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OPTIMAL CONTROL FOR STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION ON GRAPH*

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Abstract. We study the optimal control formulation for stochastic nonlinear Schrödinger equation (SNLSE) on a finite graph. By viewing the SNLSE as a stochastic Wasserstein Hamiltonian flow on density manifold, we show the global existence of a unique strong solution for SNLSE with a linear drift control or a linear diffusion control on graph. Furthermore, we provide the gradient formula, the existence of the optimal control and a description on the optimal condition via the forward and backward stochastic differential equations.

Key words. optimal control, density manifold, stochastic nonlinear Schrödinger equation on graph, Wasserstein Hamiltonian flow

MSC codes. 35R02, 30H05, 35Q55, 35Q93, 93E20

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1. Introduction.

The nonlinear Schrödinger equation (NLSE) given in the form of

$$\hbar \mathbf{i} \frac{\partial}{\partial t}\Psi(t,x) = -\frac{\hbar^2}{2}\Delta\Psi(t,x) + \Psi(t,x)\mathbb{V}(x) + \Psi(t,x)f(|\Psi(t,x)|^2)$$

has wide applications in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, and Bose–Einstein condensations (see, e.g., [34, 35, 10]). The unknown $\Psi(t,x)$ represents a complex wave function for $x \in \mathbb{R}^d, \hbar > 0$ is the Planck constant, and $\mathbb{V}(\cdot)$ and $f(\cdot)$ are real-valued functions, referred to as linear and nonlinear interaction potentials, respectively. Considering the randomness in the propagation of nonlinear dispersive waves, the stochastic nonlinear Schrödinger equation (SNLSE)

$$\begin{aligned}\hbar\mathbf{i}d\Psi(t,x) = &-\frac{\hbar^2}{2}\Delta\Psi(t,x)dt + \Psi(t,x)\mathbb{V}(x)dt + \Psi(t,x)f(|\Psi(t,x)|^2)dt \\ &- \mathbf{i}u(t,x)\mu(x)dt + u(t,x)dW(t,x)\end{aligned} \tag{1}$$

has been introduced and studied in recent years (see, e.g., [21, 8, 7, 16, 19]). Here W is a colored Wiener process (see, e.g., [20]) defined by

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$$W(t,x) = \sum_{j=1}^{N} \mu_j e_j(x)\beta_j(t), \quad t \geq 0, \quad x \in \mathbb{R}^d,$$

and

$$\mu(x) = \frac{1}{2}\sum_{j=1}^{N} |\mu_j|^2 |e_j(x)|^2, \quad x \in \mathbb{R}^d$$

with $N \in \mathbb{N} \cup \infty$, $\mu_j \in \mathbb{C}$, e_j real-valued function and β_j independent Brownian motion on a complete filtrated probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Another physical significance of SNLSE is related to the theory of measurements continuous in time in quantum mechanics and open quantum system (see, e.g., [3, 4]).

In this paper, we focus on two types of SNLSEs on a finite graph $G = (V, E, w)$ and their related stochastic control problems. Here V is the vertex set, E is the edge set, and w_{jl} is the weight of the edge $(j,l) \in E$ satisfying $\omega_{lj} = \omega_{jl} > 0$ if there is an edge between nodes j and l, and $\omega_{jl} = 0$, otherwise. Throughout this paper, we assume that G is an undirected, connected graph with no self loops or multiple edges. The first type is the nonlinear Schrödinger equation with random perturbation,

$$\mathbf{i} du_j = \Big(-\frac{1}{2}(\Delta_G u)_j + u_j \mathbb{V}_j + u_j f_j(|u|^2) \Big) dt + \sigma_j u_j \circ dW_t. \tag{2}$$

Here Δ_G is a nonlinear discretization of Laplacian operator on G introduced in [12] (see (13) for its formula), $f_j : \mathbb{R} \to \mathbb{R}$ is a continuous real-valued function, $\mathbb{V}_j$ is a given linear potential on the node j, $\sigma_j \in \mathbb{R}$ represents the diffusion coefficient, and $\{W_t\}_{t\geq 0}$ is one dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The stochastic differential $\circ dW_t$ is understood in the Stratonovich sense. A typical example of the nonlinear function f_j is that $f_j(|u|^2) = \sum_{l=1}^{N} \mathbb{W}_{jl}|u_l|^2$ with an interactive potential $\mathbb{W}_{jl} = \mathbb{W}_{lj}$ for any $(j,l) \in E$. We would like to remark that (2) can be viewed as a spatial discretization of (1) when G is a lattice obtained by discretizing a continuous domain (see, e.g., [15]). Another type is the NLSE with white noise dispersion

$$\mathbf{i} du_j = -\frac{1}{2}(\Delta_G u)_j \circ dW_t + (u_j \mathbb{V}_j + u_j f_j(|u|^2)) dt. \tag{3}$$

When G is a lattice, (3) becomes a spatial discretization of NLSE with white noise dispersion [22], which describes the propagation of a signal in an optical fiber with dispersion management.

Our current investigation is motivated by several reasons. First, the Schrödinger equation on graph and its control problem have their own interest and applications [11, 6, 32, 23]. Second, in contrast to the extensive literature on the optimal control problem and exact controllability of Schrödinger equations on continuous domain in both the deterministic and stochastic cases (see, e.g., [24, 26, 29, 27, 28, 2]), far fewer results are known when the problem settings are on graphs. One of the main difficulties lies on the weak regularization effect of free Schrödinger group and the nonlinear Laplacian operator on graph [12]. Another one arises from the compact embedding theorem in probability space. Last but not least, both NLSE and SNLSE on a lattice graph can be viewed as a semidiscretization of NLSE and SNLSE on a continuous domain, respectively (see, e.g., [15]), hence, can be used as numerical schemes to compute (stochastic) optimal control problems involving SNLSEs in practice. However, many challenging questions remain open, such as the preservation of mass, energy, and symplectic structures, and the convergence analysis of semidiscretization of SNLSEs (see, e.g., [14] for more discussions).

Inspired by the optimal control of quantum mechanical system [33, 36], we shall study an optimal control problem associated with (2) or (3). Formally, we can view their solution $u = u(j,t,\widehat{\omega}), t \geq 0, \widehat{\omega} \in \Omega$, as the quantum state or the nonlinear wave at time t. The stochastic perturbation may represent an inaccurate measurement via the quantum observation or a dispersion management in optical fiber. The optimal control problem considered here is to find an input potential $\mathbb{V}$ (or a diffusion coefficient σ) such that the state $u(T)$ is as close as possible to a target state $f^1(T)$ and a trajectory Z^1, and achieves the minimum cost (see sections 4 and 5 for more details). A different viewpoint for this problem is to recover the quantum mechanical potential $\mathbb{V}$ or a diffusion coefficient σ from the observation of the quantum state or the nonlinear wave $u(T)$ at the end of $[0,T]$. Despite many fruitful results on the continuous optimal control problems for NLSE and SNLSE [9, 5, 24, 26, 29, 27, 28, 2], a few exist for the problem defined on a graph. To the best of our knowledge, no result has been reported for stochastic control systems with (2) or (3).

In this work we study both linear drift and diffusion control. Our approach is based on two key ideas. One is used by Nelson in his derivation for NLSE [31]. The other is viewing SNLSE as a stochastic Wasserstein Hamiltonian flow [18]. By using the complex expression $u = \sqrt{\rho}e^{\mathbf{i}S}$, we obtain the equivalent Madelung systems of SNLSE on graph (see, e.g., [12, 19]). Then by exploiting the properties of Madelung systems, we obtain the existence and uniqueness of the strong solution of (2) or (3) when the control $\mathbb{V}$ or σ is admissible. When the graph is taken as a lattice, we prove that the SNLSE on graph with the nonlinear Laplacian operator preserves the stochastic dispersion relationship, while any linear discretization does not. Furthermore, for a quadratic (or convex) cost functional, we provide the gradient formula and prove the existence of the optimal control by carefully studying the probability of tail event of (2) or (3). When σ is a constant potential on every node, we derive the adjoint equation of (2) or (3) which gives a forward-backward stochastic differential equation and characterizes the necessary optimal condition for the optimal control problem on graph.

Our paper is organized as follows. In section 2, we explain why we consider the nonlinear Laplacian for the stochastic Schrödinger equation on graph. In section 3, we present some useful properties of the stochastic Schrödinger equation on graph. In section 4, we prove the existence and uniqueness result for (2) or (3) with admissible control variables and prove the existence result of the optimal control. In section 5, we derive the gradient formula and present the necessary optimal condition by deriving a forward-backward stochastic differential equation.

2. Why nonlinear Laplacian for stochastic Schrödinger equation on graph? To explain the reason, we consider the stochastic linear Schrödinger equation

$$\mathbf{i}du = -\frac{1}{2}\Delta u dt + \sigma u \circ dW_t \tag{4}$$

and the white noise dispersion linear Schrödinger equation

$$\mathbf{i}du = -\frac{1}{2}\Delta u \circ dW_t. \tag{5}$$

One can directly verify that these equations possess the stochastic dispersion relationship by Itô's formula.

LEMMA 2.1. *Let $\sigma \in \mathbb{R}$. Equation (4) (or (5)) admits infinitely many plane wave solutions given in the form of $u(x,t) = Ae^{\mathbf{i}(\mathbb{K}\cdot x - \mu t - \sigma W(t))}$ (or $Ae^{\mathbf{i}(\mathbb{K}\cdot x - \mu W(t))}$) with arbitrary $A \in \mathbb{R}^+$, any wave number $\mathbb{K} \in \mathbb{R}^d$, and frequency μ satisfying $\mu = \frac{1}{2}|\mathbb{K}|^2$.*

From the above result, we see that the stochastic dispersion relationship $\mu = \frac{1}{2}|\mathbb{K}|^2$ coincides with the classical dispersion relationship, and the argument of the plane wave contains all the information of the Wiener process. However, such a simple property may become problematic in discrete settings. To illustrate where the trouble is, let us consider a lattice G obtained by discretizing $\mathbb{R}^d$ or $\mathbb{T}^d$. Any linear discretizations of (4) and (5) can be stated

$$\mathbf{i} du_j = -\frac{1}{2} \sum_{l \in N(j)} C_{lj} u_l dt + \sigma u_j \circ dW_t \tag{6}$$

and

$$\mathbf{i} du_j = -\frac{1}{2} \sum_{l \in N(j)} C_{lj} u_l \circ dW_t, \tag{7}$$

respectively. Here $\{C_{lj}\}_{(l,j)\in E}$ are chosen to approximate the Laplacian operator in (4) and (5). For simplicity, we assume that every node has the same number of adjacent nodes, and that the weight on each edge is uniformly given by Δx. We denote the coordinate of the node j by $x_j = j\Delta x$. Regardless of how $\{C_{lj}\}_{(l,j)\in E}$ are selected, there are at most a finite discrete stochastic plane waves which satisfy the stochastic dispersion relationship.

Theorem 2.1. *For any linear discretization of* (4) *and* (5)*, there exist at most a finite number of pairs* $(\mu, \mathbb{K})$ *with* $\mu = \frac{1}{2}|\mathbb{K}|^2$ *so that the discrete stochastic plane waves, i.e.,* $u_j = Ae^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu t - \sigma W(t))}$ *for* (6) *(or* $Ae^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu W(t))}$ *for* (7)*), are the solutions.*

Proof. Consider the discrete stochastic plane waves $u_j(t) = Ae^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu t - \sigma W(t))}$ for (4) and $u_j(t) = Ae^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu W(t))}$ for (5). Substituting them into (6) and (7), we get

$$\mu A e^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu t - \sigma W(t))} dt = \frac{1}{2} \sum_{l\in N(j)} C_{lj} A e^{\mathbf{i}(\mathbb{K}\cdot x_l - \mu t - \sigma W(t))} dt$$

and

$$\mu A e^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu W(t))} \circ dW(t) = \frac{1}{2} \sum_{l\in N(j)} C_{lj} A e^{\mathbf{i}(\mathbb{K}\cdot x_l - \mu W(t))} \circ dW(t),$$

respectively. If $\mu = \frac{1}{2}|\mathbb{K}|^2$, we obtain

$$\mu = \frac{|\mathbb{K}|^2}{2} = \frac{1}{2} \sum_{l\in N(j)} C_{lj} e^{\mathbf{i}(\mathbb{K}\cdot(x_l - x_j))}.$$

Since $\frac{|\mathbb{K}|^2}{2}$ is quadratic in $\mathbb{K}$ while the trigonometric polynomial on the right-hand side is periodic and bounded with respect to $\mathbb{K}$, they only intersect in a bounded ball of the complex domain $|\mathbb{K}| \le C_N < \infty$. Besides, it can be seen that the imaginary part of $\frac{1}{2}\sum_{l\in N(j)} C_{lj} \sin(\mathbb{K}\cdot(x_l - x_j)) = 0$ has at most finite zero point. Thus, we complete the proof. ∎

To numerically preserve the stochastic dispersion relationship for any pair of $(\mu, \mathbb{K})$ with $\mu = \frac{1}{2}|\mathbb{K}|^2$, we decide to use the nonlinear Laplacian operator Δ_G constructed by using the Madelung transformation as shown in [12, 14].

3. Stochastic nonlinear Schrödinger equation on graph.

Consider a graph $G = (V, E, \omega)$. Let us denote the set of discrete probabilities on the graph by

$$\mathcal{P}(G) = \left\{ (\rho)_{j=1}^N \ : \sum_{j=1}^N \rho_j = 1, \rho_j \ge 0 \text{ for } j \in V \right\},$$

and $\mathcal{P}_o(G)$ as its interior (i.e., all $\rho_j > 0$ for $j \in V$). $\mathbb{V}_j$ is a linear potential on each node j, and $\mathbb{W}_{jl} = \mathbb{W}_{lj}$ is an interactive potential between nodes j and l. We denote $N(i) = \{j \in V : (i,j) \in E\}$ to be the adjacency set of the node i and $\theta_{ij}(\rho)$ to be a density dependent weight on the edge $(i,j) \in E$. More precisely, θ is defined by $\theta_{ij}(\rho) = \Theta(\rho_i, \rho_j)$, where Θ is a continuous differentiable function on $(0,1)^2$ satisfying $\Theta(x,y) = \Theta(y,x)$, $\Theta(x,y) \geq 0$, and $\min(x,y) \leq \Theta(x,y) \leq \max(x,y)$ for any $x, y \in (0,1)$. For example, we may take $\theta(\rho)$ as the averaged probability weight in [12], i.e., $\Theta(x,y) = \frac{1}{2}(x+y)$, or the logarithmic probability weight in [14], i.e., $\Theta(x,y) = \int_0^1 x^{1-t} y^t dt$, or the harmonic probability weight in [30], i.e., $\Theta(x,y) = \frac{2}{1/x + 1/y}$.

In this section, we present the SNLSEs on graph via the viewpoint of stochastic variational principle proposed in [18]. Define the total linear potential function $\mathcal{V}$, interaction potential function $\mathcal{W}$, and the entropy function L by

$$\mathcal{V}(\rho) = \sum_{i=1}^N \mathbb{V}_i \rho_i, \quad \mathcal{W}(\rho) = \frac{1}{2} \sum_{i,j=1}^N \mathbb{W}_{ij} \rho_i \rho_j, \quad L(\rho) = \sum_{i=1}^N (\log(\rho_i)\rho_i - \rho_i).$$

$I(\rho)$ is the discrete Fisher information on graph, i.e.,

$$I(\rho) = \frac{1}{2} \sum_{i=1}^N \sum_{j \in N(i)} \widetilde{\omega}_{ij} |\log(\rho_i) - \log(\rho_j)|^2 \widetilde{\theta}_{ij}(\rho), \tag{8}$$

where $(\widetilde{\omega}, \widetilde{\theta})$ is another pair of weight and density dependent weight on the edges G. We remark that $(\widetilde{\omega}, \widetilde{\theta})$ may be selected the same as or differently from (ω, θ). Throughout this paper, we take θ as the averaged probability weight, $\widetilde{\theta}$ as the logarithmic probability weight, and $\omega_{ij} = \widetilde{\omega}_{ij}$ for simplicity.

As given in [17], the stochastic variational principe on graph is defined as

$$\mathcal{I}\left(\rho^0, \rho^T\right) = \inf\left\{\mathcal{S}(\rho_t, \Phi_t) | \Delta_{\rho_t} \Phi_t \in \mathcal{T}_{\rho_t} \mathcal{P}_o(\mathcal{M}), \rho(0) = \rho^0, \rho(T) = \rho^T\right\}, \tag{9}$$

whose action functional is expressed in the dual coordinates,

$$\begin{aligned}\mathcal{S}(\rho_t, \Phi_t) = \langle \rho(0), \Phi(0) \rangle - \langle \rho(T), \Phi(T) \rangle + \int_0^T \langle \partial_t \Phi(t), \rho_t \rangle + \mathcal{H}_0(\rho_t, \Phi_t) dt \\ + \int_0^T \mathcal{H}_1(\rho_t, \Phi_t) \circ dW_t.\end{aligned}$$

Here $\Delta_\rho := div_G^\theta(\rho \nabla_G(S))$ defined by

$$\left(div_G^\theta(\rho \nabla_G(\cdot))\right)_i := \sum_{j \in N(i)} \theta_{ij}(\rho) \sqrt{\omega_{ij}} (S_j - S_i)$$

for any potential function $S = \{S_i\}_{i \in V}$. The vector field $\nabla_G S$ induced by S is defined by $\nabla_G(S) := (\sqrt{\omega_{ij}}(S_i - S_j))_{ij \in E}$. With the above notation, one can also introduce the inner product for the vector fields on graph defined by

$$\langle u, v \rangle_{\theta(\rho)} := \frac{1}{2} \sum_{ij \in E} u_{ij} v_{ij} \theta_{ij}(\rho) \omega_{ij}$$

for any two vector fields (skew-symmetric matrices) u, v. The kinetic energy is defined by $K(S, \rho) = \frac{1}{2} \langle \nabla_G S, \nabla_G S \rangle_{\theta(\rho)}$. Here ρ^0, ρ^T are $\mathcal{F}_0$ and $\mathcal{F}_T$ measurable functions. The dominated energy $\mathcal{H}_0$ and perturbed energy $\mathcal{H}_1$ are given by

$$\mathcal{H}_0(\rho,S)=K(S,\rho)+F(\rho)-\kappa L(\rho),$$
$$\mathcal{H}_1(\rho,S)=\eta_1 K(S,\rho)+\eta_2 I(\rho)+\eta_3\Sigma(\rho)+\eta_4\mathcal{W}(\rho)+\eta_5 L(\rho)$$

with $\kappa\in\mathbb{R}$, Σ defined by $\Sigma(\rho)=\sum_{j=1}^N\sigma_j\rho_j$ for some $\sigma_j\in\mathbb{R}$, and $F(\rho):=\frac{1}{8}I(\rho)+\mathcal{V}(\rho)+\mathcal{W}(\rho)$. In particular, when $\eta_1=0$, (9) recovers the classical variational problem with random potential in Lagrangian formalism.

By finding the critical point of the stochastic variational principle (9), we achieve the following discrete stochastic Wasserstein Hamiltonian flow on the density manifold:

$$\begin{aligned} d\rho &= \frac{\partial}{\partial S}\mathcal{H}_0(\rho,S)dt+\frac{\partial}{\partial S}\mathcal{H}_1(\rho,S)\circ dW_t,\\ dS &= -\frac{\partial}{\partial \rho}\mathcal{H}_0(\rho,S)dt-\frac{\partial}{\partial \rho}\mathcal{H}_1(\rho,S)\circ dW_t. \end{aligned}\tag{10}$$

Selecting different deterministic energy $\mathcal{H}_0$ and perturbed energy $\mathcal{H}_1$ results in various forms of SNLSEs on graph. When $\mathcal{H}_0(\rho,S)=K(S,\rho)+\mathcal{F}(\rho)-\kappa L(\rho)$, $\mathcal{H}_1(\rho,S)=\Sigma(\rho)$, the Wasserstein Hamiltonian flow becomes

$$\begin{aligned} &d\rho_i+\sum_{j\in N(i)}\omega_{ij}(S_j-S_i)\theta_{ij}(\rho)dt=0,\\ &dS_i+\frac{1}{2}\sum_{j\in N(i)}\omega_{ij}(S_i-S_j)^2\frac{\partial\theta_{ij}(\rho)}{\partial\rho_i}dt+\frac{1}{8}\frac{\partial I(\rho)}{\partial\rho_i}dt+\mathbb{V}_i dt\\ &\quad+\sum_{j=1}^N\mathbb{W}_{ij}\rho_j dt-\kappa\log(\rho_i)dt+\sigma_i dW_t=0. \end{aligned}\tag{11}$$

Its complex formulation $u(t)=\sqrt{\rho(t)}e^{\mathbf{i}S(t)}$ gives the stochastic nonlinear Schrödinger on graph,

$$\mathbf{i}du_j=\left(-\frac{1}{2}(\Delta_G u)_j+u_j\mathbb{V}_j+u_j\sum_{l=1}^N\mathbb{W}_{jl}|u_l|^2-u_j\kappa\log(|u_j|^2)\right)dt+\sigma_j u_j\circ dW_t.\tag{12}$$

Here the nonlinear Laplacian on the graph is defined by

$$\begin{aligned} (\Delta_G u)_j=-u_j\Bigg(&\frac{1}{|u_j|^2}\Bigg[\sum_{l\in N(j)}\omega_{jl}(\Im(\log(u_j))-\Im(\log(u_l))\theta_{jl})\\ &+\sum_{l\in N(j)}\widetilde{\omega}_{jl}\widetilde{\theta}_{jl}(\Re(\log(u_j))-\Re(\log(u_l)))\Bigg]\\ &+\sum_{l\in N(j)}\omega_{jl}\frac{\partial\theta_{jl}}{\partial\rho_j}|\Im(\log(u_j)-\log(u_l))|^2\\ &+\sum_{l\in N(j)}\widetilde{\omega}_{jl}\frac{\partial\widetilde{\theta}_{jl}}{\partial\rho_j}|\Re(\log(u_j)-\log(u_l))|^2\Bigg), \end{aligned}\tag{13}$$

where $\Re$ and $\Im$ are real and imaginary parts of a complex number. This is precisely the nonlinear graph Laplacian introduced in [14].

When $\mathcal{H}_0=\mathcal{V}(\rho)+\mathcal{W}(\rho)$, $\mathcal{H}_1=K(\rho,S)+\frac{1}{8}I(\rho)$, the Wasserstein Hamiltonian flow becomes

(14)

$$d\rho_i = \sum_{j\in N(i)} \omega_{ij}(S_i - S_j)\theta_{ij}(\rho) \circ dW_t;$$

$$dS_i + \left(\frac{1}{2}\sum_{j\in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}}{\partial \rho_i} + \frac{1}{8}\frac{\partial}{\partial \rho_i} I(\rho)\right) \circ dW_t + \left(\mathbb{V}_i + \sum_{j=1}^N \mathbb{W}_{ij}\rho_j\right)dt = 0,$$

whose complex formulation $u(t) = \sqrt{\rho(t)}e^{\mathbf{i}S(t)}$ satisfies the NLSEs with white noise dispersion on graph,

$$\mathbf{i}du_j = -\frac{1}{2}(\Delta_G u)_j \circ dW_t + \left(u_j\mathbb{V}_j + u_j\sum_{l=1}^N \mathbb{W}_{jl}|u_l|^2\right)dt. \tag{15}$$

Both (12) and (15) can be viewed as spatial discretization of (2) and (3), respectively, when G is a lattice graph.

Recall that in [12, 14], the global solution in deterministic case ($\eta_1 = \cdots = \eta_4 = \eta_5 = 0$, $\kappa = 0$) is obtained by using the energy conservation law if $\mathcal{F}(\rho)$ contains the Fisher information $\beta I(\rho), \beta > 0$. In the stochastic case, the existence of global solution has been studied in [17] by using the Poisson bracket $\{\cdot,\cdot\}$. In particular, when $\{\mathcal{H}_0, \mathcal{H}_1\} = 0$, for example, $\mathcal{H}_0$ is a multiple of $\mathcal{H}_1$, then $\mathcal{H}_0$ is an invariant of the stochastic Wasserstein Hamiltonian flow. Here we summarize some fundamental properties shared by the SNLSEs on graph.

PROPOSITION 3.1. *Let $T > 0$, $u(0)$ be $\mathcal{F}_0$-measurable with any finite moment, and let $u_j(0) \neq 0$ for all $j \in V$. Then* (12) *(or* (15)*) has a unique strong solution $u(t)$ on $[0,T]$. Moreover, $u(t)$ satisfies the following properties:*

(i) *It conserves the total mass*

$$\sum_{j=1}^N |u_j(t)|^2 = 1, \text{ a.s.}.$$

(ii) *The total energy satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \mathcal{E}^p(u(t))\Big] \le C(\mathcal{E}(u(0)), T, p),$$

where $\mathcal{E}$ is defined by a combination of the discrete kinetic energy $\mathcal{E}_{kin}$, linear potential $\mathcal{E}_{lin}$, interaction potential $\mathcal{E}_{int}$, and entropy $\mathcal{E}_{ent}$, i.e.,

$$\mathcal{E}(u) = \mathcal{E}_{kin}(u) + \mathcal{E}_{lin}(u) + \mathcal{E}_{int}(u) + \mathcal{E}_{ent}(u).$$

Here we have

$$\begin{aligned}\mathcal{E}_{kin}(u) = \frac{1}{4}\sum_{(j,l)\in E} &\left\{|\Re(\log u_j - \log(u_l))|^2\omega_{jl}\theta_{jl}(|u|^2)\right.\\ &\left.+|\Im(\log u_j - \log(u_l))|^2\widetilde{\omega}_{jl}\theta_{jl}(|u|^2)\right\},\end{aligned}$$

$$\mathcal{E}_{lin}(u) = \sum_{j=1}^N \mathbb{V}_j|u_j|^2,\ \mathcal{E}_{int}(u) = \frac{1}{2}\sum_{j,l=1}^N \mathbb{W}_{jl}|u_j|^2|u_l|^2,$$

$$\mathcal{E}_{ent}(u) = -\kappa\sum_{j=1}^N (\log(|u_j|^2)|u_j|^2 - |u_j|^2).$$

(iii) *It is time transverse invariant when* $\mathbb{V}$ *is independent of time: if* $u^{\alpha}(t)$ *is the solution of* (12) *(or* (15)*), where* $\mathbb{V}^{\alpha} = (\mathbb{V}_j + \alpha)_{j=1}^{N}$ *with* α *being a constant* $\mathcal{F}_0$*-measurable random variable, then*

$$u^{\alpha}(t) = u(t)e^{\mathbf{i}\alpha t}$$

is also a solution.

(iv) *It is time reversible when* $\mathbb{V}$ *is independent of time in the following sense: for* (12) *(or* (15)*) with* $\widetilde{W}(t) = W(t), t \geq 0$ *and* $\widetilde{W}(t) = -W(-t), t < 0$*, then*

$$u(t) = \bar{u}(-t).$$

Following the proof of [17, section 4], one can also obtain the lower bounds for the density trajectories as stated in the next corollary.

Corollary 3.1. *Let the conditions of Proposition* 3.1 *hold. For* (11) *and* (14)*, there exists a positive random variable which is a lower bound of the density trajectory.*

To end this section, we demonstrate that the nonlinear discretization of (4) and (5) can preserve exactly the stochastic dispersion relationship. Consider the graph version of (4),

$$\mathbf{i}du_j = -\frac{1}{2}(\Delta_G u)_j dt + \sigma u_j \circ dW_t \tag{16}$$

and that of (5),

$$\mathbf{i}du_j = -\frac{1}{2}(\Delta_G u)_j \circ dW_t. \tag{17}$$

Proposition 3.2. *Given a lattice graph* G *with* $|x_j - x_l| = \Delta x$ *for* $l \in N(j)$*,* $\omega_{ij} = (\frac{\partial \theta_{ij}}{\partial \rho_i}\mathcal{N}(\Delta x)^2)^{-1}$*, where* $\mathcal{N}$ *is the total number of nodes in* $N(j)$ *and* θ_{ij} *is the symmetric probability weight. The nonlinear discretizations of* (16) *and* (17) *preserve the stochastic dispersion relationship.*

Proof. The discrete stochastic plane waves read $u_j(t) = Ae^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu t - \sigma W(t))}$ for (4) and $u_j(t) = Ae^{\mathbf{i}(\mathbb{K}\cdot x_j - \mu W(t))}$ for (5) with $\mu = \frac{1}{2}|\mathbb{K}|^2$. By the Madelung transformation $u_j = \sqrt{\rho_j}e^{\mathbf{i}S_j(t)}$, $\rho_j = A$ is constant. As a consequence, the partial derivative of Fisher information $\frac{\partial I(\rho)}{\partial \rho_i} = 0$. On the other hand, since $S_i = \mathbb{K}\cdot x_i - \mu t - \sigma W(t)$, one can verify that $\frac{1}{2}\sum_{j\in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i} = \frac{1}{2}|\mathbb{K}|^2 = \mu$. This implies that

$$dS_i + \frac{1}{2}\sum_{j\in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i}dt + \frac{1}{8}\frac{\partial I(\rho)}{\partial \rho_i}dt + \sigma dW_t = 0$$

is satisfied. Thus (4) preserves all the stochastic dispersion relationship.

Similar calculations can show that (5) satisfies

$$dS_i + \frac{1}{2}\sum_{j\in N(i)} \omega_{ij}(S_i - S_j)^2 \frac{\partial \theta_{ij}(\rho)}{\partial \rho_i} \circ dW_t = 0,$$

which implies that (5) preserves all the stochastic dispersion relationship. ∎

4. Stochastic control problem on density manifold of finite graph. In this section, we propose two stochastic optimal control formulations corresponding to SNLSEs (2) and (3) on graph, respectively.

4.1. Stochastic control problem with linear potential control. We first assume that the linear potential term $\{\mathbb{V}_j\}_{j\in V}$ is a control variable depending on t. It can be seen that this will not affect the well-posedness of (2) and (3). For convenience, we denote the corresponding solution by $u_j^{\mathbb{V}}$ in the complex function representation and $(\rho_j^{\mathbb{V}}, S_j^{\mathbb{V}})$ on Wasserstein manifold. The admissible control set $\mathcal{U}$ is defined by

$$\begin{aligned}\mathcal{U} := \Big\{ &\mathbb{V} : \Omega \times [0,T] \to \mathbb{R}^N \,\big|\, \mathbb{V}(t) \text{ is } \mathcal{F}_t\text{-adapted}, \mathbb{V}_j \in L^2([0,T]),\\ &\text{there exists } \alpha > 0, \text{ such that } |\mathbb{V}_j| \leq \alpha \text{ a.s. for } j \in V \Big\}\end{aligned}$$

for some $\alpha > 0$. Our first optimal control problem is to minimize the cost functional

$$J(\mathbb{V}) := \gamma \mathbb{E}\Bigg[\sum_{i=1}^N |u_j^{\mathbb{V}}(T) - f_j^1|^2\Bigg] + \beta \mathbb{E}\Bigg[\int_0^T \sum_{i=1}^N |\mathbb{V}_j(t) - Z_j(t)|^2 dt\Bigg], \tag{18}$$

subject to the constraint given by either (11) or (14) with given $(\rho(0), S(0))$. Here $\gamma, \beta \geq 0$, f^1 is $\mathcal{F}_T$-adapted satisfying $\|f^1\|_{L^2(\Omega;\mathbb{C}^N)} < \infty$, and $Z \in \mathcal{U}$. The above optimal control problem may be viewed as the graph version of the stochastic control problem in [1, 26, 27, 28]. The following lemma (see, e.g., [25, Chapter 3]) is very useful to show the existence and uniqueness of the optimal control.

LEMMA 4.1. *Let $\mathcal{B}$ be a uniformly convex Banach space, and let $\mathcal{S}$ be a bounded closed subset of $\mathcal{B}$. Furthermore, let $F : \mathcal{S} \to \overline{\mathbb{R}}$ be a lower semicontinuous functional which is bounded from below and $p \geq 1$. Then there exists a dense subset $\mathcal{D} \subset \mathcal{B}$ such that for each $x \in \mathcal{D}$, the functional $F(s) + \|s - x\|_{\mathcal{B}}^p$ attains its minimum over $\mathcal{S}$, which implies that there exists an $s(x) \in \mathcal{S}$ such that*

$$F(s(x)) + \|s(x) - x\|_{\mathcal{B}}^p = \inf_{s\in\mathcal{S}} \{F(s) + \|s - x\|_{\mathcal{B}}^p\}.$$

In particular, if $p > 1$, then $s(x)$ is unique. Besides, each minimizing sequence converges strongly and the function $x \mapsto s(x)$ is continuous in $\mathcal{D}$.

In our case, we take $\mathcal{B} := L^2(\Omega \times [0,T]; \mathbb{C}^N)$ which is uniformly convex, and choose $\mathcal{S}$ as the admission control set. The functional $F = \gamma\mathbb{E}[\sum_{i=1}^N |u_j^{\mathbb{V}}(T) - f_j^1|^2]$ is bounded from below and $p = 2$. According to Lemma 4.1, if we can verify the lower semicontinuity of F, then there exists a dense subset $\mathcal{D}$ of $\mathcal{B}$ such that for each $Z \in \mathcal{D}$ the functional $J(\mathbb{V}) = F(\mathbb{V}) + \beta\|\mathbb{V} - Z\|_{\mathcal{B}}^2$ attains its unique minimum over $\mathcal{U}$. In other words, there exists a unique $\mathbb{V}^* \in \mathcal{U}$ such that

$$J(\mathbb{V}^*) = F(\mathbb{V}^*) + \beta\|\mathbb{V}^* - Z\|_{\mathcal{B}}^2 = \inf_{\mathbb{V}} J(\mathbb{V}).$$

To prove the lower semicontinuity of $u^{\mathbb{V}}$ with respect to $\mathbb{V}$, we show a strong convergence result first.

PROPOSITION 4.1. *Let $u(0)$ be $\mathcal{F}_0$-adapted with any finite moment satisfying $u_j(0) \neq 0, j \leq N$. Let the sequence $\{\mathbb{V}^n\}_{n\geq 1} \subset \mathcal{U}$ be convergent to $\mathbb{V}$, and let $u^{\mathbb{V}^n}$ be the corresponding solution of the stochastic nonlinear Schrödinger equation* (11) *(or* (14)*) with respect to the control $\mathbb{V}^n$ and the initial value $u^{\mathbb{V}^n}(0) = u(0)$. Then the sequence $(u^{\mathbb{V}^n}) \in L^2(\Omega; \mathcal{C}([0,T]; \mathbb{C}^N)), n \geq 1$, converges strongly to the solution of stochastic nonlinear Schrödinger equation* (11) *(or* (14)*) with respect to the control $\mathbb{V} \in \mathcal{U}$.*

Proof. In this proof, we only show the details when the constraint is (11). A similar argument can lead to the strong convergence result for the case of (14). By Proposition 3.1, the Itô formula, and the Burkholder's inequality, we have the following a priori estimates:

(19)
$$\sum_{i=1}^N \left|u_i^{\mathbb{V}_n}(t)\right|^2 = \sum_{i=1}^N |u_i(0)|^2 = 1, \text{ a.s.}$$
(20)
$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\langle \nabla_G S^{\mathbb{V}_n}(t), \nabla_G S^{\mathbb{V}_n}(t)\rangle_{\theta(\rho^{\mathbb{V}_n}(t))} + \frac{1}{8}I(\rho^{\mathbb{V}_n}(t))\Big)^p\Big] \le C(u(0),T,\alpha,p),\ p\ge 1.$$

To show the strong convergence of $u^{\mathbb{V}_n}$, we introduce a stopping time τ_c defined by

$$\tau_c^n := \inf\{t\in[0,T] : \|S^{\mathbb{V}_n}\|_{\mathcal{C}([0,t];\mathbb{R}^N)} \ge c\} \wedge \inf\left\{t\in[0,T] : \min_{i=1}^N \min_{s\in[0,t]} \rho_i^{\mathbb{V}_n}(s) \le \frac{1}{c}\right\}.$$

By Corollary 3.1, we have that $\lim_{c\to\infty}\tau_c = T,$a.s. Introduce the truncated sample subspace Ω_c^n defined by

$$\Omega_c^n = \left\{\sup_{t\in[0,T]} \|S^{\mathbb{V}_n}\|_{\mathcal{C}([0,t];\mathbb{R}^N)} \le c, \min_{i=1}^N \min_{s\in[0,T]} \rho_i^{\mathbb{V}_n}(s) \ge \frac{1}{c}\right\}.$$

Similarly, we denote Ω_c as the truncated sample subspace with respect to $u^{\mathbb{V}}$. Our goal is to show the error estimate in $\Omega_c^n\cap\Omega_c$ and $\Omega/\{\Omega_c^n\cap\Omega_c\}$. First, we prove the convergence in $\Omega/\{\Omega_c^n\cap\Omega_c\}$. Due to the mass conservation law (19) of the SNLSE, by applying the Chebyshev's inequality, we get

$$\begin{aligned}
&\|1_{\Omega/\{\Omega_c^n\cap\Omega_c\}}(u^{\mathbb{V}^n} - u^{\mathbb{V}})\|_{\mathcal{B}}^2 \\
&\quad\le \int_0^T \mathbb{E}\Big[1_{\Omega/\{\Omega_c^n\cap\Omega_c\}}(|u^{\mathbb{V}^n}(s)|^2 + |u^{\mathbb{V}}(s)|^2)\Big]ds \\
&\quad\le CT\Big[\mathbb{P}\Big(\sup_{s\in[0,T]}|S^{\mathbb{V}^n}|\ge c\Big) + \mathbb{P}\Big(\sup_{s\in[0,T]}|S^{\mathbb{V}}|\ge c\Big) \\
&\qquad + \mathbb{P}\Big(\min_{i=1}^N\min_{s\in[0,T]}\rho_i^{\mathbb{V}^n}(s)\le\frac{1}{c}\Big) + \mathbb{P}\Big(\min_{i=1}^N\min_{s\in[0,T]}\rho_i^{\mathbb{V}}(s)\le\frac{1}{c}\Big)\Big].
\end{aligned}$$

It suffices to prove all the above probabilities converges to 0 as $c\to\infty$. Indeed, since G is connected, by applying the lower bound estimate in [14, section 3], there exists a positive random variable $C(\omega)$ such that

$$\inf_{t\ge 0}\min_{i\le N}\rho_i^{\mathbb{V}^n}(t) \ge c_2\exp(-c_1 C(\omega)). \tag{21}$$

Here $c_2, c_1 > 0$ are constants depending on the structure of G, and $C(\omega)$ is the positive random variable in Corollary 3.1. More precisely, the positive random variable $C(\omega)$ is bounded by the upper bound of $\mathbb{V}^n$ and $\mathbb{V}$ plus

$$\sup_{t\in[0,T]}\Big(\langle\nabla_G S^{\mathbb{V}^n}(t), \nabla_G S^{\mathbb{V}^n}(t)\rangle_{\theta(\rho^{\mathbb{V}^n}(t))} + \frac{1}{8}I(\rho^{\mathbb{V}^n}(t))\Big), \tag{22}$$

which possess any finite moment by (20). Thus, by (21), Chebyshev's inequality and the monotonicity of the logarithmic function, we get

$$\begin{aligned}\mathbb{P}\Big(\min_{s\in[0,T]}\min_{i=1}^{N}\rho_i^{\mathbb{V}^n}(s)\le\frac{1}{c}\Big)&\le\mathbb{P}\Big(c_2\exp(-c_1C(\omega))\le\frac{1}{c}\Big)\\&=\mathbb{P}\Big(C(\omega)\ge\frac{1}{c_1}(\log(c)+\log(c_2))\Big)\\&\le\frac{c_1^p\mathbb{E}\big[C(\omega)^p\big]}{(\log(c)-\log(c_2))^p},\ p\ge1.\end{aligned}\tag{23}$$

When $c\to\infty$, by the dominated convergence theorem, we have that

$$\lim_{c\to\infty}\Big[\mathbb{P}(\min_{i=1}^{N}\min_{s\in[0,T]}\rho_i^{\mathbb{V}^n}(s)\le\frac{1}{c})+\mathbb{P}(\min_{i=1}^{N}\min_{s\in[0,T]}\rho_i^{\mathbb{V}}(s)\le\frac{1}{c})\Big]=0.$$

For the tail estimate of $S^{\mathbb{V}^n}$, we make use of the differential equation of $S^{\mathbb{V}^n}$ and get that

$$\begin{aligned}|S_i^{\mathbb{V}^n}(t)|\le|S_i^{\mathbb{V}^n}(0)|&+\int_0^T\sum_{j\in N(i)}\frac{1}{4}|S_i-S_j|^2\omega_{ij}\Big|\frac{\partial}{\partial\rho_i}I(\rho)\Big|ds\\&+\int_0^T|\mathbb{V}_i^n|+\sum_{j=1}^{N}|\mathbb{W}_{ij}|\rho_jds+\sup_{t\in[0,T]}\Big|\int_0^t\sigma_idW(s)\Big|.\end{aligned}$$

The Burkholder's inequality yields that $\mathbb{E}[\sup_{t\in[0,T]}|\int_0^t\sigma_idW(s)|^p]\le C(p,\sigma)$. Notice that (20) and (21) implies that

$$\begin{aligned}&\max_{ij\in E}|S_i-S_j|^2\le\frac{2}{\min_{ij\in E}\omega_{ij}(\rho_i+\rho_j)}C(\omega)\le\frac{1}{\min_{ij\in E}\omega_{ij}c_2}\exp(c_1C(\omega))C(\omega),\ \text{a.s.}\\&\max_i\Big|\frac{\partial}{\partial\rho_i}I(\rho)\Big|\le\max_{ij}\omega_{ij}\max_i\Big[\frac{2}{\rho_i}+2|\log(\rho_i)|\Big]\\&\qquad\le\max_{ij}\omega_{ij}2\Big(\frac{1}{c_2}\exp(c_1C(\omega))+|\log(c_2)|+c_1C(\omega)\Big)<\infty,\ \text{a.s.}\end{aligned}$$

Combining with the fact that $|\mathbb{V}_i^n(t)|\le\alpha$, we conclude that for c large enough,

$$\begin{aligned}&\mathbb{P}\left(\sup_{s\in[0,T]}|S^{\mathbb{V}^n}|\ge c\right)\\&\le\mathbb{P}\left(\min_{ij\in E}\frac{1}{\omega_{ij}c_2}\exp(c_1C(\omega))C(\omega)\ge\frac{c}{4T}\right)\\&+\mathbb{P}\left(\max_{ij\in E}\omega_{ij}2\left(\frac{1}{c_2}\exp(c_1C(\omega))+|\log(c_2)|+c_1C(\omega)\right)\ge\frac{c}{4T}\right)\\&+\mathbb{P}\left(\sup_{i\le N}\sup_{t\in[0,T]}\Big|\int_0^t\sigma_idW(s)\Big|\ge\frac{c}{4T}\right)\\&+\mathbb{P}\left(\sup_{i\le N}|S_i^{\mathbb{V}^n}(0)|+\max_{ij\in E}\mathbb{W}_{ij}+\alpha T\ge\frac{c}{4T}\right).\end{aligned}$$

Using the moment estimate of $C(\omega)$ and Chebyshev's inequality, we obtain that

$$\lim_{c\to\infty}\mathbb{P}\left(\sup_{s\in[0,T]}|S^{\mathbb{V}^n}|\ge c\right)=0.$$

Similarly, we can get $\lim_{c\to\infty}\mathbb{P}(\sup_{s\in[0,T]}|S^{\mathbb{V}}|\ge c)=0$.

On $\Omega_c^n \cap \Omega_c$, we use the stopping time technique to show the strong convergence. By the definition of τ_c^n and τ_c we can see that $\tau_c^n = T$ on Ω_c^n and $\tau_c = T$ on Ω_c. According to the complex form of $u^{\mathbb{V}^n} = \sqrt{\rho^{\mathbb{V}^n}} e^{\mathbf{i}S^{\mathbb{V}^n}}$, we have that

$$\begin{aligned}
&\int_0^T \mathbb{E}[1_{\Omega_c^n\cap\Omega_c}|u^{\mathbb{V}^n} - u^{\mathbb{V}}|^2]ds \\
&\le \int_0^T \sum_{i=1}^N 2\Big(\mathbb{E}[1_{\Omega_c^n\cap\Omega_c}|\sqrt{\rho_i^{\mathbb{V}^n}} - \sqrt{\rho^{\mathbb{V}}}_i|^2] + \mathbb{E}[1_{\Omega_c^n\cap\Omega_c}|\sqrt{\rho_i^{\mathbb{V}}}(e^{\mathbf{i}S_i^{\mathbb{V}^n}} - e^{\mathbf{i}S_i^{\mathbb{V}}})|^2]\Big)ds \\
&\le C\int_0^T \sum_{i=1}^N \Big(\mathbb{E}[1_{\Omega_c^n\cap\Omega_c}|\sqrt{\rho_i^{\mathbb{V}^n}} - \sqrt{\rho_i^{\mathbb{V}}}|^2] + \mathbb{E}[1_{\Omega_c^n\cap\Omega_c}|S_i^{\mathbb{V}^n} - S_i^{\mathbb{V}}|^2]\Big)ds.
\end{aligned}$$

By applying the Itô formula before $\tau_c^n \cap \tau_c$ and Hölder's inequality, we obtain that

$$\begin{aligned}
&\Big|\sqrt{\rho^{\mathbb{V}^n}(t)} - \sqrt{\rho^{\mathbb{V}}(t)}\Big|^2 \\
&= \int_0^t 2\sum_{i=1}^N \sum_{j\in N(i)} \Big(\frac{1}{\sqrt{\rho_i^{\mathbb{V}^n}}}(S_i^{\mathbb{V}^n} - S_j^{\mathbb{V}^n})\theta_{ij}(\rho^{\mathbb{V}^n}) - \frac{1}{\sqrt{\rho_i^{\mathbb{V}}}}(S_i^{\mathbb{V}} - S_j^{\mathbb{V}})\theta_{ij}(\rho^{\mathbb{V}})\Big) \\
&\quad \Big(\sqrt{\rho_i^{\mathbb{V}^n}} - \sqrt{\rho_i^{\mathbb{V}}}\Big) ds \\
&\le \int_0^t C(1+c)\sum_{i=1}^N \Big(|S_i^{\mathbb{V}^n} - S_i^{\mathbb{V}}||\sqrt{\rho_i^{\mathbb{V}^n}} - \sqrt{\rho_i^{\mathbb{V}}}| + |\sqrt{\rho_i^{\mathbb{V}^n}} - \sqrt{\rho_i^{\mathbb{V}}}|^2\Big)ds
\end{aligned}$$

and that

$$\begin{aligned}
&|S^{\mathbb{V}^n}(t) - S^{\mathbb{V}}(t)|^2 \\
&= \int_0^t 2\sum_{i=1}^N \sum_{j\in N(i)} \Big(-\frac{1}{4}(S_i^{\mathbb{V}^n} - S_j^{\mathbb{V}^n})^2 + \frac{1}{4}(S_i^{\mathbb{V}} - S_j^{\mathbb{V}})^2)(S_i^{\mathbb{V}^n} - S_i^{\mathbb{V}}\Big)ds \\
&+ \int_0^t 2\sum_{i=1}^N (-\mathbb{V}_i^n + \mathbb{V}_i)(S_i^{\mathbb{V}^n} - S_i^{\mathbb{V}})ds \\
&+ \int_0^t 2\sum_{i=1}^N \sum_{j=1^N} (-\mathbb{W}_{ij}\rho_j^{\mathbb{V}^n} + \mathbb{W}_{ij}\rho_j^{\mathbb{V}})(S_i^{\mathbb{V}^n} - S_i^{\mathbb{V}})ds \\
&\le \int_0^t C(1+c)\Big(|S^{\mathbb{V}^n} - S^{\mathbb{V}}|^2 + |\sqrt{\rho^{\mathbb{V}^n}} - \sqrt{\rho^{\mathbb{V}}}|^2 + |\mathbb{V}^n - \mathbb{V}|^2\Big)ds.
\end{aligned}$$

The Gronwall's inequality, together with the above estimates, leads to

$$\mathbb{E}\Big[\Big|\sqrt{\rho^{\mathbb{V}^n}(t)} - \sqrt{\rho^{\mathbb{V}}(t)}\Big|^2 + |S^{\mathbb{V}^n}(t) - S^{\mathbb{V}}(t)|^2\Big] \le \exp^{\int_0^t C(1+c)ds} \int_0^t \mathbb{E}\big[|\mathbb{V}^n - \mathbb{V}|^2\big]ds.$$

Taking $n \to \infty$ and then $c \to \infty$, we achieve that

$$\begin{aligned}
&\lim_{c\to\infty}\lim_{n\to\infty} \int_0^T \mathbb{E}[1_{\Omega_c^n\cap\Omega_c}|u^{\mathbb{V}^n} - u^{\mathbb{V}}|^2]ds \\
&\quad \le \lim_{c\to\infty}\lim_{n\to\infty} \int_0^T \exp^{\int_0^t C(1+c)ds} \int_0^t \mathbb{E}\big[|\mathbb{V}^n - \mathbb{V}|^2\big]dsdt = 0.
\end{aligned}$$

Combining the estimate on $\Omega_c^n \cap \Omega_c$ and $\Omega/(\Omega_c^n \cap \Omega_c)$, we obtain the desired result. Similarly, one could also obtain the strong convergence of $u^{\mathbb{V}^n}$ in the topology $L^2(\Omega; \mathcal{C}([0,T]; \mathbb{C}^N))$. ∎

THEOREM 4.1. *Let $\beta \geq 0$. For the control problem* (18) *with the constraint* (11) *or* (14)*. there always exists an optimal control $\mathbb{V}^* \in \mathcal{U}$ which minimizes the objective functional J.*

Proof. By Lemma 4.1, to get the unique existence of an optimal control, it suffices to show the lower semicontinuity of F if $\beta > 0$, which can be obtained by using Proposition 4.1 and the Fatou lemma.

In the following, we show the existence of an optimal control when $\beta = 0$. Since $\gamma \sum_{i=1}^{N} |u_i^{\mathbb{V}}(T) - f_i^1|^2$ is bounded from below and $|\mathbb{V}_i| \leq \alpha$ in $\mathcal{U}$, the infimum of F exists. Let $(u^{\mathbb{V}^n}, \mathbb{V}^n)$ be a minimizing sequence. By the a priori estimate in Proposition 3.1, there exists a subsequence, still denoted by $\mathbb{V}^n$, such that $\mathbb{V}^n \to \mathbb{V}^*$ weakly in $L^2(\Omega \times [0,T]; \mathbb{R}^N)$. By Mazur's theorem, we have a sequence of convex combinations denoted by $\widetilde{\mathbb{V}}^m : \sum_{n\geq 1} \alpha_{nm} u_{n+m}$ with $\alpha_{nm} \geq 0, \sum_{n\geq 1} \alpha_{nm} = 1$ such that

$$\widetilde{\mathbb{V}}^m \to \mathbb{V}^*, \text{ strongly in } L^2(\Omega \times [0,T]; \mathbb{R}^N).$$

Using the fact that $|\widetilde{\mathbb{V}}_i^m| \leq \alpha$, it follows that $\mathbb{V}^* \in \mathcal{U}$. By Proposition 4.1, we also have the strong convergence, $u^{\widetilde{\mathbb{V}}^m} \to u^{\mathbb{V}^*}$ in $L^2(\Omega; \mathcal{C}([0,T]; \mathbb{C}^N))$. Therefore, $(u^{\mathbb{V}^*}, \mathbb{V}^*)$ is admissible. By making use of the convexity of $|u_i^{\mathbb{V}}(T) - f_i^1|^2, i \leq N$ and the Fatou lemma, we conclude that

$$J(u^{\mathbb{V}^*}) \leq \lim_{m\to\infty} J(\widetilde{\mathbb{V}}^m) \leq \lim_{m\to\infty} \sum_{n\geq 1} \alpha_{nm} J(\widetilde{\mathbb{V}}^m) \leq \inf_{\mathbb{V}\in\mathcal{U}} J(\mathbb{V}),$$

which completes the proof. ☐

From the above procedures, it can be seen that all the results in this subsection still hold as long as the cost functional has a lower bound and is lower semicontinuous convex.

4.2. Stochastic control problem with diffusion control. Similar to the linear potential control problem on graph, we can also obtain the existence of an optimal control problem with diffusion control which has not been reported even in the continuous case. Since the proof is similar to that of Theorem 4.1, we omit the details and only present the main result here.

Consider the constraint (11) with the control variable $\sigma \in \mathbb{R}^N$. The admissible control set $\widetilde{\mathcal{U}}$ is defined by

$$\widetilde{\mathcal{U}} := \Big\{ \sigma : \Omega \times [0,T] \to \mathbb{R}^N \;\Big|\; \sigma(t) \text{ is } \mathcal{F}_t\text{-adapted}, \sigma \in L^2([0,T]),$$
$$\text{there exists } \alpha > 0, \text{ such that } |\sigma_j| \leq \alpha \text{ a.s.} \Big\}.$$

Here the optimal control problem is to minimize the cost functional

$$J(\sigma) := \gamma \mathbb{E}\left[\sum_{i=1}^{N} |u_i^{\sigma}(T) - f_i^1|^2\right] + \beta \mathbb{E}\left[\int_0^T \sum_{i=1}^{N} |\sigma_i(t) - Z_i(t)|^2 dt\right], \tag{24}$$

where $\gamma, \beta \geq 0$, f^1 is $\mathcal{F}_T$-adapted satisfying $\|f^1\|_{L^2(\Omega;\mathbb{C}^N)} < \infty$, and $Z \in \widetilde{\mathcal{U}}$, u^{σ} is the solution of (11) with the control σ.

THEOREM 4.2. *For the control problem* (24) *with the constraint* (11)*, there always exists an optimal control $\sigma^* \in \widetilde{\mathcal{U}}$ which minimizes the objective functional J.*

Proof. By applying Proposition 3.1 and repeating the steps in the proof of Proposition 4.1, the lower continuity of J when $\beta = 0$ can be established. Therefore, the existence of optimal control is ensured by the convexity of $|u_j^\sigma(T) - f_j^1|^2$. When $\beta > 0$, the existence of optimal control can be guaranteed by Lemma 4.1. ∎

Similar to the linear potential control problem on graph, it can be seen that all the results in this subsection still hold as long as the cost functional has a lower bound and is lower semicontinuous convex.

5. Optimal condition for the stochastic control on graph. As has been pointed out in [27], compared to NLSEs driven by additive noise, it is more difficult to investigate the multiplicative noise case. Beyond that, for the NLSE on graph, the appearance of the nonlinear Laplacian Δ_G makes it more challenging to characterize the optimal condition than the continuous control problem.

In this section we mainly consider the following control problem:

$$\begin{aligned} J(\mathbb{V}) := \gamma \mathbb{E}\Big[\sum_{i=1}^N |u_i^{\mathbb{V}}(T) - f_i^1|^2\Big] + \beta_1 \mathbb{E}\Big[\int_0^T \sum_{i=1}^N |u_i^{\mathbb{V}}(t) - Z_i^1(t)|^2\Big] dt \\ + \beta \mathbb{E}\Big[\int_0^T \sum_{i=1}^N |\mathbb{V}_i(t) - Z_i(t)|^2 dt\Big] \end{aligned} \tag{25}$$

with the constraint (11) to illustrate how to derive the optimal condition on graph. Here $\gamma \geq 0, \beta_1 \geq 0, \beta \geq 0$, and Z^1 is an $\mathcal{F}_t$-adapted and L^2-integrable process. When $\beta_1 = 0$, (25) degenerates into (18). Our approach can be also extended to a more general smooth convex functional setting.

5.1. Gradient formula. In section 4, we have shown the existence of optimal potential and diffusion controls. Furthermore, in this part we study the necessary optimal condition near the minimizer $\mathbb{V}^*$ of (25) which is also called the gradient formula.

PROPOSITION 5.1. *Let* $(u^{\mathbb{V}^*}, \mathbb{V}^*)$ *be the solution and optimal control of* (25)*. Then for*

$$\sup_{t\in[0,T]} |\mathbb{V}^\epsilon(t) - \mathbb{V}^*(t)| \leq \epsilon, \quad \mathbb{V}^\epsilon \in \mathcal{U},$$

it holds that

$$\mathbb{E}\Big[1_{\Omega_c} \sup_{t\in[0,T]} |u^{\mathbb{V}^*}(t) - u^{\mathbb{V}^\epsilon}|^p\Big] \leq C(c, u(0), T, p)\epsilon^p, \tag{26}$$

where $p \geq 2$ *and* $\Omega_c = \{\sup_{i\leq N}\sup_{s\in[0,T]} \frac{1}{\rho_i^{\mathbb{V}^*}} + \sup_{i\leq N}\sup_{s\in[0,T]} \frac{1}{\rho_i^{\mathbb{V}^\epsilon}} \leq c\}$.

Furthermore, suppose there exists $c(\epsilon) \to \infty$ *such that the random variable* $C(\omega)$*, defined by* (22) *with* $\mathbb{V} \in \mathcal{U}$*, satisfies*

$$\lim_{\epsilon\to 0}\Big[C(c(\epsilon), u(0), T, 2)\epsilon + \frac{1}{\epsilon}\mathbb{P}\Big(C(\omega) \geq \frac{1}{c_1}(\log(c(\epsilon)) + \log(c_2))\Big)\Big] = 0; \tag{27}$$

then for any $\mathbb{V} \in \mathcal{U}$*, the following variational inequality holds:*

(28)

$$\lim_{c(\epsilon)\to\infty} \mathbb{E}\Big[1_{\Omega_{c(\epsilon)}} \Re\Big\{ \int_0^T \sum_{i=1}^N \Big((u_i^{\mathbb{V}^*}(t) - Z_i^1(t))\overline{X_i(t)} + (\mathbb{V}_i^*(t) - Z_i(t))(\mathbb{V}_i(t) - \mathbb{V}_i^*(t)) \Big) dt + \sum_{i=1}^N (u_i^{\mathbb{V}^*}(T) - f_i^1(T))\overline{X_i(T)} \Big\}\Big] \geq 0,$$

where X *is the solution of the following equation:*

(29)

$$\begin{aligned} dX_i(t) = &\Big\{ \frac{\mathbf{i}}{2} \sum_{j\in N(i)} \frac{\partial(\Delta_G u)_i}{\partial u_j}\Big|_{u=u^{\mathbb{V}^*}} X_j - \mathbf{i}\mathbb{V}_i^* X_i - \mathbf{i}\sum_{l=1}^N \mathbb{W}_{il} |u_l^{\mathbb{V}^*}|^2 X_i \\ &- 2\mathbf{i}\sum_{l=1}^N \mathbb{W}_{il} \Re(\bar{u}_l^{\mathbb{V}^*} X_l) u_i^{\mathbb{V}^*} \Big\} dt + \Big\{ -\mathbf{i}u_i^{\mathbb{V}^*}(\mathbb{V}_i - \mathbb{V}_i^*) \Big\} dt + \Big\{ -\mathbf{i}\sigma_i X_i \Big\} \circ dW(t) \\ X(0) = &\ 0. \end{aligned}$$

Proof. Since the admission control set $\mathcal{U}$ is convex, we can use a convex perturbation to illustrate the procedures. Consider $\mathbb{V}^\epsilon = (1-\epsilon)\mathbb{V}^* + \epsilon\mathbb{V}$. Define two processes $\xi(t) := \frac{u^{\mathbb{V}^\epsilon} - u^{\mathbb{V}}}{\epsilon}$ and $\delta\mathbb{V} := \mathbb{V} - \mathbb{V}^\epsilon$. Before the stopping time τ^c, according to the proof of Proposition 4.1, the equation of X_i is well-posed since the coefficients of (29) are globally Lipschitz. By the mean value theorem, ξ will satisfy

$$\begin{aligned} d\xi_i(t) = &\Big\{ \frac{\mathbf{i}}{2} \sum_{j\in N(i)} \int_0^1 \frac{\partial(\Delta_G u)_i}{\partial u_j}\Big|_{u=u^{\mathbb{V}^*}+\kappa\epsilon\xi} d\kappa \xi_j - \int_0^1 \mathbf{i}(\mathbb{V}_i^* + \kappa\epsilon\delta\mathbb{V}_i) d\kappa \xi_i \\ &- \sum_{l=1}^N \mathbf{i}\mathbb{W}_{il} \Big(\int_0^1 |u_l^{\mathbb{V}^*} + \kappa\epsilon\xi_i|^2 d\kappa \Big) \xi_i \\ &- 2\sum_{l=1}^N \mathbf{i}\mathbb{W}_{il} \int_0^1 \Re((\overline{u_l^{\mathbb{V}^*} + \epsilon\kappa\xi_l})\xi_l)(u_i^{\mathbb{V}^*} + \epsilon\kappa\xi_i) d\kappa \Big\} dt \\ &+ \int_0^1 \Big\{ -\mathbf{i}(u_i^{\mathbb{V}^*} + \epsilon\kappa\xi_i)(\mathbb{V}_i - \mathbb{V}_i^*) \Big\} d\kappa dt + \Big\{ -\mathbf{i}\sigma_i \xi_i \Big\} \circ dW(t). \end{aligned}$$

Using the similar steps in the proof of Proposition 4.1, on Ω_c, it holds that for any $p \geq 2$,

$$\mathbb{E}\Big[1_{\Omega_c} \sup_{t\in[0,T]} |\xi(t)|^p\Big] + \mathbb{E}\Big[1_{\Omega_c} \sup_{t\in[0,T]} |X(t)|^p\Big] \leq C(c, u(0), T)\mathbb{E}\Big[\Big(\int_0^T |\delta\mathbb{V}|^2 ds \Big)^{\frac{p}{2}}\Big]$$

and that for $p \geq 2$,

$$\mathbb{E}\Big[1_{\Omega_c} \sup_{t\in[0,T]} |\xi(t) - X(t)|^p\Big] \leq C(c, u(0), T, p).$$

Thus, (26) follows. Here $C(c, u(0), T, p)$ is increasing with respect to c satisfying $\lim_{c\to\infty} C(c, u(0), T, p) = +\infty$.

For convenience, let us denote

$$J_{\Omega_c}(\mathbb{V}) := \gamma \mathbb{E}\Big[1_{\Omega_c} \sum_{i=1}^N |u_i^{\mathbb{V}}(T) - f_i^1|^2\Big] + \beta_1 \mathbb{E}\Big[1_{\Omega_c} \int_0^T \sum_{i=1}^N |u^{\mathbb{V}}(t) - Z^1(t)|^2\Big] dt$$
$$+ \beta \mathbb{E}\Big[1_{\Omega_c} \int_0^T \sum_{i=1}^N |\mathbb{V}_i(t) - Z_i(t)|^2 dt\Big].$$

Due to the fact that $J(\mathbb{V}^*) \le J(\mathbb{V}^\epsilon)$, we obtain that

$$\begin{aligned} 0 &\le J(\mathbb{V}^\epsilon) - J(\mathbb{V}^*) \\ &= J_{\Omega_c}(\mathbb{V}^\epsilon) - J_{\Omega_c}(\mathbb{V}^*) + J_{\Omega/\Omega_c}(\mathbb{V}^\epsilon) - J_{\Omega/\Omega_c}(\mathbb{V}^*). \end{aligned}$$

Using the tail estimate of $1_{\Omega/\Omega_c}$ by the arguments in the proof of Proposition 4.1, we get

$$\lim_{c\to\infty} \lim_{\epsilon \to 0} J_{\Omega/\Omega_c}(\mathbb{V}^\epsilon) - J_{\Omega/\Omega_c}(\mathbb{V}^*) = 0.$$

To derive a necessary optimal condition, we need to consider the speed of the convergence for c and ϵ. By (23), we have that

$$J_{\Omega/\Omega_c}(\mathbb{V}^\epsilon) \le C\mathbb{P}\Big(C(\omega) \ge \frac{1}{c_1}(\log(c) + \log(c_2))\Big).$$

By the Taylor expansion and (26), we have

$$\begin{aligned} 0 &\le \frac{1}{\epsilon}\Big[J_{\Omega_c}(\mathbb{V}^\epsilon) - J_{\Omega_c}(\mathbb{V}^*)\Big] + \frac{1}{\epsilon}\Big[J_{\Omega/\Omega_c}(\mathbb{V}^\epsilon) - J_{\Omega/\Omega_c}(\mathbb{V}^*)\Big] \\ &\le \mathbb{E}\Big[1_{\Omega_c} \Re\Big\{ \int_0^T \sum_{i=1}^N \Big((u_i^{\mathbb{V}^*}(t) - Z_i^1(t))\overline{X_i(t)} + (\mathbb{V}_i^*(t) - Z_i(t))(\mathbb{V}_i(t) - \mathbb{V}_i^*(t))\Big) dt \\ &\quad + \sum_{i=1}^N (u_i^{\mathbb{V}^*}(T) - f_i^1(T))\overline{X_i(T)}\Big\}\Big] \\ &\quad + C(c, u(0), T, 2)\epsilon + \frac{1}{\epsilon} C\mathbb{P}\Big(C(\omega) \ge \frac{1}{c_1}(\log(c) + \log(c_2))\Big). \end{aligned}$$

Using the condition (27), there exists $c(\epsilon) \to \infty$ such that

$$\begin{aligned} &\lim_{c\to\infty} \mathbb{E}\Big[1_{\Omega_c} \Re\Big\{ \int_0^T \sum_{i=1}^N \Big((u_i^{\mathbb{V}^*}(t) - Z_i^1(t))\overline{X_i(t)} + (\mathbb{V}_i^*(t) - Z_i(t))(\mathbb{V}_i(t) - \mathbb{V}_i^*(t))\Big) dt \\ &\quad + \sum_{i=1}^N (u_i^{\mathbb{V}^*}(T) - f_i^1(T))\overline{X_i(T)}\Big\}\Big] \ge 0, \end{aligned}$$

which implies (28). ∎

Remark 5.1. If $\mathbb{V}^*$ is in the interior of $\mathcal{U}$, then (28) becomes the equality. In general, the limit with respect to c in (28) does not commute with the expectation since the variational equation (28) may not have a global estimate in the expectation sense and the coefficient is singular near boundary of $\mathcal{P}(G)$.

Our approach is also applicable for the cost functional

$$J(\mathbb{V}) = \mathbb{E}\Big[\int_0^T g(u^{\mathbb{V}}(t), \mathbb{V}(t))dt + h(u^{\mathbb{V}}(T))\Big] \tag{30}$$

or

$$J(\sigma) = \mathbb{E}\Big[\int_0^T g(u^\sigma(t), \sigma(t))dt + h(u^\sigma(T))\Big], \tag{31}$$

where g and h are continuous convex and differentiable with bounded first derivatives. Due to the page limitation, we omit the details here.

5.2. Backward SDE. In this subsection, we aim to give a more in-depth description on the optimal condition via the forward and backward stochastic differential equations. To better illustrate the procedure while clearly explaining the main idea, we use the control problem (25) with $\gamma = \beta = \beta_1 = 1$ as an example. To this end, we need a priori estimate of the variational solution X of (29) such that the limit with respect to c commutes with the expectation in (28).

PROPOSITION 5.2. *Let σ be a constant potential, i.e., $\sigma_i = \sigma_j$, and $\rho(0) \in \mathcal{P}_o(G)$, $S(0) \in \mathbb{R}^N$. Assume that $\mathbb{V} \in \mathcal{U}$. Then it holds that for $p \geq 2$,*

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^{\mathbb{V}}(t)\|^p\Big] \leq C(u(0), T, p, \alpha), \\ &\mathbb{E}\Big[\sup_{t\in[0,T]} \|X(t)\|^p\Big] \leq C(u(0), T, p, \alpha). \end{aligned} \tag{32}$$

Proof. According to Proposition 3.1 and the proof of Proposition 4.1, it suffices to prove a uniform lower bound estimate of the density function $\rho^{\mathbb{V}}(t) = |u^{\mathbb{V}}(t)|^2$. Since $\sigma_i = \sigma_j$, we denote $\sigma_i = \widetilde{\sigma}$. Introducing $\widetilde{S}_i = S_i + \widetilde{\sigma} W(t)$, (11) can be rewritten as

$$\begin{aligned} &d\rho_i = \sum_{j\in N(i)} \omega_{ij}(\widetilde{S}_i - \widetilde{S}_j)\theta_{ij}(\rho)dt; \\ &d\widetilde{S}_i + \Bigg(\frac{1}{2}\sum_{j\in N(i)} \omega_{ij}(\widetilde{S}_i - \widetilde{S}_j)^2 \frac{\partial \theta_{ij}}{\partial \rho_i} + \frac{1}{8}\frac{\partial}{\partial \rho_i} I(\rho) + \mathbb{V}_j + \sum_{j\in N(i)} \mathbb{W}_{ij}\rho_j\Bigg)dt = 0, \end{aligned}$$

which is a nonlinear Schrödinger equation with random inputs. Thus it follows that

$$\begin{aligned} \mathcal{H}(\rho(t), \widetilde{S}(t)) &:= \frac{1}{2}\langle \nabla_G \widetilde{S}, \nabla_G \widetilde{S}\rangle_{\theta(\rho(t))} + \mathcal{V}(\rho(t)) + \mathcal{W}(\rho(t)) + \frac{1}{8} I(\rho(t)) \\ &= \mathcal{H}(\rho(0), S(0)) < \infty, \text{ a.s.} \end{aligned}$$

The property of Fisher information yields that there exists a constant $c_{low} > 0$ such that

$$\inf_{t\geq 0} \min_{i\leq N} \rho_i(t) \geq c_{low} > 0, \text{a.s.}$$

Therefore, we have $\Omega_{\frac{1}{c_{low}}} = \Omega$ and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u^{\mathbb{V}}(t)\|^p\Big] \leq C(u(0), T, p, \alpha, c_{low}).$$

The lower bound of the density function also implies that the coefficient of (29) are bounded and Lipschitz. By repeating similar steps in the proof of Proposition 5.1, we complete the proof. ∎

Thanks to the lower bound estimate of the density function, we are also able to derive the corresponding backward stochastic differential equation, which is also called the adjoint equation of (29).

COROLLARY 5.1. *Let the condition of Proposition* 5.1 *hold. Let* $(u^{\mathbb{V}^*}, \mathbb{V}^*)$ *be an optimal control of* (25). *Then there exists an adapted solution* $(Y, \mathbb{Z})$ *of the following system:*

$$
\begin{aligned}
dY_i(t) = &-\Bigg\{ \frac{\mathbf{i}}{2} \sum_{i\in N(j)} \overline{\frac{\partial (\Delta_G u)_j}{\partial u_i}}\Big|_{u=u^{\mathbb{V}^*}} Y_j - \mathbf{i}\mathbb{V}_i^* Y_i - \mathbf{i}\sum_{l=1}^N \mathbb{W}_{il}|u_l^{\mathbb{V}^*}|^2 Y_i \\
&-2\sum_{l=1}^N \mathbb{W}_{il}\Re(\mathbf{i}u_l^{\mathbb{V}^*}\bar{Y}_l)u_i^{\mathbb{V}^*}\Bigg\} dt \\
&+\frac{1}{2}\sigma_i^2 Y_i(t)dt + \mathbf{i}\sigma_i\mathbb{Z}_i dt + 2(u_i^{\mathbb{V}^*} - Z_i^1)dt + \mathbb{Z}_i(t)dW(t), \\
Y(T) = &-2u^{\mathbb{V}^*}(T) + 2f_1(T).
\end{aligned} \tag{33}
$$

Proof. Thanks to Proposition 5.2, the coefficients of (33) are Lipschitz and bounded. Then the standard arguments in [37, section 3] yield the well-posedness of the linear BSDE (33), that is, there exists a unique adapted solution $(Y, \mathbb{Z})$. ◻

Based on the above results, we are ready to characterize the optimal condition by a coupled forward-backward SDE system.

THEOREM 5.1. *Let the condition of Proposition* 5.2 *hold. Then the optimal control pair* $(u^{\mathbb{V}^*}, \mathbb{V}^*)$ *satisfies the generalized stochastic Hamiltonian system consisting of* (11), (33) *with* $u(0) = \sqrt{\rho(0)}e^{\mathbf{i}S(0)}$, $Y(T) = -2u^{\mathbb{V}^*}(T) + 2f_1(T)$ *and the stationary condition, i.e., for arbitrary* $\mathbb{V}$,

$$
\Re\langle -\mathbf{i}u^{\mathbb{V}^*}Y + 2(\mathbb{V}^* - Z), \mathbb{V} - \mathbb{V}^*\rangle \geq 0, \text{ a.e. } t\in[0,T], \text{ a.s.}
$$

Proof. For convenience, let us denote $\Re\langle X, Y\rangle := \Re(\sum_{i=1}^N \bar{X}_i Y_i)$ and $\Re(a,b) = \Re(\bar{a}b)$. Applying Itô's formula, we obtain that

$$
\begin{aligned}
&d\Re\langle X(t), Y(t)\rangle \\
&= \sum_{i=1}^N \Bigg\{ \Re\Bigg(\frac{\mathbf{i}}{2}\sum_{j\in N(i)} \frac{\partial(\Delta_G u)_i}{\partial u_j}\Big|_{u=u^{\mathbb{V}^*}} X_j, Y_i\Bigg) - \Re(\mathbf{i}\mathbb{V}_i^* X_i, Y_i) \\
&\quad - \Re\Bigg(\mathbf{i}\sum_{l=1}^N \mathbb{W}_{il}|u_l^{\mathbb{V}^*}|^2 X_i, Y_i\Bigg) \\
&\quad - 2\Re\Bigg(\sum_{l=1}^N \mathbf{i}\mathbb{W}_{il}\Re(\bar{u}_l^{\mathbb{V}^*}X_l)u_i^{\mathbb{V}^*}, Y_i\Bigg)\Bigg\}dt + \sum_{i=1}^N \Bigg\{\Re\Bigg(-\frac{1}{2}\sigma_i^2 X_i, Y_i\Bigg) \\
&\quad + \sum_{i=1}^N \Re\Big(-\mathbf{i}u_i^{\mathbb{V}^*}(\mathbb{V}_i - \mathbb{V}_i^*), Y_i\Big)\Bigg\} dt \\
&\quad + \sum_{i=1}^n \Bigg\{\Re\Bigg(-\frac{\mathbf{i}}{2}\sum_{i\in N(j)} \overline{\frac{\partial(\Delta_G u)_j}{\partial u_i}\Big|_{u=u^{\mathbb{V}^*}}} Y_j, X_i\Bigg) + \Re\Big(\mathbf{i}\mathbb{V}_i^* Y_i, X_i\Big) \\
&\quad + \Re\Bigg(\mathbf{i}\sum_{l=1}^N \mathbb{W}_{il}|u_l^{\mathbb{V}^*}|^2 Y_i + 2\sum_{l=1}^N \mathbb{W}_{il}\Re(\mathbf{i}u_l^{\mathbb{V}^*}\bar{Y}_l)u_i^{\mathbb{V}^*}, X_i\Bigg)\Bigg\}dt \\
&\quad + \sum_{l=1}^N \Bigg\{\Re\Bigg(\frac{1}{2}\sigma_i^2 Y_i, X_i\Bigg) + \Re(\mathbf{i}\sigma_i\mathbb{Z}_i, X_i) + 2\Re\Big(u_i^{\mathbb{V}^*} - Z_i^1, X_i\Big)\Bigg\}dt
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{N}\Big\{\Re\Big(\mathbb{Z}_i(t),Y_i(t)\Big)+\Re\Big(-\mathbf{i}\sigma_i X_i,Z_i\Big)\Big\}dW(t)+\sum_{l=1}^{N}\Re\Big(-\mathbf{i}\sigma_i X_i,\mathbb{Z}_i\Big)dt\\
&=\sum_{i=1}^{N}\Re\Big(-\mathbf{i}u_i^{\mathbb{V}^*}(\mathbb{V}_i-\mathbb{V}_i^*),Y_i\Big)dt+\sum_{i=1}^{N}2\Re\Big(u_i^{\mathbb{V}^*}-Z_i^1,X_i\Big)dt+\sum_{l=1}^{N}\Big\{\Re\Big(\mathbb{Z}_i(t),Y_i(t)\Big)\\
&\quad+\Re\Big(-\mathbf{i}\sigma_i X_i,Z_i\Big)\Big\}dW(t).
\end{aligned}
$$

Taking expectation yields that

$$
\begin{aligned}
&-\mathbb{E}[2\Re\langle u^{\mathbb{V}}(T)-f^1(T),X(T)\rangle]=\mathbb{E}[\Re\langle X(T),Y(T)\rangle]-\mathbb{E}[\Re\langle X(0),Y(0)\rangle]\\
&=\int_0^T\mathbb{E}\Big[\Re\langle\mathbf{i}u^{\mathbb{V}^*}(\mathbb{V}-\mathbb{V}^*),Y\rangle+2\Re\langle u^{\mathbb{V}^*}-Z^1,X(t)\rangle\Big]dt.
\end{aligned}
$$

By using (28), Proposition 5.2, and Corollary 5.1, we obtain

$$
\begin{aligned}
0&\le\mathbb{E}\Big[\Big\{\int_0^T\sum_{i=1}^{N}2\Big((u_i^{\mathbb{V}^*}(t)-Z_i^1(t))\overline{X_i(t)}+(\mathbb{V}_i^*(t)-Z_i(t))(\mathbb{V}_i(t)-\mathbb{V}_i^*(t))\Big)dt\\
&\quad+\sum_{i=1}^{N}2(u_i^{\mathbb{V}^*}(T)-f_i^1(T))\overline{X_i(T)}\Big\}\Big]\\
&=\int_0^T\mathbb{E}\Big[-\Re\langle\mathbf{i}u^{\mathbb{V}^*}Y,\mathbb{V}-\mathbb{V}^*\rangle+2\Re\langle\mathbb{V}^*-Z,\mathbb{V}-\mathbb{V}^*\rangle\Big]dt.
\end{aligned}
$$

Thus for arbitrary $\mathbb{V}$, we conclude that

$$
\Re\langle-\mathbf{i}u^{\mathbb{V}^*}Y+2(\mathbb{V}^*-Z),\mathbb{V}-\mathbb{V}^*\rangle\ge 0,\ \text{a.e.}\ t\in[0,T],\ \text{a.s.}\qquad\square
$$

Theorem 5.1 can be also viewed as the Pontryagin's maximum principle. Based on the above theorem, we propose the corresponding forward-backward stochastic differential equation (FBSDE) for (25),

(34)

$$
\begin{aligned}
&\mathbf{i}du_j=-\frac{1}{2}(\Delta_G u)_j dt+u_j\mathbb{V}_j dt+u_j\sum_{l=1}^{N}\mathbb{W}_{jl}|u_l|^2dt+\sigma_j u_j\circ dW_t,\\
&dY_i(t)=-\Bigg\{\frac{\mathbf{i}}{2}\sum_{i\in N(j)}\frac{\partial(\Delta_G u)_j}{\partial u_i}\Big|_{u=u^{\mathbb{V}}}Y_j-\mathbf{i}\mathbb{V}_iY_i-\mathbf{i}\sum_{l=1}^{N}\mathbb{W}_{il}|u_l^{\mathbb{V}}|^2Y_i\\
&\qquad\qquad-2\mathbf{i}\sum_{l=1}^{N}\mathbb{W}_{il}\Re(\bar{u}_l^{\mathbb{V}}Y_l)u_i^{\mathbb{V}}\Bigg\}dt\\
&\qquad+\frac{1}{2}\sigma_i^2Y_i(t)dt+\mathbf{i}\sigma_i\mathbb{Z}_idt+2(u_i^{\mathbb{V}}-Z_i^1)dt+\mathbb{Z}_i(t)dW(t),\\
&u(0)=\sqrt{\rho(0)}e^{\mathbf{i}S(0)},\ Y(T)=-2u^{\mathbb{V}^*}(T)+2f_1(T),\ \Re\langle\mathbf{i}u^{\mathbb{V}^*}Y+2(\mathbb{V}^*-Z),\mathbb{V}-\mathbb{V}*\rangle=0.
\end{aligned}
$$

If the control problem (25) admits a unique optimal control, and the stochastic generalized FBSDE also admits a unique adapted solution $(u,Y,\mathbb{Z})$, then u is the optimal state process and the corresponding control $\mathbb{V}$ is optimal.

We could also present the Pontryagin's maximum principle for (24) with the constraint (11) and the diffusion control $\sigma_i = \sigma_j$, $i,j \le N$.

THEOREM 5.2. *Let the condition of Proposition* 5.2 *hold. Then the optimal control pair* (u^{σ^*}, σ^*) *satisfies the generalized stochastic Hamiltonian system consisting of* (11), *and*

$$
\begin{aligned}
(35) \qquad dY_i(t) = &-\Bigg\{ \frac{\mathbf{i}}{2} \sum_{i\in N(j)} \overline{\frac{\partial (\Delta_G u)_j}{\partial u_i}\Big|_{u=u^{\sigma^*}}} Y_j - \mathbf{i}\mathbb{V}_i Y_i - \mathbf{i}\sum_{l=1}^{N} \mathbb{W}_{il} |u_l^{\sigma^*}|^2 Y_i \\
&\quad -2\sum_{l=1}^{N} \mathbb{W}_{il} \Re(\mathbf{i} u_l^{\sigma^*} \bar{Y}_l) u_i^{\sigma^*} \Bigg\} dt \\
&+ \frac{1}{2}\sigma_i^2 Y_i(t)dt + \mathbf{i}\sigma_i \mathbb{Z}_i dt + 2(u_i^{\sigma^*} - Z_i^1)dt + \mathbb{Z}_i(t) dW(t), \\
Y(T) = &-2u^{\sigma^*}(T) + 2f_1(T),
\end{aligned}
$$

with $u(0) = \sqrt{\rho(0)} e^{\mathbf{i}S(0)}$, $Y(T) = -2u^{\sigma^*}(T) + 2f_1(T)$, *and the stationary condition*

$$
\Re\langle -\sigma u^\sigma Y - \mathbf{i} u^\sigma Z + 2(\sigma - Z), \sigma - \sigma* \rangle \ge 0 \text{ a.e. } t \in [0,T], \text{ a.s.}
$$

Proof. The proof is similar to that of Theorem 5.1. By applying Propositions 5.2 and using Itô's formula to $\Re\langle X(t), Y(t)\rangle$, we have that

$$
\int_0^T \mathbb{E}\Big[\Re\langle -\sigma^* u^{\sigma^*} Y - \mathbf{i} u^{\sigma*} Z^*, \sigma - \sigma^* \rangle + 2\Re\langle \sigma^* - Z, \sigma - \sigma^* \rangle \Big] dt \ge 0,
$$

which completes the proof. ∎

In general, if the cost functional is (30) or (31), the similar result still holds. We omit this part due to the page limitation. Besides, it can be seen that if the V^* (or σ^*) is achieved in the interior of $\mathcal{U}$ (or $\widetilde{\mathcal{U}}$), then the stationary condition could be simplified to an equality.

6. Conclusion. In this paper, we propose the stochastic control problem subject to stochastic nonlinear Schrödinger equation on graph with either a linear potential or diffusion control. From the numerical viewpoint, we demonstrate the particular features, such as the stochastic dispersion relationship, mass conservation law, and moment bounds of energy of stochastic nonlinear Schrödinger on graph. Furthermore, we provide the gradient formula and the Pontryagin's maximum principle for stochastic nonlinear Schrödinger equation on graph driven by multiplicative noise. These may serve as a foundation of the numerical computation for stochastic control of stochastic nonlinear Schrödinger equation in a continuous domain as well (see, e.g., [13]).

There are many interesting questions that remain to be tackled. For instance, it will be more difficult to investiagete the nonlinear potential and diffusion controls of the stochastic nonlinear Schrödinger equation driven by general multiplicative noise. Given the solutions of the FBSDEs, can this stationary condition uniquely determine the optimal control for stochastic nonlinear Schrödinger equation on graph? The stochastic control problem over density manifold, such as the mean-field game involved with the Fisher information or nonmonotone coefficient, is challenging. Besides, the numerical computation has not been addressed in the current work. We plan to explore these issues in the future work.

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