le C\int_{\mathcal{O}} |u|^{2\gamma-2\eta}|u-u^{\epsilon_n}|^{1-\gamma}|u|^{1-\gamma}dx\\
&\quad + C\int_{\mathcal{O}}(1+|u|^{2\gamma+2\eta})|u-u^{\epsilon_n}|^{1-\gamma}|u|^{1-\gamma}dx\\
&\le C\|u-u^{\epsilon_n}\|^{1-\gamma}\Big(\|u\|^{1-\gamma}_{L^{\frac{2(1-\gamma)}{1+\gamma}}} + \|u\|^{1+\gamma-2\eta}_{L^{\frac{2(1-\gamma+2\eta)}{1+\gamma}}} + \|u\|^{1+\gamma+2\eta}_{L^{\frac{2(1+\gamma+2\eta)}{1+\gamma}}}\Big)\\
&\le C\|u-u^{\epsilon_n}\|^{1-\gamma}\Big(\|u\|^{1-\gamma-\frac{d\gamma}{\alpha_2}}\|u\|^{\frac{d\gamma}{\alpha_2}}_{L^2_\alpha} + \|u\|^{\frac{d(\gamma-\eta)}{\alpha}}_{L^2_\alpha}\|u\|^{1+\gamma-2\eta-\frac{d(\gamma-\eta)}{\alpha}}\\
&\quad + \|u\|^{(1+\gamma+2\eta)(1-\alpha_3)}\|\nabla u\|^{(1+\gamma+2\eta)\alpha_3}\Big).
\end{aligned}$$

For the integral term of $f_{\epsilon_n}(|u|^2)(u^{\epsilon_n}-u)$, according to the property that $\log(1+y) \le C_{\eta'}y^{\eta'}$, $\eta' \in (0,1)$ for $y \ge 0$ and $|\log(x)| \le C_\eta x^{-\eta}$, $\eta > 0$ for $x \in (0,1)$, using (3.3) and Lemma 6, it follows that for $\frac{\eta' d}{2\eta'+2} \in (0,1)$ and $\eta < \frac{2\alpha}{2\alpha+d}$,

$$\begin{aligned}
&\int_{|u|>|u^{\epsilon_n}|} |f_{\epsilon_n}(|u|^2)(u^{\epsilon_n}-u)|^2 dx\\
&\le (C_{\frac{\eta}{4}})^2\int_{|u|>|u^{\epsilon_n}|,\epsilon_n+|u|^2\le 1}(\epsilon_n+|u|^2)^{-\frac{\eta}{2}}|u^{\epsilon_n}-u|^2dx\\
&\quad + (C_{\frac{\eta'}{4}})^2\int_{|u|>|u^{\epsilon_n}|,\epsilon_n+|u|^2\ge 1}(\epsilon_n+|u|^2)^{\frac{\eta'}{2}}|u^{\epsilon_n}-u|^2dx\\
&\quad + (C_{\frac{\eta'}{4}})^2\int_{|u|>|u^{\epsilon_n}|}\epsilon_n^{\frac{\eta'}{2}}|u|^{\eta'}|u^{\epsilon_n}-u|^2dx
\end{aligned}$$

$$\leq C\|u^{\epsilon_n}-u\|\Big(\|u^{\epsilon_n}-u\|+\|u\|^{1+\eta'}_{L^{2+2\eta'}}+\|u\|^{1-\eta}_{L^{2-2\eta}}\Big)$$
$$\leq C\|u^{\epsilon_n}-u\|\Big(\|u^{\epsilon_n}-u\|+\|\nabla u\|^{\frac{\eta' d}{2}}\|u\|^{1+\eta'-\frac{\eta' d}{2}}+\|u\|^{\frac{d\eta}{2\alpha}}_{L^2_\alpha}\|u\|^{1-\eta-\frac{d\eta}{2\alpha}}\Big).$$

For the integral term of $(f_{\epsilon_n}(|u|^2)-\log(|u|^2))u$, making use of similar arguments as above and (3.3) with $q=2\eta'+2$ and $\gamma=\frac{\eta' d}{2\eta'+2}$ yield that for η' such that $\gamma\in[0,1)$ and $\eta<\frac{\alpha}{2\alpha+d}$,

$$\begin{aligned}&\int_{\mathcal{O}}|(f_{\epsilon_n}(|u|^2)-\log(|u|^2))u|^2dx\\&\quad\leq C\epsilon_n^{\eta'}\|u\|^{2\eta'+2}_{L^{2\eta'+2}}+C\epsilon_n^{\eta}\|u\|^{2-2\eta}_{L^{2-2\eta}}\\&\quad\leq C\Big(\epsilon_n^{\eta'}\|\nabla u\|^{\eta' d}\|u\|^{2+2\eta'-\eta' d}+\epsilon_n^{\eta}\|u^{\epsilon_n}\|^{\frac{d\eta}{\alpha}}_{L^2_\alpha}\|u^{\epsilon_n}\|^{2-2\eta-\frac{d\eta}{\alpha}}\Big).\end{aligned}$$

Combining the above estimates, using the a priori estimate of u^{ϵ_n} and u in Lemmas 3 and 4 and Proposition 2, and applying the strong convergence (4.3) of u^{ϵ_n}, we obtain that

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)\big(f_{\epsilon_n}(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)-\log(|u(s)|^2)u(s)\big)ds\Big\|^2\Big]=0.$$

The Minkowski inequality and Assumption 1 yield that

$$\begin{aligned}&\Big\|-\int_0^t S(t-s)\frac{1}{2}(g(|u^{\epsilon_n}(s)|^2))^2u^{\epsilon_n}(s)\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2ds-V_2\Big\|\\&\quad\leq\sum_{k\in\mathbb{N}^+}\|Q^{\frac12}e_k\|^2_{L^\infty}\int_0^T\|g(|u^{\epsilon_n}(s)|^2))^2u^{\epsilon_n}(s)-g(|u(s)|^2))^2u(s)\|ds\\&\quad\leq C_gT\sum_{k\in\mathbb{N}^+}\|Q^{\frac12}e_k\|^2_{L^\infty}\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)-u(t)\|\end{aligned}$$

and

$$\begin{aligned}&\Big\|-\mathbf{i}\int_0^t S(t-s)(g'(|u^{\epsilon_n}(s)|^2)g(|u^{\epsilon_n}(s)|^2)|u^{\epsilon_n}(s)|^2u^{\epsilon_n}(s)\sum_{k\in\mathbb{N}^+}Im(Q^{\frac12}e_k)Q^{\frac12}e_kds-V_3\Big\|\\&\leq C_gT\sum_{k\in\mathbb{N}^+}\|Q^{\frac12}e_k\|^2_{L^\infty}\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)-u(t)\|.\end{aligned}$$

The Burkholder inequality and the unitary property of $S(\cdot)$ yield that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\mathbf{i}\int_0^t S(t-s)g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)dW(s)-V_4\Big\|^2\Big]\\&\quad\leq C\mathbb{E}\Big[\int_0^T\sum_{k\in\mathbb{N}^+}\|Q^{\frac12}e_k\|^2_{L^\infty}\|g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)-g(|u(s)|^2)u(s)\|^2ds\Big]\\&\quad\leq C\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)-u(t)\|^2\Big].\end{aligned}$$

Combining the above estimates and the strong convergence of u^{ϵ_n}, we complete the proof of Step 2.

Step 3*:* u is independent of the choice of the sequence of $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$. Assume that $\widetilde{u}$ and u are two different limit processes of two different sequences of $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ and

$\{u^{\epsilon_m}\}_{m\in\mathbb{N}^+}$, respectively. Then by Step 2, they both satisfy (1.1). By repeating the procedures in Steps 1 and 2, one can verify the uniqueness of the mild solution. ∎

The procedures in the above proof immediately yield the following convergence rate result for u^ϵ in the regularized problem (1.4) and the Hölder regularity estimate of u^ϵ and u^0.

COROLLARY 3. *Let the conditions of Theorem* 1 *hold. Assume that* u^ϵ *is the mild solution in Proposition* 5, $\epsilon\in(0,1)$. *For* $p\geq 2$, *there exists* $C(Q,T,\lambda,p,u_0)>0$ *such that for any* $\frac{\eta' d}{2\eta'+2}\in(0,1)$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)-u^\epsilon(t)\|^p\Big]\leq C(Q,T,\lambda,p,u_0)(\epsilon^{\frac{p}{2}}+\epsilon^{\frac{\eta' p}{2}})$$

when $\mathcal{O}$ *is a bounded domain, and*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)-u^\epsilon(t)\|^p\Big]\leq C(Q,T,\lambda,p,u_0,\alpha)(\epsilon^{\frac{\alpha p}{2\alpha+d}}+\epsilon^{\frac{\eta' p}{2}})$$

when $\mathcal{O}=\mathbb{R}^d$.

COROLLARY 4. *Let the conditions of Theorem* 1 *hold. Assume that* u^ϵ *is the mild solution in Proposition* 5, $\epsilon\in(0,1)$, *and* u^0 *is the mild solution of* (1.1). *For* $p\geq 2$, *there exists* $C(Q,T,\lambda,p,u_0)>0$ *such that for* $\epsilon\in[0,1)$,

$$\mathbb{E}\Big[\|u^\epsilon(t)-u^\epsilon(s)\|^p\Big]\leq C(Q,T,\lambda,p,u_0)|t-s|^{\frac{p}{2}}.$$

Proof. By means of the mild form of u^ϵ, $\epsilon\in[0,1)$, the a priori estimate of u^ϵ in $\mathbb{H}^1\cap L^2_\alpha$ in Lemmas 2 and 4, and in Step 2 of the proof of Theorem 1, and the Burkholder inequality, we obtain the desired result. ∎

4.2. Well-posedness of the SlogS equation with superlinearly growing diffusion coefficients. In this part, we extend the scope of $\widetilde{g}$, which allows the diffusion with superlinear growth, for the well-posedness of the SlogS equation driven by conservative multiplicative noise. For instance, it includes the example $\widetilde{g}(x)=\mathbf{i}g(|x|^2)x=\mathbf{i}x\log(c+|x|^2)$ for $c>0$.

THEOREM 2. *Let* $W(t)$ *be* $L^2(\mathcal{O};\mathbb{R})$ *valued and* $g\in\mathcal{C}^1_b([0,+\infty))\cap\mathcal{C}([0,+\infty))$ *satisfy the growth condition and the embedding condition,*

$$\sup_{x\in[0,\infty)}|g'(x)x|\leq C_g,$$
$$\|vg(|v|^2)\|\leq C_d(1+\|v\|_{\mathbb{H}^1}+\|v\|_{L^2_\alpha})$$

for some $\alpha\in[0,1]$, *where* $C_g>0$ *depends on* g, $C_d>0$ *depends on* $\mathcal{O}$, d, $\|v\|$, *and* $v\in\mathbb{H}^1\cap L^2_\alpha$. *Assume that* $d=1$, $u_0\in\mathbb{H}^1\cap L^2_\alpha$, $\alpha\in(0,1]$, *with any finite pth moment, and* $\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|^2_{\mathbb{H}^1}+\|Q^{\frac12}e_i\|^2_{W^{1,\infty}}<\infty$. *Then there exists a unique mild solution* u *in* $C([0,T];\mathbb{H})$ *for* (1.1) *satisfying*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)\|^p_{\mathbb{H}^1}\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)\|^p_{L^2_\alpha}\Big]\leq C(Q,T,\lambda,p,u_0).$$

Proof. By Proposition 5 and Lemma 4, we can introduce the truncated sample space

$$\Omega_R(t):=\Big\{\omega:\sup_{s\in[0,t]}\|u^{\epsilon_m}(s)\|_{L^\infty}\leq R,\ \sup_{s\in[0,t]}\|u^{\epsilon_n}(s)\|_{L^\infty}\leq R\Big\},$$

where $n \le m$. The Gagliardo–Nirenberg interpolation inequality (3.3) with $d=1$, the a priori estimate in $\mathbb{H}^1$, and the continuity in L^2 of u^{ϵ_n} imply that u^{ϵ_n} are continuous in L^∞ a.s. Define a stopping time

$$\tau_R := \inf\{t \ge 0 : \min(\sup_{s\in[0,t]} \|u^{\epsilon_m}(s)\|_{L^\infty}, \sup_{s\in[0,t]} \|u^{\epsilon_n}(s)\|_{L^\infty}) \ge R\} \wedge T.$$

Then on $\Omega_R(T)$, we have $\tau_R = T$. Let us take $f_\epsilon(x) = \log(x+\epsilon), x > 0$, for convenience. It is obvious that $\Omega_R(t) \to \Omega$ as $R \to \infty$ and that for any $p \ge 1$,

$$\begin{aligned} \mathbb{P}\Big(\sup_{t\in[0,T]} \min(\|u^{\epsilon_m}(t)\|_{L^\infty}, \|u^{\epsilon_n}(t)\|_{L^\infty}) \ge R\Big) \\ \le C\frac{1}{R^p}\Big(\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^{\epsilon_m}(t)\|^p_{L^\infty}\Big] + \sup_{t\in[0,T]} \mathbb{E}\Big[\|u^{\epsilon_m}(t)\|^p_{L^\infty}\Big]\Big). \end{aligned} \tag{4.4}$$

Step 1: $\{u^{\epsilon_n}\}_{n\in\mathbb{N}^+}$ forms a Cauchy sequence in $\mathbb{M}^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{H}))$. Following the same steps as in the proof of Theorem 1, applying the Itô formula for $t \in (0, \tau_R]$ yields that

$$\begin{aligned} &\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2 \\ &\le \int_0^t 4|\lambda| \|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2 ds + 4|\lambda|\epsilon_n^{\frac{1}{2}} \int_0^t \|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|_{L^1} ds \\ &\quad + \int_0^t \sum_{k\in\mathbb{N}^+} \langle |Q^{\frac{1}{2}} e_k|^2 (g(|u^{\epsilon_n}(s)|^2) - g(|u^{\epsilon_m}(s)|^2))^2 u^{\epsilon_m}(s), u^{\epsilon_n}(s)\rangle ds \\ &\quad + \int_0^t \langle u^{\epsilon_m} - u^{\epsilon_n}, \mathbf{i}\Big(g(|u^{\epsilon_m}(s)|^2) u^{\epsilon_m}(s) - g(|u^{\epsilon_n}(s)|^2) u^{\epsilon_n}(s)\Big) dW(s)\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} &\mathbb{E}\Big[\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2\Big] \\ &\le \int_0^t 4|\lambda| \mathbb{E}\Big[\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2\Big] ds + 4|\lambda|\epsilon_n^{\frac{1}{2}} \int_0^t \mathbb{E}\Big[\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|_{L^1}\Big] ds \\ &\quad + \int_0^t \sum_{k\in\mathbb{N}^+} \|Q^{\frac{1}{2}} e_k\|^2_{L^\infty} \mathbb{E}\Big[\langle (g(|u^{\epsilon_n}(s)|^2) - g(|u^{\epsilon_m}(s)|^2))^2 u^{\epsilon_m}(s), u^{\epsilon_n}(s)\rangle\Big] ds. \end{aligned}$$

Making use of the assumptions on g, we get for $t \le \tau_R$,

$$\begin{aligned} &\mathbb{E}\Big[\|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2\Big] \\ &\le \int_0^t 4|\lambda| \mathbb{E}\Big[\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2\Big] ds + 4|\lambda|\epsilon_n^{\frac{1}{2}} \int_0^t \mathbb{E}\Big[\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|_{L^1}\Big] ds \\ &\quad + C(Q,g) \int_0^t \mathbb{E}\Big[\int_{\mathcal{O}} (|u^{\epsilon_m}(s)|^2 + |u^{\epsilon_n}(s)|^2) |u^{\epsilon_n}(s)| |u^{\epsilon_m}(s)| |u^{\epsilon_m}(s) - u^{\epsilon_n}(s)|^2 dx\Big] ds \\ &\le \int_0^t 4|\lambda| \mathbb{E}\Big[\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2\Big] ds + 4|\lambda|\epsilon_n^{\frac{1}{2}} \int_0^t \mathbb{E}\Big[\|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|_{L^1}\Big] ds \\ &\quad + C(Q,g) \int_0^t \mathbb{E}\Big[(1+R^4) \|u^{\epsilon_m}(s) - u^{\epsilon_n}(s)\|^2\Big] ds. \end{aligned}$$

If $\mathcal{O}$ is bounded, then Hölder's inequality and Gronwall's inequality yield that

$$\mathbb{E}\Big[\mathbb{I}_{\{t\le\tau_R\}} \|u^{\epsilon_m}(t) - u^{\epsilon_n}(t)\|^2\Big] \le C(u_0, Q, T, \lambda, |\mathcal{O}|) e^{(1+R^4)T} \epsilon_n.$$

On the other hand, the Chebyshev inequality and the a priori estimate lead to

$$\begin{aligned}\mathbb{E}\Big[\mathbb{I}_{\{t>\tau_R\}}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big] &\le \mathbb{E}\Big[\mathbb{I}_{\Omega_R^c(t)}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\\ &\le (\mathbb{P}(\Omega_R^c(t)))^{\frac{1}{p_1}}\Big(\mathbb{E}\Big[\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^{2q_1}\Big]\Big)^{\frac{1}{q}_1},\end{aligned}$$

where $\frac{1}{p_1}+\frac{1}{q_1}=1$. Combining the above estimates on $\Omega_R(t)$ and $\Omega_R^c(t)$, applying (4.4) with $p\gg p_1$, we conclude that

$$\mathbb{E}\Big[\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\le C(u_0,Q,T,\lambda,|\mathcal{O}|,p,p_1)\Big(e^{(1+R^4)T}\epsilon_n+R^{-\kappa}\Big),$$

where $\kappa=\frac{p}{p_1}$. Then one may take $R=(\frac{c_0}{T}|\log(\epsilon_n)|)^{\frac14}$ for $c_0\in(0,1)$ and get

$$\mathbb{E}\Big[\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\le C(u_0,Q,T,\lambda,|\mathcal{O}|,p,p_1)(\epsilon_n^{1-c_0}+(\frac{c_0}{T}|\log(\epsilon_n)|)^{-\frac{\kappa}{4}}).$$

By further applying the Burkholder inequality to the stochastic integral, we achieve that for any $\kappa>0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\le C(u_0,Q,T,\lambda,|\mathcal{O}|,p,p_1)|\log(\epsilon_n)|^{-\frac{\kappa}{4}}.$$

When $\mathcal{O}=\mathbb{R}^d$, using Lemma 6, we obtain that for $\eta\in(0,\frac{2\alpha}{2\alpha+d})$ and $\alpha\in(0,1]$,

$$\begin{aligned}&\mathbb{E}\Big[\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\\ &\le \int_0^t 4|\lambda|\mathbb{E}\Big[\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2\Big]ds+C\epsilon_n^{\eta}\int_0^t\mathbb{E}\Big[\|u^{\epsilon_n}(s)\|_{L^2_\alpha}^{\frac{d\eta}{\alpha}}\|u^{\epsilon_n}(s)\|^{2-2\eta-\frac{d\eta}{\alpha}}\Big]ds\\ &\quad+\ C(Q,g)\int_0^t\mathbb{E}\Big[(1+R^4)\|u^{\epsilon_m}(s)-u^{\epsilon_n}(s)\|^2\Big]ds.\end{aligned}$$

By using Gronwall's inequality and the estimate of $\mathbb{P}(\Omega_R^c(t))$, we immediately have that for $\eta\in(0,\frac{2\alpha}{2\alpha+d}),\alpha\in(0,1]$, and any $\kappa>0$,

$$\mathbb{E}\Big[\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\le C(u_0,Q,T,\lambda,p,p_1)\Big(e^{(1+R^4)T}\epsilon_n^{\eta}+R^{-\kappa}\Big).$$

Taking $R=(\frac{\eta c_0}{T}|\log(\epsilon_n)|)^{\frac14}$ for $c_0\in(0,1)$ and using the Burkholder inequality, we have for $\eta\in(0,\frac{2\alpha}{2\alpha+d}),\alpha\in(0,1]$ and any small $\kappa>0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_m}(t)-u^{\epsilon_n}(t)\|^2\Big]\le C(u_0,Q,T,\lambda,p,p_1,\eta)|\log(\epsilon_n)|^{-\frac{\kappa}{4}}.$$

Step 2*:* u is the mild solution.

Let us use the same notations and procedures as in Step 2 of the proof in Theorem 1. To show that the mild form u^{ϵ_n} converges to $S(t)u_0+V_1+V_2+V_3+V_4$, we only need to estimate V_2 and V_4 since $V_3=0$. Define

$$\Omega_{R_1}(t):=\{\omega:\sup_{s\in[0,t]}\|u^{\epsilon_n}(s)\|_{L^\infty}\le R_1,\ \sup_{s\in[0,t]}\|u(s)\|_{L^\infty}\le R_1\},$$

and a stopping time

$$\tau_{R_1}:=\inf\{t\ge0:\max(\sup_{s\in[0,t]}\|u(s)\|_{L^\infty},\ \sup_{s\in[0,t]}\|u^{\epsilon_n}(s)\|_{L^\infty})\ge R_1\}\wedge T.$$

Then on $\Omega_{R_1}(T)$, we have $\tau_{R_1}=T$. The Minkowski inequality and the properties of g yield that on $\Omega_{R_1}(t)$,

$$\begin{aligned}&\Big\|-\frac{1}{2}\int_0^t S(t-s)(g(|u^{\epsilon_n}(s)|^2))^2u^{\epsilon_n}(s)\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2ds-V_2\Big\|\\&\le\sum_{k\in\mathbb{N}^+}\|Q^{\frac12}e_k\|^2_{L^\infty}\int_0^T\|(g(|u^{\epsilon_n}(s)|^2))^2u^{\epsilon_n}(s)-(g(|u(s)|^2))^2u(s)\|ds\\&\le C_gT\sum_{k\in\mathbb{N}^+}\|Q^{\frac12}e_k\|^2_{L^\infty}\Big(1+R_1^2\Big)\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)-u(t)\|.\end{aligned}$$

On the other hand, for any $p_2>0$,

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{I}_{\Omega^c_{R_1}(t)}\Big\|-\frac{1}{2}\int_0^t S(t-s)(g(|u^{\epsilon_n}(s)|^2))^2u^{\epsilon_n}(s)\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2ds-V_2\Big\|^2\Big]\\&\le C(u_0,Q,T,p_2)R_1^{-p_2}.\end{aligned}$$

Taking $R_1=\mathscr{O}(|\log(\epsilon_n)|^{\frac{\kappa_1}{4(4+p_2)}}),\kappa_1<\kappa$, we have that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\mathbb{E}\Big[\Big\|\int_0^t-\frac{1}{2}S(t-s)(g(|u^{\epsilon_n}(s)|^2))^2u^{\epsilon_n}(s)\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2ds-V_2\Big\|^2\Big]=0,$$

which immediately implies that

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|-\frac{1}{2}\int_0^t S(t-s)(g(|u^{\epsilon_n}(s)|^2))^2u^{\epsilon_n}(s)\sum_{k\in\mathbb{N}^+}|Q^{\frac12}e_k|^2ds-V_2\Big\|^2\Big]=0.$$

The Burkholder inequality and the unitary property of $S(\cdot)$ yield that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,\tau_R]}\Big\|\int_0^t\mathbf{i}S(t-s)g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)dW(s)-V_4\Big\|^2\Big]\\&\le C\mathbb{E}\Big[\int_0^T\sum_{k\in\mathbb{N}^+}\|Q^{\frac12}e_k\|^2_{L^\infty}\|g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)-g(|u(s)|^2)u(s)\|^2ds\Big]\\&\le C(1+R_1^2)\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^{\epsilon_n}(t)-u(t)\|^2\Big].\end{aligned}$$

On the other hand, the Chebyshev inequality, together with the a priori estimate of u^{ϵ_n}, implies that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t>\tau_{R_1}}\Big\|\int_0^t\mathbf{i}S(t-s)g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)dW(s)-V_4\Big\|^2\Big]\\&\le\mathbb{E}\Big[\sup_{t\in[0,T]}\mathbb{I}_{\Omega^c_{R_1}(t)}\Big\|\int_0^t\mathbf{i}S(t-s)g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)dW(s)-V_4\Big\|^2\Big]\\&\le C(u_0,Q,T,p)R_1^{-p_2}.\end{aligned}$$

Taking $R_1=\mathscr{O}(|\log(\epsilon_n)|^{\frac{\kappa_1}{4(2+p_2)}}),\kappa_1<\kappa$, we have that

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}\Big\|\int_0^t\mathbf{i}S(t-s)g(|u^{\epsilon_n}(s)|^2)u^{\epsilon_n}(s)dW(s)-V_4\Big\|^2\Big]=0.$$

Combining the above estimates and strong convergence of u^{ϵ_n}, we complete the proof. ∎

Remark 2. One may extend the scope of $\widetilde{g}$ to an abstract framework by similar arguments. Here the assumption $d=1$ lies on the fact that $\mathbb{H}^1$ is an algebra by the Sobolev embedding theorem. When considering the case $d\geq 2$, one may use $\mathbb{H}^{\mathbf{s}}, \mathbf{s}>\frac{d}{2}$, as the underlying space for the local well-posedness. However, as stated in Lemma 7, it seems impossible to get the uniform bound of u^ϵ in $\mathbb{H}^{\mathbf{s}}$ for $\mathbf{s}\geq 2$.

Remark 3. As mentioned in Remark 1, to prove the existence of the global solution of (3.7) with $d=1$, one could make some modifications to the proof of Theorem 2. For example, one may need to take a suitable sequence with (ϵ_n, c_n) for (3.6), then deduce the relationship between c_n and ϵ_n to get an explicit convergence rate. However, the convergence rate analysis of (3.6) is beyond the scope of this current work. We will study this problem in the future.

Appendix A. The original problem and the regularized problem can be rewritten into the equivalent evolution forms

$$\begin{aligned} du &= Audt + F(u)dt + G(u)dW(t), \\ u(0) &= u_0, \end{aligned} \tag{A.1}$$

where $A=\mathbf{i}\Delta$, F is the Nemystkii operator of the drift coefficient function, and G is the Nemystkii operator of the diffusion coefficient function. Then the mild solution of the above evolution is defined as follows (see, e.g., [21, Appendix F]).

DEFINITION 1. *A continuous $\mathbb{H}$-valued $\mathcal{F}_t$ adapted process u is a solution to* (A.1) *if it satisfies $\mathbb{P}$-a.s. for all $t\in[0,T]$,*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t G(u(s))dW(s),$$

where $S(\cdot)$ is the C_0-group generated by A.

DEFINITION 2. *A local mild solution of* (A.1) *is $(u,\tau) := (u,\tau_n,\tau)$ satisfying $\tau_n \nearrow \tau$ a.s., as $n\to\infty$, $u\in\mathbb{M}^p_{\mathcal{F}}(\Omega; C([0,\tau);\mathbb{H}^{\mathbf{s}}), \mathbf{s}\geq 0, p\geq 1$, and that*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)G(u(s))dW(s) \quad a.s.$$

for $t\leq \tau_n$ in $\mathbb{H}^2$ for $n\in\mathbb{N}^+$. Solutions of (A.1) *are called unique if*

$$\mathbb{P}\Big(u_1(t) = u_2(t), \forall t\in[0,\sigma_1\wedge\sigma_2)\Big) = 1$$

for all local mild solutions (u_1,σ_1) and (u_2,σ_2). The local solution (u,τ) is called a global mild solution if $\tau=T$ a.s. and $u\in\mathbb{M}^p_{\mathcal{F}}(\Omega; C([0,T];\mathbb{H}^{\mathbf{s}})$.

LEMMA 8. *Let $\epsilon\in(0,1)$. Then $f_\epsilon(x)=\log(|x|^2+\epsilon), x\in\mathbb{C}$, satisfies*

$$|Im[(f_\epsilon(x_1)x_1 - f_\epsilon(x_2)x_2)(\bar{x}_1-\bar{x}_2)]| \leq 4|x_1-x_2|^2.$$

Proof. Without loss of generality, we assume that $0<|x_2|\leq|x_1|$. Notice that

$$\begin{aligned} &Im[(f_\epsilon(x_1)x_1 - f_\epsilon(x_2)x_2)(\bar{x}_1-\bar{x}_2)] \\ &\quad = \frac{1}{2}(\log(\epsilon+|x_1|^2) - \log(\epsilon+|x_2|^2))Im(\bar{x}_1x_2 - \bar{x}_2x_1). \end{aligned}$$

Direct calculation yields that

$$|Im(\bar{x}_1 x_2 - \bar{x}_2 x_1)| \le 2|x_2||x_1 - x_2|.$$

Using the fact that

$$|\log(\epsilon + |x_1|^2) - \log(\epsilon + |x_2|^2)| = 2|\log((\epsilon + |x_1|^2)^{\frac{1}{2}}) - \log((\epsilon + |x_2|^2)^{\frac{1}{2}})|,$$

we obtain

$$\begin{aligned}
&|Im[(f_\epsilon(x_1)x_1 - f_\epsilon(x_2)x_2)(\bar{x}_1 - \bar{x}_2)]| \\
&\quad \le 2|\log((\epsilon + |x_1|^2)^{\frac{1}{2}}) - \log((\epsilon + |x_2|^2)^{\frac{1}{2}})||x_2||x_1 - x_2|.
\end{aligned}$$

The mean-value theorem leads to the desired result. ∎

Proof of Proposition 4. Due to Lemma 4, it suffices to prove

$$\mathbb{E}\Big[\sup_{t\in[0,T]} (F_\epsilon(|u^\epsilon(t)|^2))^p\Big] \le C(u_0, T, Q, p).$$

Let us take $f_\epsilon(|x|^2) = \log(|x|^2 + \epsilon)$ as an example to illustrate the procedure. The desired estimate in the case that $f_\epsilon(|x|^2) = \log(\frac{|x|^2+\epsilon}{1+\epsilon|x|^2})$ can be obtained similarly. Using the property of the logarithmic function, we have that for small $\eta > 0$,

$$\begin{aligned}
&|F_\epsilon(|u^\epsilon(t)|^2)| \\
&= \Big|\int_{\mathcal{O}} \Big((\epsilon + |u^\epsilon(t)|^2)\log(\epsilon + |u^\epsilon(t)|^2) - |u^\epsilon(t)|^2 - \epsilon\log(\epsilon)\Big)dx\Big| \\
&\le \|u^\epsilon(t)\|^2 + \int_{\mathcal{O}} |u^\epsilon(t)|^2 \log(\epsilon + |u^\epsilon(t)|^2)dx + \Big|\int_{\mathcal{O}} \epsilon(\log(\epsilon + |u^\epsilon(t)|^2) - \log(\epsilon))dx\Big| \\
&\le 2\|u^\epsilon(t)\|^2 + C\|u^\epsilon(t)\|^{2-2\eta}_{L^{2-2\eta}} + C\|u^\epsilon(t)\|^{2+2\eta}_{L^{2+2\eta}},
\end{aligned}$$

where we have used the following estimation, for any small enough $\eta > 0$,

$$\begin{aligned}
&\int_{\mathcal{O}} |u^\epsilon(t)|^2 \log(\epsilon + |u^\epsilon(t)|^2)dx \\
&= \int_{\epsilon+|u^\epsilon(t)|^2\ge 1} f_\epsilon(|u^\epsilon(t)|^2)|u^\epsilon(t)|^2 dx + \int_{\epsilon+|u^\epsilon(t)|^2\le 1} f_\epsilon(|u^\epsilon(t)|^2)|u^\epsilon(t)|^2 dx \\
&\le \int_{\epsilon+|u^\epsilon(t)|^2\ge 1} (\epsilon + |u^\epsilon(t)|^2)^{2\eta}|u^\epsilon(t)|^2 dx + \int_{\epsilon+|u^\epsilon(t)|^2\le 1} (\epsilon + |u^\epsilon(t)|^2)^{-2\eta}|u^\epsilon(t)|^2 dx \\
&\le C\|u^\epsilon(t)\|^{2-2\eta}_{L^{2-2\eta}} + C\|u^\epsilon(t)\|^{2+2\eta}_{L^{2+2\eta}}.
\end{aligned}$$

Then by (3.3), we have that

$$\begin{aligned}
&F_\epsilon(|u^\epsilon(s)|^2) \\
&\quad \le \Big|\int_{\mathcal{O}} \Big((\epsilon + |u^\epsilon(s)|^2)\log(\epsilon + |u^\epsilon(s)|^2) - |u^\epsilon(s)|^2 - \epsilon\log\epsilon\Big)dx\Big| \\
&\quad \le C(\|u^\epsilon(s)\|^2 + \|u^\epsilon(s)\|^{2-2\eta}_{L^{2-2\eta}} + \|u^\epsilon(s)\|^{2+2\eta}_{L^{2+2\eta}}) \\
&\quad \le C(\|u^\epsilon(s)\|^2 + \|u^\epsilon(s)\|^{2-2\eta}_{L^{2-2\eta}} + \|\nabla u^\epsilon(s)\|^{\frac{d\eta}{2\eta+2}}\|u^\epsilon(s)\|^{1-\frac{d\eta}{2\eta+2}}).
\end{aligned}$$

Taking the pth moment, and applying (3.3) with $q = 2\eta + 2$, Lemma 4, and Corollary 1, we complete the proof for the case that $\mathcal{O}$ is a bounded domain.

When $\mathcal{O} = \mathbb{R}^d$, we need to control $\|u^\epsilon\|_{L^{2-2\eta}}$ separately. By using the weighted interpolation inequality in Lemma 6 with $\alpha > \frac{d\eta}{2-2\eta}$ and $\alpha \in (0,1)$ and Lemmas 4 and 2, we complete the proof by using the Young inequality and taking the pth moment. ∎

Sketch Proof of Lemma 7. Due to the loss of the regularity of the solution in time, we cannot establish the bound in $\mathbb{H}^2$ through $\frac{\partial u^\epsilon}{\partial t}$ like in the deterministic case. According to Lemma 4, it suffices to bound $\|\Delta u^\epsilon(t)\|^2$. We present the procedures of the estimation of $\mathbb{E}[\|\Delta u(t)\|^2]$ for the conservative multiplicative noise case. One can easily follow the procedures to obtain the estimate of $\mathbb{E}[\sup_{t\in[0,\tau]} \|\Delta u(t)\|^2]$ for both additive and multiplicative noises.

By using the Itô formula to $\|\Delta u^\epsilon(t)\|^2$ we obtain that

$$\|\Delta u^\epsilon(t)\|^2 = \|\Delta u_0\|^2 + 2\int_0^t \langle \Delta u^\epsilon(s), II_{det}\rangle ds + 2\int_0^t \langle \Delta u^\epsilon(s), II_{mod}\rangle ds + 2II_{Sto},$$

where

$$\begin{aligned}
II_{Sto} :=& \int_0^t \langle \Delta u^\epsilon(s), \mathbf{i}g(|u^\epsilon(s)|^2)\nabla u^\epsilon(s)\nabla dW(s)\rangle \\
&+ \int_0^t \langle \Delta u^\epsilon(s), \mathbf{i}g(|u^\epsilon(s)|^2)u^\epsilon(s)\Delta dW(s)\rangle \\
&+ \int_0^t \langle \Delta u^\epsilon(s), \mathbf{i}2g'(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))\nabla u^\epsilon(s)dW(s) \\
&+ \int_0^t \langle \Delta u^\epsilon(s), \mathbf{i}2g'(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))u^\epsilon(s)\nabla dW(s)\rangle \\
&+ \int_0^t \langle \Delta u^\epsilon(s), \mathbf{i}4g''(|u^\epsilon(s)|^2)(Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s)))^2u^\epsilon(s)dW(s) \\
&+ \int_0^t \langle \Delta u^\epsilon(s), \mathbf{i}2g'(|u^\epsilon(s)|^2)(Re(\bar{u}^\epsilon(s)\Delta u^\epsilon(s)))u^\epsilon(s)dW(s)\rangle \\
&+ \int_0^t \langle \Delta u^\epsilon(s), 2g'(|u^\epsilon(s)|^2)|\nabla u|^2u^\epsilon(s)dW(s)\rangle, \\
II_{det} :=&\ \mathbf{i}\Delta^2u^\epsilon(s) + \mathbf{i}\lambda f_\epsilon(|u^\epsilon(s)|^2)\Delta u^\epsilon(s)dt + \mathbf{i}4\lambda f'_\epsilon(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s))\nabla u^\epsilon(s) \\
&+ \mathbf{i}4\lambda f''_\epsilon(|u^\epsilon(s)|^2)(Re(\bar{u}^\epsilon(s)\nabla u^\epsilon(s)))^2u^\epsilon(s) \\
&+ \mathbf{i}2\lambda f'_\epsilon(|u^\epsilon(s)|^2)Re(\bar{u}^\epsilon(s)\Delta u^\epsilon(s))u^\epsilon(s),
\end{aligned}$$

and II_{mod} is the summation of all terms involving the second derivative of the Itô modified term produced by the Stratonovich integral. Here for simplicity, we omit the presentation of the explicit form for II_{mod}.

Taking the expectation, using (2.2), the fact that $f''_\epsilon(x)x^3 \le C\epsilon^{-\frac{1}{2}}$, and the Gagliardo–Nirenberg interpolation inequality $\|\nabla v\|_{L^4} \le C\|\Delta v\|^{\frac{1}{4}}\|\nabla v\|^{\frac{3}{4}}$ in $d=1$, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big[\|\Delta u^\epsilon(t)\|^2\Big] \\
&\le \mathbb{E}\Big[\|\Delta u^\epsilon(0)\|^2\Big] + C(\lambda,p)\epsilon^{-\frac{1}{2}}\mathbb{E}\Big[\int_0^t \|\Delta u^\epsilon(r)\|(1+\|\nabla u^\epsilon(r)\|_{L^4}^2)dr\Big]
\end{aligned}$$

$$
\begin{aligned}
&+C(\lambda,p)\mathbb{E}\Big[\int_0^t \|\Delta u^\epsilon(r)\|\Big(\|\sum_{i\in\mathbb{N}^+}\Delta Q^{\frac12}e_iQ^{\frac12}e_i(g(|u^\epsilon(r)|^2))^2u^\epsilon(r)\|\\
&+\|\sum_{i\in\mathbb{N}^+}|\nabla Q^{\frac12}e_i|^2(g(|u^\epsilon(r)|^2))^2u^\epsilon(r)\|\\
&+\|\sum_{i\in\mathbb{N}^+}\nabla Q^{\frac12}e_iQ^{\frac12}e_ig(|u^\epsilon(r)|^2)g'(|u^\epsilon(r)|^2)|u^\epsilon(r)|^2\nabla u^\epsilon(r)\|\\
&+\|\sum_{i\in\mathbb{N}^+}|Q^{\frac12}e_i|^2g(|u^\epsilon(r)|^2)g'(|u^\epsilon(r)|^2)|\nabla u^\epsilon(r)|^2u^\epsilon(r)\|\\
&+\|\sum_{i\in\mathbb{N}^+}|Q^{\frac12}e_i|^2(g(|u^\epsilon(r)|^2)g''(|u^\epsilon(r)|^2)+(g'(|u^\epsilon(r)|^2))^2)|\nabla u^\epsilon(r)|^2|u^\epsilon(r)|^3\|\\
&+\|\sum_{i\in\mathbb{N}^+}\nabla Q^{\frac12}e_iQ^{\frac12}e_i(g(|u^\epsilon(r)|^2))^2\nabla u^\epsilon(r)\|\Big)dr\Big]\\
&+C(\lambda,p)\mathbb{E}\Big[\int_0^t\sum_{i\in\mathbb{N}}\Big(\|g(|u^\epsilon(r)|^2)\nabla u^\epsilon(r)\nabla Q^{\frac12}e_i\|^2+\|g(|u^\epsilon(r)|^2)u^\epsilon(r)\Delta Q^{\frac12}e_i\|^2\\
&+\|g'(|u^\epsilon(r)|^2)|\nabla u^\epsilon(r)|^2u^\epsilon(r)Q^{\frac12}e_i\|^2+\|g'(|u^\epsilon(r)|^2)\nabla u^\epsilon(r)|u^\epsilon(r)|^2Q^{\frac12}e_i\|^2\\
&+\|g''(|u^\epsilon(r)|^2)|\nabla u^\epsilon(r)|^2|u^\epsilon(r)|^3Q^{\frac12}e_i\|^2+\|g'(|u^\epsilon(r)|^2)|\nabla u^\epsilon(r)|^2u^\epsilon(r)Q^{\frac12}e_i\|^2\Big)dr\Big]\\
=:&\ \mathbb{E}\Big[\|\Delta u^\epsilon(0)\|^2\Big]+C(\lambda,p)\epsilon^{-\frac12}\mathbb{E}\Big[\int_0^t\|\Delta u^\epsilon(r)\|(1+\|\nabla u^\epsilon(r)\|_{L^4}^2)dr\Big]\\
&+C(\lambda,p)\mathbb{E}\Big[\int_0^t\|\Delta u^\epsilon(r)\|A(r)dr\Big]+C(\lambda,p)\mathbb{E}\Big[\int_0^tB(r)dr\Big].
\end{aligned}
$$

Now applying the Hölder inequality, using the properties of g, using the Gagliardo–Nirenberg interpolation inequality, we obtain that

$$
\begin{aligned}
A(r)\le&\sum_{i\in\mathbb{N}^+}\|\Delta Q^{\frac12}e_i\|\|Q^{\frac12}e_i\|_{L^\infty}\|(g(|u^\epsilon(r)|^2))^2u^\epsilon(r)\|_{L^\infty}\\
&+\sum_{i\in\mathbb{N}^+}\|\nabla Q^{\frac12}e_i\|_{L^4}^2\|(g(|u^\epsilon(r)|^2))^2u^\epsilon(r)\|_{L^\infty}\\
&+\sum_{i\in\mathbb{N}^+}\|\nabla Q^{\frac12}e_i\|_{L^4}\|Q^{\frac12}e_i\|_{L^\infty}\|g(|u^\epsilon(r)|^2)\|_{L^\infty}\|\nabla u^\epsilon(r)\|_{L^4}\\
&+\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|_{L^\infty}^2\|\nabla u^\epsilon(r)\|_{L^4}^2\|g(|u^\epsilon(r)|^2)g'(|u^\epsilon(r)|^2)u^\epsilon(r)\|_{L^\infty}\\
&+\sum_{i\in\mathbb{N}^+}\|Q^{\frac12}e_i\|_{L^\infty}^2\|\nabla u^\epsilon(r)\|_{L^4}^2\|(g(|u^\epsilon(r)|^2)g''(|u^\epsilon(r)|^2)\\
&+(g'(|u^\epsilon(r)|^2))^2)|u^\epsilon(r)|^3\|_{L^\infty}\\
&+\sum_{i\in\mathbb{N}^+}\|\nabla Q^{\frac12}e_i\|_{L^4}\|Q^{\frac12}e_i\|_\infty\|g(|u^\epsilon(r)|^2)\|_{L^\infty}^2\|\nabla u^\epsilon(r)\|_{L^4}\\
\le&\sum_{i\in\mathbb{N}^+}\Big(\|\nabla Q^{\frac12}e_i\|_{L^4}^2+\|\nabla Q^{\frac12}e_i\|_{L^\infty}^2+\|\Delta Q^{\frac12}e_i\|^2+\|Q^{\frac12}e_i\|_{L^\infty}^2\Big)\\
&\times\Big(1+\|\nabla u^\epsilon\|_{L^4}+\|\nabla u^\epsilon\|_{L^4}^2\Big)\\
&\times\Big(\|(g(|u^\epsilon(r)|^2))^2u^\epsilon(r)\|_{L^\infty}+\|g(|u^\epsilon(r)|^2)\|_{L^\infty}+\|g(|u^\epsilon(r)|^2)g'(|u^\epsilon(r)|^2)u^\epsilon(r)\|_{L^\infty}\\
&+\|(g(|u^\epsilon(r)|^2)g''(|u^\epsilon(r)|^2)+(g'(|u^\epsilon(r)|^2))^2)|u^\epsilon(r)|^3\|_{L^\infty}+\|g(|u^\epsilon(r)|^2)\|_{L^\infty}^2\Big)
\end{aligned}
$$

$$\le C \sum_{i\in\mathbb{N}^+} \Big(\|\nabla Q^{\frac12} e_i\|_{L^4}^2 + \|\nabla Q^{\frac12} e_i\|_{L^\infty}^2 + \|\Delta Q^{\frac12} e_i\|^2 + \|Q^{\frac12} e_i\|_{L^\infty}^2 \Big) \times \Big(1 + \|\nabla u^\epsilon(r)\|_{L^4} + \|\nabla u^\epsilon(r)\|_{L^4}^2 \Big).$$

Similarly, we have that

$$\begin{aligned} B(r) \le \sum_{i\in\mathbb{N}^+} \Big(&\|g(|u^\epsilon(r)|^2)\|_{L^\infty}^2 \|\nabla u^\epsilon(r)\|_{L^4}^2 \|\nabla Q^{\frac12} e_i\|_{L^4}^2 + \|g(|u^\epsilon(r)|^2)u^\epsilon(r)\|_{L^\infty}^2 \|\Delta Q^{\frac12} e_i\|^2 \\ &+ \|g'(|u^\epsilon(r)|^2)u^\epsilon(r)\|_{L^\infty}^2 \|\nabla u^\epsilon(r)\|_{L^4}^4 \|Q^{\frac12} e_i\|_{L^\infty}^2 \\ &+ \|g'(|u^\epsilon(r)|^2)|u^\epsilon(r)|^2\|_{L^\infty}^2 \|\nabla u^\epsilon(r)\|_{L^4}^2 \|Q^{\frac12} e_i\|_{L^4}^2 \\ &+ \|g''(|u^\epsilon(r)|^2)|u^\epsilon(r)|^3\|_{L^\infty}^2 \|\nabla u^\epsilon(r)\|_{L^4}^4 \|Q^{\frac12} e_i\|_{L^\infty}^2 \\ &+ \|g'(|u^\epsilon(r)|^2)u^\epsilon(r)\|_{L^\infty}^2 \|\nabla u^\epsilon(r)\|_{L^4}^4 \|Q^{\frac12} e_i\|_{L^\infty}^2 \Big) \\ \le C \sum_{i\in\mathbb{N}^+} &\Big(\|\nabla Q^{\frac12} e_i\|_{L^4}^2 + \|\nabla Q^{\frac12} e_i\|_{L^\infty}^2 + \|\Delta Q^{\frac12} e_i\|^2 + \|Q^{\frac12} e_i\|_{L^\infty}^2 \Big) \\ &\times \Big(1 + \|\nabla u^\epsilon(r)\|_{L^4}^4 \Big). \end{aligned}$$

Combining the above estimates, and using the Young inequality and Gronwall inequality, we obtain

$$\mathbb{E}\Big[\|\Delta u^\epsilon(t)\|^2 \Big] \le C(u_0, T, Q, p)(1 + \epsilon^{-2}).$$

Now, taking the supremum over t, then taking the expectation, and applying the Burkholder inequality to the term II_{sto}, we achieve that

$$\mathbb{E}\Big[\sup_{t\in[0,\tau]} \|\Delta u^\epsilon(t)\|^2 \Big] \le C(u_0, T, Q)(1 + \epsilon^{-2}).$$ ∎

Proof of Proposition 3. We follow the steps in the proof of Proposition 2 to present the proof in the case of $p = 2$. For convenience, we present the proof for the multiplicative noise case. Applying the Itô formula to $\|u^\epsilon(t)\|_{L^2_\alpha}^2 = \int_{\mathbb{R}^d} (1+|x|^2)^\alpha |u^\epsilon(t)|^2 dx$, using integration by parts, then taking the supremum over t, and applying the Burkholder inequality, we deduce that

$$\begin{aligned} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u^\epsilon(t)\|_{L^2_\alpha}^2 \Big] &\le \mathbb{E}\Big[\|u_0\|_{L^2_\alpha}^2 \Big] + 2\alpha \mathbb{E}\Big[\int_0^T \Big| \langle (1+|x|^2)^{\alpha-1} x u^\epsilon(s), \mathbf{i}\nabla u^\epsilon(s) \rangle \Big| ds \Big] \\ &\quad + \mathbb{E}\Big[\sup_{t\in[0,T]} \Big| \int_0^t \langle (1+|x|^2)^\alpha u^\epsilon(s), \mathbf{i} g(|u^\epsilon(s)|^2) u^\epsilon(s) dW(s) \rangle \Big| \Big] \\ &\le \mathbb{E}\Big[\|u_0\|_{L^2_\alpha}^2 \Big] + C_\alpha \mathbb{E}\Big[\int_0^T \Big| \langle (1+|x|^2)^{\alpha-1} x u^\epsilon(s), \mathbf{i}\nabla u^\epsilon(s) \rangle \Big| ds \Big] \\ &\quad + C\mathbb{E}\Big[\int_0^T \sum_{i\in\mathbb{N}^+} \Big| \langle (1+|x|^2)^\alpha u^\epsilon(s), \mathbf{i} g(|u^\epsilon(s)|^2) u^\epsilon(s) Q^{\frac12} e_i \rangle \Big|^2 \Big| ds \Big]. \end{aligned}$$

By Hölder's inequality, for $\alpha \in (1, 2]$, we have that

$$\Big| \langle (1+|x|^2)^{\alpha-1} x u^\epsilon(s), \nabla u^\epsilon(s) \rangle \Big| \le C \|u^\epsilon(s)\|_{L^2_\alpha} \|(1+|x|^2)^{\frac{\alpha}{2}-\frac12} \nabla u^\epsilon(s)\|.$$

Integration by parts and Young's and Hölder's inequalities yield that for $\alpha \le 2$ and small $\eta > 0$,

$$\begin{aligned}
&\|(1+|x|^2)^{\frac{\alpha}{2}-\frac{1}{2}}\nabla u^\epsilon(s)\|^2 = \langle (1+|x|^2)^{\alpha-1}\nabla u^\epsilon(s), \nabla u^\epsilon(s)\rangle \\
&= -\langle (1+|x|^2)^{\alpha-1}u^\epsilon(s), \Delta u^\epsilon(s)\rangle - 2(\alpha-1)\langle (1+|x|^2)^{\alpha-2}x\nabla u^\epsilon(s), u^\epsilon(s)\rangle \\
&\le \|u^\epsilon(s)\|_{L^2_{\max(2\alpha-2,0)}}\|\Delta u^\epsilon(s)\| + C(\eta)|\alpha-1|\|u^\epsilon(s)\|^2_{L^2_\alpha} + \eta|\alpha-1|\|\nabla u^\epsilon(s)\|^2_{L^2_{\max(\alpha-3,0)}}.
\end{aligned}$$

Combining the above estimates, Proposition 2, Lemma 7, and using Young's inequality, we achieve that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^\epsilon(t)\|^2_{L^2_\alpha}\Big] \le e^{CT}(1+\epsilon^{-1}) \text{ if } \alpha \in \Big(1, \frac{3}{2}\Big],\\
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^\epsilon(t)\|^2_{L^2_\alpha}\Big] \le e^{CT}(1+\epsilon^{-\frac{3}{2}}) \text{ if } \alpha \in \Big[\frac{3}{2}, 2\Big),\\
&\mathbb{E}\Big[\sup_{t\in[0,T]}\|u^\epsilon(t)\|^2_{L^2_\alpha}\Big] \le e^{CT}(1+\epsilon^{-2}) \text{ if } \alpha = 2.
\end{aligned}$$

∎

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